M.-A. Inaba
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ON THE MODULI OF STABLE SHEAVES ON A REDUCIBLE PROJECTIVE SCHEME AND EXAMPLES ON A REDUCIBLE QUADRIC SURFACE

MICHI-AKI INABA

Abstract. We study the moduli space of stable sheaves on a reducible projective scheme by use of a suitable stratification of the moduli space. Each stratum is the moduli space of "triples", which is the main object investigated in this paper. As an application, we can see that the relative moduli space of rank two stable sheaves on quadric surfaces gives a nontrivial example of the relative moduli space which is not flat over the base space.

Introduction

Simpson has constructed the moduli scheme of stable sheaves on an arbitrary projective scheme ([17]). This result causes us to study many examples of moduli spaces of stable sheaves on degenerate varieties. A typical example is the moduli space of sheaves on a nodal curve studied by Seshadri ([16]). He showed that the moduli space has the singularity similar to that of base curve. On the other hand, Gieseker and Li used the moduli space of stable sheaves on a reducible surface in order to prove the irreducibility of the moduli space of rank 2 stable bundles on a smooth surface ([3]). This result tells us an importance of the study of the moduli spaces of stable sheaves on reducible schemes.

In this paper we shall study the moduli space of stable sheaves on a reducible projective scheme $X = X_1 \cup X_2$ such that X_1 and X_2 are purely d-dimensional and $Y := X_1 \cap X_2$ is a Cartier divisor of X_1 and X_2. In order to study the moduli space, we shall use a generalization of the method of Nagaraj and Seshadri ([11]), which was used on reducible curves. We shall show that there is a bijective correspondence between the purely d-dimensional coherent sheaves on X and the triples $(E_1, \tilde{E}_2, f)$ on X, where E_1 is a purely d-dimensional sheaf on X_1, $\tilde{E}_2$ a purely d-dimensional sheaf on X_2 and $f : E_1|_Y \to \tilde{E}_2|_Y$ a homomorphism. In Theorem 1.10, we will

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show that the moduli space of triples exists and that the moduli spaces of triples give a stratification of the moduli space of stable sheaves on X. This stratification is a valuable tool for the study of the moduli space of stable sheaves on a reducible scheme. In fact we can investigate a more precise structure of the moduli space of triples from the construction given in Theorem 2.1. From this construction, one sees that the moduli space of triples has a fibration in étale topology whose fibers are open subschemes of projective spaces. Although the dimension of the fibers of this fibration is constant in the case of reducible curves, the dimension of fibers may jump in higher dimensional case (Remark 2.2). On the other hand, there is a bijective correspondence between the purely d-dimensional coherent sheaves on X and the "parabolic triples" defined in Definition 3.1. The moduli space of parabolic triples is an intersection of two moduli spaces of triples.

As an application of the study of the moduli space of triples, we can construct a non-trivial example of the relative moduli space of stable sheaves which is not flat over the base scheme. We shall see an example of the decomposition of the moduli space of rank 2 stable sheaves on a reducible quadric surface by the moduli spaces of triples (Theorem 5.1). From this, one sees that there are components of the moduli space whose dimension is jumping. Moreover we shall study the deformations of sheaves on reducible surfaces and apply it to degenerations of quadric surfaces. Then we can see that "general" points of the moduli space on a reducible quadric surface are contained in the limits of stable sheaves on smooth quadric surfaces (Theorem 5.4 and Conclusion 5.5). From this point of view, one recognizes that the concept of stable sheaf introduced by [17] makes a good sense in this case.

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Notation and convention

Let X be a projective scheme over a noetherian scheme S, $\mathcal{O}_X(1)$ an S-very ample invertible sheaf and E a coherent sheaf on X. (Sch/S) denotes the category of locally noetherian schemes over S and (Sets) the category of sets. For an integer m, $E(m)$ denotes $E \otimes \mathcal{O}_X(m)$. If s is a point of S, then we denote the fiber of X over s by X_s, $E \otimes k(s)$ by $E(s)$, $\dim H^i(X_s, E(s))$ by $h^i(E(s))$ and $\sum_{i \geq 0}(-1)^i h^i(E(s))$ by $\chi(E(s))$. For a morphism $g : T \to S$ of schemes, E_T denotes the sheaf $(1_X \times g)^*(E)$ on $X \times_S T$. If F is a coherent sheaf on S, $\mathbf{P}(F)$ means $\operatorname{Proj} S(F)$ and $\mathbf{V}(F)$ means $\operatorname{Spec} S(F)$, where $S(F)$ is the symmetric algebra of F over $\mathcal{O}_S$. For a polynomial $H(x)$ and an integer m, $H[m](x)$ is the shift of $H(x)$ by m.

§1. Fundamental properties of the moduli spaces of stable sheaves on reducible projective schemes

Throughout this paper, we fix an algebraically closed field k.

Definition 1.1. Let E be a non-zero coherent sheaf on an algebraic scheme S over k. Then E is said to be of pure dimension d if $\dim(\operatorname{Supp} F) = d$ for any non-zero coherent subsheaf F of E.

In this section we will consider a projective scheme X over k with the following properties:

(†) $X = X_1 \cup X_2$ where X_i $(i = 1, 2)$ are closed subschemes of X such that $\mathcal{O}_{X_i}$ are of pure dimension d for $i = 1, 2$ and $I_{X_1} \cap I_{X_2} = 0$, where I_{X_i} is the ideal sheaf of $\mathcal{O}_X$ corresponding to the closed subscheme X_i, and $Y := X_1 \cap X_2$ is a Cartier divisor of X_1 and X_2 at the same time.

Note that there is a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Since $\mathcal{O}_{X_1}$ and $\mathcal{O}_{X_2}$ are of pure dimension d, $\mathcal{O}_X$ is also of pure dimension d.

We will give a description of purely d-dimensional sheaves on X by data on X_1, X_2. Let E be a coherent sheaf of pure dimension d on X. Put

$$E^{(1)} := (E|_{X_1})/(E|_{X_1})_{\mathrm{tor}}\,, \qquad E^{(2)} := (E|_{X_2})/(E|_{X_2})_{\mathrm{tor}},$$

where $(E|_{X_i})_{\mathrm{tor}}$ is the coherent subsheaf of $E|_{X_i}$ such that $\dim \operatorname{Supp}(E|_{X_i})_{\mathrm{tor}} < d$ and $(E|_{X_i})/(E|_{X_i})_{\mathrm{tor}}$ is of pure dimension d. There is a canonical injection $i : E \to E^{(1)} \oplus E^{(2)}$. If we put $E^{(0)} := \operatorname{coker} i$, then we have the following exact sequence;

$$0 \longrightarrow E \xrightarrow{\ i\ } E^{(1)} \oplus E^{(2)} \longrightarrow E^{(0)} \longrightarrow 0.$$

Lemma 1.2. *$E^{(0)}$ is an $\mathcal{O}_Y$-module.*

Proof. The composite $I_{X_2} \hookrightarrow \mathcal{O}_X \to \mathcal{O}_{X_1}$ is injective and the image is just the ideal sheaf $\mathcal{O}_{X_1}(-Y)$ corresponding to the Cartier divisor Y of X_1. Tensoring E to the injection $I_{X_2} \hookrightarrow \mathcal{O}_X$, we have a homomorphism

$$E|_{X_1} \otimes \mathcal{O}_{X_1}(-Y) \cong E \otimes I_{X_2} \longrightarrow E.$$

Since E is of pure dimension d, the above homomorphism induces the homomorphism $E^{(1)} \otimes \mathcal{O}_{X_1}(-Y) \to E$. The composition $E^{(1)} \otimes \mathcal{O}_{X_1}(-Y) \to E \to E^{(1)}$ is just the canonical homomorphism. Note that $E^{(1)} \otimes \mathcal{O}_{X_1}(-Y) \to E^{(1)}$ is injective since $E^{(1)}$ is of pure dimension d. We denote $E^{(1)} \otimes \mathcal{O}_{X_1}(-Y)$ by $E^{(1)}(-Y)$. Similarly we have a canonical injection $E^{(2)}(-Y) \hookrightarrow E$. Let (a, b) be a local section of $E^{(1)} \oplus E^{(2)}$ and c a local section of I_Y, where I_Y is the ideal sheaf of $\mathcal{O}_X$ corresponding to the closed subscheme Y of X. Then c can be written as $c = c_1 + c_2$, where $c_1 \in I_{X_1}$ and $c_2 \in I_{X_2}$. So $c \cdot (a, b) = (c_2 a, c_1 b)$. Since $c_2 a \in E^{(1)}(-Y) \subset E$ and $c_1 b \in E^{(2)}(-Y) \subset E$, we have $c \cdot (a, b) \in E$. Hence $I_Y(E^{(1)} \oplus E^{(2)}) \subset E$ and so $I_Y E^{(0)} = 0$. $\square$

Let p be the composition:

$$\begin{array}{cccccc} p : E^{(2)} & \hookrightarrow & E^{(1)} \oplus E^{(2)} & \longrightarrow E^{(0)}. \\ a & \mapsto & (0, a) & \end{array}$$

Then we can easily see that p is surjective. Now we put $\tilde{E}^{(2)} := \ker p \otimes \mathcal{O}_{X_2}(Y)$. Since $E^{(0)}$ is an $\mathcal{O}_Y$-module, the composite

$$E^{(2)}(-Y) \hookrightarrow E^{(2)} \xrightarrow{p} E^{(0)}$$

is zero. So we have a factorization

$$E^{(2)}(-Y) \hookrightarrow \tilde{E}^{(2)}(-Y) = \ker p \hookrightarrow E^{(2)}.$$

Tensoring $\mathcal{O}_{X_2}(Y)$ to the injection $E^{(2)}(-Y) \hookrightarrow \tilde{E}^{(2)}(-Y)$, we have an injection $E^{(2)} \hookrightarrow \tilde{E}^{(2)}$ and the composition $\tilde{E}^{(2)}(-Y) \hookrightarrow E^{(2)} \hookrightarrow \tilde{E}^{(2)}$ is just the canonical injection obtained from the injection $\mathcal{O}_{X_2}(-Y) \hookrightarrow \mathcal{O}_{X_2}$. Since the composite $\ker p = \tilde{E}^{(2)}(-Y) \to E^{(2)} \to \tilde{E}^{(2)} \to \tilde{E}^{(2)}|_Y$ is zero, there exists a homomorphism $j : E^{(0)} \to \tilde{E}^{(2)}|_Y$ such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 \longrightarrow & \ker p & \longrightarrow & E^{(2)} & \xrightarrow{p} & E^{(0)} & \longrightarrow 0 \\ & & & \downarrow & & \downarrow j & \\ & & & \tilde{E}^{(2)} & \to & \tilde{E}^{(2)}|_Y. & \end{array}$$

Let us consider the composition $f_E : E^{(1)}|_Y \to E^{(0)} \xrightarrow{j} \tilde{E}^{(2)}|_Y$ and put

$$\varphi_{f_E} : E^{(1)} \oplus \tilde{E}^{(2)} \to \tilde{E}^{(2)}|_Y; \quad \varphi_{f_E}(a,b) := f_E(a|_Y) - b|_Y.$$

We have a canonical injection $E \hookrightarrow E^{(1)} \oplus E^{(2)} \hookrightarrow E^{(1)} \oplus \tilde{E}^{(2)}$.

LEMMA 1.3. *With respect to the above injection $E \hookrightarrow E^{(1)} \oplus \tilde{E}^{(2)}$, we have $E = \ker \varphi_{f_E}$.*

Proof. Since the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & E & \longrightarrow & E^{(1)} \oplus E^{(2)} & \longrightarrow & E^{(0)} & \longrightarrow 0 \\ & & & \downarrow & & \downarrow j & \\ & & & E^{(1)} \oplus \tilde{E}^{(2)} & \xrightarrow{\varphi_{f_E}} & \tilde{E}^{(2)}|_Y & \end{array}$$

commutes, the composite $E \hookrightarrow E^{(1)} \oplus \tilde{E}^{(2)} \xrightarrow{\varphi_{f_E}} \tilde{E}^{(2)}|_Y$ is zero. So we have the inclusion $E \subset \ker \varphi_{f_E}$. Conversely let $(a,b) \in \ker \varphi_{f_E} \subset E^{(1)} \oplus \tilde{E}^{(2)}$ be a local section. There exists a local section $\alpha \in E$ such that $\alpha|_{X_1} = a$ in $E^{(1)}$. If we put $b' := b - \alpha|_{X_2} (\in \tilde{E}^{(2)})$, then $(0,b') = (a,b) - i(\alpha) \in \ker \varphi_{f_E}$ and so we have only to prove that $(0,b') \in E$. Since $0 = \varphi_E(0,b') = b'|_Y$ in $\tilde{E}^{(2)}|_Y$, $b' \in \tilde{E}^{(2)}(-Y) = \ker p \subset E^{(2)}$. Hence $(0,b')$ is in the kernel of the homomorphism $E^{(1)} \oplus E^{(2)} \to E^{(0)}$ and so $(0,b') \in E$. ∎

DEFINITION 1.4. Let E_1 (resp. $\tilde{E}_2$) be a coherent sheaf of pure dimension d on X_1 (resp. X_2). Let $f : E_1|_Y \to \tilde{E}_2|_Y$ be a homomorphism. Then we call $(E_1, \tilde{E}_2, f)$ a triple. Two triples $(E_1, \tilde{E}_2, f)$, $(E_1', \tilde{E}_2', f')$ are said to be isomorphic if there exist isomorphisms $g_1 : E_1 \xrightarrow{\sim} E_1'$, $g_2 : \tilde{E}_2 \xrightarrow{\sim} \tilde{E}_2'$ such that the diagram

$$\begin{array}{ccc} E_1|_Y & \xrightarrow{f} & \tilde{E}_2|_Y \\ g_1|_Y \downarrow \wr & & g_2|_Y \downarrow \wr \\ E_1'|_Y & \xrightarrow{f'} & \tilde{E}_2'|_Y \end{array}$$

commutes.

The following proposition is a generalization of [[11], Lemma 2.3].

PROPOSITION 1.5. *$E \mapsto (E^{(1)}, \tilde{E}^{(2)}, f_E)$ is a bijective correspondence between the isomorphism classes of coherent sheaves of pure dimension d on X and the isomorphism classes of triples.*

Proof. Take a triple $(E_1, \tilde{E}_2, f)$. Put

$$\varphi_f : E_1 \oplus \tilde{E}_2 \to \tilde{E}_2|_Y; \quad \varphi_f(a,b) := f(a|_Y) - b|_Y.$$

We will show that $(E_1, \tilde{E}_2, f) \mapsto \ker \varphi_f$ gives the inverse map. If we put $E := \ker \varphi_f$, then we can construct the triple $(E^{(1)}, \tilde{E}^{(2)}, f_E)$. Since we know Lemma 1.3, we have only to prove that $(E^{(1)}, \tilde{E}^{(2)}, f_E) \cong (E_1, \tilde{E}_2, f)$. From the definition of $E^{(1)}$, the homomorphism $E \to E_1$ factors through $E^{(1)}$:

$$E \longrightarrow E^{(1)} \overset{\sigma_1}{\to} E_1.$$

By construction $E \to E_1$ is surjective. On the other hand, σ_1 is injective since it is injective on $X_1 \setminus Y$ and $E^{(1)}$ is of pure dimension d. Thus $\sigma_1 : E^{(1)} \to E_1$ is an isomorphism. Put $\psi : \tilde{E}_2(-Y) \hookrightarrow \tilde{E}_2 \hookrightarrow E_1 \oplus \tilde{E}_2$. Then ψ factors through E. Let $\psi' : \tilde{E}_2(-Y) \to E$ be the induced homomorphism. Then

$$0 \longrightarrow \tilde{E}_2(-Y) \overset{\psi'}{\longrightarrow} E \longrightarrow E_1 \longrightarrow 0$$

becomes an exact sequence. On the other hand

$$0 \longrightarrow \tilde{E}^{(2)}(-Y) \longrightarrow E \longrightarrow E^{(1)} \longrightarrow 0$$

is also exact. So we have an isomorphism $\tilde{E}^{(2)}(-Y) \overset{\sim}{\to} \tilde{E}_2(-Y)$ with the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \tilde{E}^{(2)}(-Y) & \longrightarrow & E & \longrightarrow & E^{(1)} & \longrightarrow 0 \\
& \downarrow \wr & & \| & & \wr \downarrow \sigma_1 & \\
0 \longrightarrow & \tilde{E}_2(-Y) & \longrightarrow & E & \longrightarrow & E_1 & \longrightarrow 0.
\end{array}$$

This isomorphism induces the isomorphism $\tilde{E}^{(2)} \overset{\sigma_2}{\to} \tilde{E}_2$ and the diagram

$$\begin{array}{ccccc}
\tilde{E}^{(2)}(-Y) & \longrightarrow & E & \longrightarrow & \tilde{E}^{(2)} \\
\downarrow \wr & & \| & & \wr \downarrow \sigma_2 \\
\tilde{E}_2(-Y) & \longrightarrow & E & \longrightarrow & \tilde{E}_2
\end{array}$$

commutes. Since the diagrams

$$\begin{array}{ccccc}
E & \longrightarrow & E^{(2)} & \to & \tilde{E}_2 \\
& \searrow & & & \downarrow \\
& & E_1|_Y & \overset{f}{\longrightarrow} & \tilde{E}_2|_Y
\end{array}
\qquad
\begin{array}{ccccc}
E & \longrightarrow & E^{(2)} & \to & \tilde{E}^{(2)} \\
& \searrow & & & \downarrow \\
& & E^{(1)}|_Y & \overset{f_E}{\longrightarrow} & \tilde{E}^{(2)}|_Y
\end{array}$$

both commute, we can see that the diagram

$$\begin{array}{ccc} E^{(1)}|_Y & \xrightarrow{f_E} & \tilde{E}^{(2)}|_Y \\ \wr \downarrow \sigma_1 & & \wr \downarrow \sigma_2 \\ E_1|_Y & \xrightarrow{f} & \tilde{E}_2|_Y. \end{array}$$

commutes. Thus we have an isomorphism $(E^{(1)}, \tilde{E}^{(2)}, f_E) \cong (E_1, \tilde{E}_2, f)$. □

Now let us recall the definition of stable sheaf on X. We fix a very ample line bundle $\mathcal{O}_X(1)$ on X. Then the Hilbert polynomial of a coherent sheaf E on X with respect to $\mathcal{O}_X(1)$ can be written as

$$\chi(E(m)) = \sum_{i=0}^{d} a_i(E) \binom{m+d-i}{d-i},$$

with $a_i(E)$ integers. We put $\mu^S(E) := a_1(E)/a_0(E)$.

Definition 1.6. Let E be a coherent sheaf of pure dimension d on X. E is said to be stable (resp. semi-stable) if for any coherent subsheaf F of E with $0 < a_0(F) < a_0(E)$,

$$\chi(F(m))/a_0(F) < \chi(E(m))/a_0(E) \\ \text{(resp. } \leq)$$

for all sufficiently large integers m.

Let (Sch$/k$) be the category of locally noetherian schemes over k. Let $H, H_1, H_2,\ H'_1, H'_2$ be numerical polynomials of degree d such that $H = H_1 + H'_2$.

Definition 1.7. We define a functor $\mathcal{M}_X^H : (\mathrm{Sch}/k) \to (\mathrm{Sets})$ by

$$\mathcal{M}_X^H(T) := \left\{ E \,\middle|\, \begin{array}{l} E \text{ is a } T\text{-flat coherent sheaf on } X \times T \\ \text{such that } \chi(E(t)(m)) = H(m) \text{ and} \\ E(t) \text{ is a stable sheaf for all } t \in T \end{array} \right\} \Big/ \sim$$

where $E \sim E'$ if and only if $E \cong E' \otimes L$ for some line bundle L on T.

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We define a functor $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2} : (\mathrm{Sch}/k) \to (\mathrm{Sets})$ by

$$\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}(T) := \left\{ (E_1,\tilde{E}_2,f) \;\middle|\; \begin{array}{l} E_1 \text{ is a } T\text{-flat coherent } \mathcal{O}_{X_1\times T}\text{-module,} \\ \tilde{E}_2 \text{ is a } T\text{-flat coherent } \mathcal{O}_{X_2\times T}\text{-module} \\ \text{and } f : E_1|_{Y\times T} \to \tilde{E}_2|_{Y\times T} \text{ is a homo-} \\ \text{morphism with the property } (*) \end{array} \right\} \Big/ \sim$$

where $(E_1,\tilde{E}_2,f) \sim (E'_1,\tilde{E}'_2,f')$ if and only if there exist a line bundle L on T and isomorphisms $g_1 : E_1 \xrightarrow{\sim} E'_1 \otimes L$, $g_2 : \tilde{E}_2 \xrightarrow{\sim} \tilde{E}'_2 \otimes L$ with the following commutative diagram:

$$\begin{array}{ccc} E_1|_{Y\times T} & \xrightarrow{f} & \tilde{E}_2|_{Y\times T} \\ {\scriptstyle g_1|_{Y\times T}} \downarrow \wr & & {\scriptstyle g_2|_{Y\times T}} \downarrow \wr \\ E'_1|_{Y\times T} \otimes L & \xrightarrow{f'} & \tilde{E}'_2|_{Y\times T} \otimes L. \end{array}$$

$(*)$ For any $t \in T$, $E_1(t)$ and $\tilde{E}_2(t)$ are of pure dimension d, $\chi(E_1(t)(m)) = H_1(m)$, $\chi(\tilde{E}_2(t)(m)) = H_2(m)$, $\chi(E_1(t)(-Y)(m)) = H'_1(m)$, $\chi(\tilde{E}_2(t)(-Y)(m)) = H'_2(m)$ and $\ker \varphi_{f(t)}$ is a stable sheaf on $X \times k(t)$, where $\varphi_{f(t)} : E_1(t) \oplus \tilde{E}_2(t) \to \tilde{E}_2(t)|_{Y\times k(t)}$ is the homomorphism defined by $\varphi_{f(t)}(a,b) := f(t)(a|_Y) - b|_Y$.

We can similarly define a functor $\mathcal{M}^{(2),H_1,H_2}_{X,H'_1,H'_2}$:

$$\mathcal{M}^{(2),H_1,H_2}_{X,H'_1,H'_2}(T) := \left\{ (\tilde{E}_1,E_2,f) \;\middle|\; \begin{array}{l} E_2 \text{ is a } T\text{-flat coherent } \mathcal{O}_{X_2\times T}\text{-module,} \\ \tilde{E}_1 \text{ is a } T\text{-flat coherent } \mathcal{O}_{X_1\times T}\text{-module} \\ \text{and } f : E_2|_{Y\times T} \to \tilde{E}_1|_{Y\times T} \text{ is a homo-} \\ \text{morphism with the property } (*') \end{array} \right\} \Big/ \sim$$

where $\sim$ is the equivalence relation defined similarly to that of $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}$.

$(*')$ For any $t \in T$, $E_2(t)$ and $\tilde{E}_1(t)$ are of pure dimension d, $\chi(E_2(t)(m)) = H_2(m)$, $\chi(\tilde{E}_1(t)(m)) = H_1(m)$, $\chi(E_2(t)(-Y)(m)) = H'_2(m)$, $\chi(\tilde{E}_1(t)(-Y)(m)) = H'_1(m)$ and $\ker \varphi_{f(t)}$ is a stable sheaf on $X \times k(t)$, where $\varphi_{f(t)} : E_2(t) \oplus \tilde{E}_1(t) \to \tilde{E}_1(t)|_{Y\times k(t)}$ is the homomorphism defined by $\varphi_{f(t)}(a,b) := f(t)(a|_Y) - b|_Y$.

Let $(E_1, \tilde{E}_2, f)$ be an element of $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}(T)$. If we put

$$\varphi_f : E_1 \oplus \tilde{E}_2 \longrightarrow \tilde{E}_2|_{Y\times T}; \quad \varphi_f(a,b) := f(a|_{Y\times T}) - b|_{Y\times T},$$

then $\ker\varphi_f$ is an element of $\mathcal{M}^H_X(T)$. So we can define a morphism of functors:

$$\Phi : \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2} \longrightarrow \mathcal{M}^H_X; \quad (E_1, \tilde{E}_2, f) \mapsto \ker\varphi_f.$$

PROPOSITION 1.8. *The morphism $\Phi : \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2} \to \mathcal{M}^H_X$ is a monomorphism. Moreover for any noetherian scheme S over k and for any morphism of functors $h_S \to \mathcal{M}^H_X$, $h_S \times_{\mathcal{M}^H_X} \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}$ is representable by a subscheme S_0 of S.*

Proof. First we will prove that Φ is a monomorphism. Take T-valued points $(E_1, \tilde{E}_2, f)$, $(E'_1, \tilde{E}'_2, f')$ of $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1 H'_2}$ such that $\Phi((E_1, \tilde{E}_2, f)) = \Phi((E'_1, \tilde{E}'_2, f'))$. Let $\varphi_f : E_1 \oplus \tilde{E}_2 \to \tilde{E}_2|_{Y\times T}$ and $\varphi_{f'} : E'_1 \oplus \tilde{E}'_2 \to \tilde{E}'_2|_{Y\times T}$ be the homomorphisms induced by f and f' respectively. Put $E := \ker\varphi_f$ and $E' := \ker\varphi_{f'}$. Then there exist a line bundle L on T and an isomorphism $\sigma : E \xrightarrow{\sim} E' \otimes_{\mathcal{O}_T} L$. The injection

$$\begin{array}{ccc} E_1(-Y) & \hookrightarrow & E_1 \oplus \tilde{E}_2 \\ a & \mapsto & (a, 0) \end{array}$$

factors through E:

$$E_1(-Y) \hookrightarrow E \hookrightarrow E_1 \oplus \tilde{E}_2. \tag{1}$$

On the other hand, the canonical commutative diagram

$$\begin{array}{ccc} I_{X_2} \otimes E & \longrightarrow & E \\ \downarrow & & \downarrow \\ I_{X_2} \otimes (E_1 \oplus \tilde{E}_2) & \longrightarrow & E_1 \oplus \tilde{E}_2 \end{array} \tag{2}$$

induces the following commutative diagram:

$$\begin{array}{ccc} I_{X_2} \otimes E & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ E_1(-Y) & \longrightarrow & E_1 \oplus \tilde{E}_2. \end{array} \tag{3}$$

The homomorphism $I_{X_2} \otimes E \to E_1(-Y)$ is surjective since the canonical homomorphism $E \to E_1$ is surjective. Hence the sequence

$$0 \longrightarrow E_1(-Y) \longrightarrow E \longrightarrow E \otimes \mathcal{O}_{X_2} \longrightarrow 0 \tag{4}$$

is exact. The injection

$$\begin{array}{ccc} \tilde{E}_2(-Y) & \hookrightarrow & E_1 \oplus \tilde{E}_2 \\ b & \mapsto & (0,b) \end{array}$$

factors through E:

$$\tilde{E}_2(-Y) \hookrightarrow E \hookrightarrow E_1 \oplus \tilde{E}_2. \tag{5}$$

We can easily check that

$$0 \longrightarrow \tilde{E}_2(-Y) \longrightarrow E \longrightarrow E_1 \longrightarrow 0 \tag{6}$$

is an exact sequence. Similarly to (4) and (6), there are exact sequences:

$$0 \longrightarrow E'_1(-Y) \longrightarrow E' \longrightarrow E' \otimes \mathcal{O}_{X_2} \longrightarrow 0 \tag{7}$$

$$0 \longrightarrow \tilde{E}'_2(-Y) \longrightarrow E' \longrightarrow E'_1 \longrightarrow 0. \tag{8}$$

From (4) and (7), there is an isomorphism $\sigma'_1 : E_1(-Y) \xrightarrow{\sim} E'_1(-Y) \otimes L$ such that the following diagram commutes:

$$\begin{array}{ccccc} I_{X_2} \otimes E & \longrightarrow & E_1(-Y) & \hookrightarrow & E \\ {\scriptstyle 1\otimes\sigma}\downarrow\wr & & {\scriptstyle \sigma'_1}\downarrow\wr & & {\scriptstyle \sigma}\downarrow\wr \\ I_{X_2} \otimes E' \otimes L & \longrightarrow & E'_1(-Y) \otimes L & \hookrightarrow & E' \otimes L. \end{array} \tag{9}$$

Let $\sigma_1 : E_1 \xrightarrow{\sim} E'_1 \otimes L$ be the isomorphism obtained by tensoring $\mathcal{O}_{X_1}(Y)$ to σ'_1. Then we have the following commutative diagram:

$$\begin{array}{ccc} E & \longrightarrow & E_1 \\ {\scriptstyle \sigma}\downarrow\wr & & {\scriptstyle \sigma_1}\downarrow\wr \\ E' \otimes L & \longrightarrow & E'_1 \otimes L. \end{array} \tag{10}$$

Taking the kernels of the horizontal homomorphisms of the diagram (10) and using (6),(8), we get an isomorphism $\sigma'_2 : \tilde{E}_2(-Y) \xrightarrow{\sim} \tilde{E}'_2(-Y) \otimes L$ such that the diagram

$$\begin{array}{ccccc} I_{X_1} \otimes E & \longrightarrow & \tilde{E}_2(-Y) & \longrightarrow & E \\ {\scriptstyle 1\otimes\sigma}\downarrow\wr & & {\scriptstyle \sigma'_2}\downarrow\wr & & {\scriptstyle \sigma}\downarrow\wr \\ I_{X_1} \otimes E' \otimes L & \longrightarrow & \tilde{E}'_2(-Y) \otimes L & \longrightarrow & E' \otimes L \end{array} \tag{11}$$

commutes. Note that there is a commutative diagram

$$\begin{array}{ccc} I_{X_1} \otimes E & \longrightarrow & \tilde{E}_2(-Y) \\ \downarrow & \swarrow & \downarrow \\ E & \longrightarrow & \tilde{E}_2. \end{array}$$

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Let $\sigma_2 : \tilde{E}_2 \xrightarrow{\sim} \tilde{E}'_2 \otimes L$ be the isomorphism obtained by tensoring $\mathcal{O}_{X_2}(Y)$ to σ'_2. Then a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & \tilde{E}_2 \\ \sigma \downarrow \wr & & \sigma_2 \downarrow \wr \\ E' \otimes L & \longrightarrow & \tilde{E}'_2 \otimes L \end{array} \tag{12}$$

is obtained. From the definition of E and E', the diagrams

$$\begin{array}{ccccc} E & \longrightarrow & E_1 & \longrightarrow & E_1|_{Y\times T} \\ & \searrow & & & \downarrow f \\ & & \tilde{E}_2 & \longrightarrow & \tilde{E}_2|_{Y\times T} \end{array} \qquad \begin{array}{ccccc} E' & \longrightarrow & E'_1 & \longrightarrow & E'_1|_{Y\times T} \\ & \searrow & & & \downarrow f' \\ & & \tilde{E}'_2 & \longrightarrow & \tilde{E}'_2|_{Y\times T} \end{array} \tag{13}$$

both commute. Hence from (10) and (12), the following commutative diagram is obtained:

$$\begin{array}{ccc} E_1|_{Y\times T} & \xrightarrow{f} & \tilde{E}_2|_{Y\times T} \\ \sigma_1|_{Y\times T} \downarrow \wr & & \sigma_2|_{Y\times T} \downarrow \wr \\ E'_1|_{Y\times T} \otimes L & \xrightarrow{f'} & \tilde{E}'_2|_{Y\times T} \otimes L. \end{array} \tag{14}$$

These mean that $(E_1, \tilde{E}_2, f) \sim (E'_1, \tilde{E}'_2, f')$ and so Φ is a monomorphism.

Next we prove the second assertion of the proposition. Let S be a noetherian scheme over k and $\phi : h_S \to \mathcal{M}_X^H$ a morphism of functors. ϕ is given by an element $E \in \mathcal{M}_X^H(S)$. From the flattening stratification theorem, there exists a subscheme S_1 of S such that for any $T \in (\mathrm{Sch}/k)$ and any morphism $f : T \to S$, f factors through S_1 if and only if $(1 \times f)^*(E \otimes \mathcal{O}_{X_2})$ is flat over T and $\chi((E \otimes \mathcal{O}_{X_2}) \otimes k(t)(m)) = H(m) - H'_1(m)$ for all $t \in T$. Let $q : E_{S_1} \to E_{S_1} \otimes \mathcal{O}_{X_2}$ be the canonical surjection. Since $\ker q$ is the image of the homomorphism $I_{X_2} \otimes E_{S_1} \to E_{S_1}$, it is an $\mathcal{O}_{X_1 \times S_1}$-module. If we put $E^{(1)} := \ker q \otimes \mathcal{O}_{X_1}(Y)$, then $E^{(1)}$ becomes an S_1-flat $\mathcal{O}_{X_1\times S_1}$-module. The composition

$$g : E_{S_1} \longrightarrow E|_{X_1\times S_1} \xrightarrow{\sim} I_{X_2} \otimes E|_{X_1 \times S_1} \otimes \mathcal{O}_{X_1}(Y) \longrightarrow \ker q \otimes \mathcal{O}_{X_1}(Y) \xrightarrow{\sim} E^{(1)}$$

is surjective. Since the diagram

$$\begin{array}{cccc} & I_{X_2} \otimes E_{S_1} & \longrightarrow & E_{S_1} \\ & \downarrow & & \downarrow g \\ I_{X_2} \cdot E_{S_1} = & E^{(1)}(-Y) & \hookrightarrow & E^{(1)} \end{array} \tag{15}$$

commutes, the composition $E^{(1)}(-Y) \hookrightarrow E_{S_1} \xrightarrow{g} E^{(1)}$ is just the canonical injection induced by the inclusion $\mathcal{O}_{X_1}(-Y) \hookrightarrow \mathcal{O}_{X_1}$. The image of the homomorphism $I_{X_2} \otimes \ker g \to E_{S_1}$ is contained in $E^{(1)}(-Y)$ and the composition $E^{(1)}(-Y) \to E_{S_1} \xrightarrow{g} E^{(1)}$ is injective. Hence $I_{X_2} \ker g = 0$ and so $\ker g$ is an $\mathcal{O}_{X_2 \times S_1}$-module. Put $\tilde{E}^{(2)} := \ker g \otimes \mathcal{O}_{X_2}(Y)$. Note that $E^{(1)}$ and $\tilde{E}^{(2)}$ are both flat over S_1. So there exists an open and closed subscheme S_0 of S_1 such that a point $s \in S_1$ is in S_0 if and only if the following equalities hold:

$$\begin{aligned} &\chi(E^{(1)} \otimes k(s)(m)) = H_1(m), \quad \chi(\tilde{E}^{(2)} \otimes k(s)(m)) = H_2(m) \\ &\chi(\tilde{E}^{(2)} \otimes k(s)(-Y)(m)) = H_2'(m). \end{aligned} \tag{16}$$

Since the composite $I_{X_1} \otimes E_{S_1} \to E_{S_1} \xrightarrow{g} E^{(1)}$ is zero, the image of the canonical homomorphism $I_{X_1} \otimes E_{S_1} \to E_{S_1}$ is contained in $\tilde{E}^{(2)}(-Y)$. Then we have the following composition:

$$g' : E_{S_1} \to E|_{X_2 \times S_1} \xrightarrow{\sim} I_{X_1} \otimes E_{S_1} \otimes \mathcal{O}_{X_2}(Y) \to \tilde{E}^{(2)}.$$

There exists a homomorphism $\alpha : E^{(1)}_{S_1} \to \tilde{E}^{(2)}|_{Y \times S_1}$ satisfying the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{E}^{(2)}(-Y) & \longrightarrow & E_{S_1} & \xrightarrow{g} & E^{(1)} & \longrightarrow & 0 \\ & & \| & & \downarrow g' & & \downarrow \alpha & & \\ 0 & \longrightarrow & \tilde{E}^{(2)}(-Y) & \longrightarrow & \tilde{E}^{(2)} & \longrightarrow & \tilde{E}^{(2)}|_{Y \times S_1} & \longrightarrow & 0. \end{array} \tag{17}$$

Let $f : E^{(1)}|_{Y \times S_1} \to \tilde{E}^{(2)}|_{Y \times S_1}$ be the homomorphism induced by α. Put

$$\begin{array}{cccc} \varphi_f : & E^{(1)} \oplus \tilde{E}^{(2)} & \longrightarrow & \tilde{E}^{(2)}|_{Y \times S_1} \\ & (a, b) & \mapsto & f(a|_{Y \times S_1}) - b|_{Y \times S_1}. \end{array} \tag{18}$$

Let $i : E_{S_1} \to E^{(1)} \oplus \tilde{E}^{(2)}$ be the homomorphism induced by the homomorphisms $g : E_{S_1} \to E^{(1)}$ and $g' : E_{S_1} \to \tilde{E}^{(2)}$. i is injective since its restriction to every fiber over S_1 is injective ([4], IV Proposition (11.3.7)). Moreover we can check that the sequence

$$0 \longrightarrow E_{S_1} \xrightarrow{i} E^{(1)} \oplus \tilde{E}^{(2)} \xrightarrow{\varphi_f} \tilde{E}^{(2)}|_{Y \times S_1} \longrightarrow 0 \tag{19}$$

is exact.

The triple $(E^{(1)}_{S_0}, \tilde{E}^{(2)}_{S_0}, f_{S_0})$ defines an element of $\mathcal{M}^{(1),H_1,H_2}_{X,H_1',H_2'}(S_0)$. Then we have a morphism $\phi' : h_{S_0} \to \mathcal{M}^{(1),H_1,H_2}_{X,H_1',H_2'}$. From the exactness of (19), the following diagram commutes:

$$\begin{array}{ccc} h_{S_0} & \hookrightarrow & h_S \\ \phi' \downarrow & & \downarrow \phi \\ \mathcal{M}^{(1),H_1,H_2}_{X,H_1',H_2'} & \hookrightarrow & \mathcal{M}^H_X. \end{array} \tag{20}$$

In particular we have a monomorphism

$$h_{S_0} \hookrightarrow \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2} \times_{\mathcal{M}^H_X} h_S \hookrightarrow h_S. \tag{21}$$

Let $((E_1,\tilde{E}_2,f),\psi)$ be any element of $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}(T)\times_{\mathcal{M}^H_X(T)} S(T)$. Put

$$\varphi_f : E_1 \oplus \tilde{E}_2 \longrightarrow \tilde{E}_2|_{Y\times T}; \quad \varphi_f(a,b) := f(a|_{Y\times T}) - b|_{Y\times T}. \tag{22}$$

Then we have $E_T \cong \ker\varphi_f \otimes L$ for some line bundle L on T. Replacing $(E_1,\tilde{E}_2,f)$ by $(E_1,\tilde{E}_2,f)\otimes L$, we may assume that $E_T \cong \ker\varphi_f$.

From (4),(6), there are exact sequences:

$$0 \longrightarrow E_1(-Y) \longrightarrow E_T \longrightarrow E|_{X_2\times T} \longrightarrow 0 \tag{23}$$

$$0 \longrightarrow \tilde{E}_2(-Y) \longrightarrow E_T \longrightarrow E_1 \longrightarrow 0. \tag{24}$$

Since $E_1(-Y)(t) \to E(t)$ is injective for all $t \in T$, $E|_{X_2\times T}$ is flat over T ([4], IV Proposition 11.3.7) and $\chi(E|_{X_2\times T}(t)(m)) = H(m) - H'_1(m)$ for all $t \in T$. Hence the morphism $\psi : T \to S$ factors through S_0. This proves that $h_{S_0} \longrightarrow \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2} \times_{\mathcal{M}^H_X} h_S$ is an isomorphism. □

Let us recall the following well-known result. The proof is in [17] or [9].

Theorem 1.9. (Simpson) *There exists a coarse moduli scheme M^H_X of $\mathcal{M}^H_X$.*

As a corollary of Proposition 1.8, we have the following result.

Theorem 1.10. *There exists a coarse moduli scheme $M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ of $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}$. Moreover $M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ is a subscheme of M^H_X.*

Proof. From the arguments in [17] or [9], we can see that the moduli scheme M^H_X for $\mathcal{M}^H_X$ is obtained as a quotient of an open subscheme R of a Quot-scheme by an action of $PGL(V)$ for some vector space V over k. Moreover R is a principal $PGL(V)$-bundle over M^H_X. Note that the notion of e-stable sheaf is now needless because the boundedness of semistable sheaves has been proven ([7]). From Proposition 1.8, there exists a subscheme R' of R such that $h_{R'} \cong h_R \times_{\mathcal{M}^H_X} \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}$. From the construction, $R' \hookrightarrow R$ is $PGL(V)$-equivariant and R' descends to a subscheme $M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ of M^H_X. Since $R' \to M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ is a principle $PGL(V)$-bundle, we can easily see that $M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ is a coarse moduli scheme of $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}$. □

The following Proposition means that $M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ and M^H_X have the same scheme structure at "general points" of $M^{(1),H_1,H_2}_{X,H'_1,H'_2}$.

Proposition 1.11. *Let R be a noetherian local ring and s be the closed point of* $\operatorname{Spec} R$. *Let E be an R-valued point of $\mathcal{M}^H_X$. Assume that the triple $(F_1, \tilde{F}_2, f)$ corresponding to $E(s)$ is contained in $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}(k(s))$ and that $f : F_1|_{Y\times k(s)} \to \tilde{F}_2|_{Y\times k(s)}$ is surjective. Then E is contained in $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}(R)$.*

Proof. The canonical homomorphism $E(s) \to \tilde{F}_2$ is surjective since f is surjective. Hence $\tilde{F}_2(-Y) \subset E(s)$ is the image of the canonical homomorphism $I_{X_1} \otimes E(s) \to E(s)$. On the other hand there is an exact sequence;

$$0 \longrightarrow \tilde{F}_2(-Y) \longrightarrow E(s) \longrightarrow F_1 \longrightarrow 0.$$

So we have an isomorphism $E(s) \otimes \mathcal{O}_{X_1} \xrightarrow{\sim} F_1$. Hence the canonical homomorphism $I_{X_2} \otimes E(s) \xrightarrow{\sim} F_1(-Y) \to E(s)$ is injective. Since E is flat over S, $I_{X_2} \otimes E \to E$ is injective and $E \otimes \mathcal{O}_{X_2}$ is flat over S ([4], IV Proposition (11.3.7)). From the proof of Proposition 1.8, E is contained in $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}(R)$. ∎

Remark 1.12. Take a member $E \in M^H_X(k)$. Let $(E^{(1)}, \tilde{E}^{(2)}, f_E) \in M^{(1),H_1,H_2}_{X,H'_1,H'_2}(k)$ and $(\tilde{E}^{(1)}, E^{(2)}, g_E) \in M^{(2),G_1,G_2}_{X,G'_1,G'_2}(k)$ be the corresponding triples. Then we can check that $f_E : E^{(1)}|_Y \to \tilde{E}^{(2)}|_Y$ is surjective if and only if $g_E : E^{(2)}|_Y \to \tilde{E}^{(1)}|_Y$ is injective. If a point $p = [E] \in M^H_X(k)$ satisfies this equivalence condition, then we can see from the proof of Proposition 1.11 that $M^{(2),G_1,G_2}_{X,G'_1,G'_2}$ and M^H_X have the same scheme structure at p.

The following proposition means that we can fix the rank of the sheaf restricted to X_1 or X_2 in considering the moduli space of stable sheaves on X.

Proposition 1.13. *Let S be a connected locally noetherian scheme and E be an element of $\mathcal{M}^H_X(S)$. Then $a_0(E|_{X_1\times S}(s))$ and $a_0(E|_{X_1\times S}(s))$ are constant on S, where $a_0(E|_{X_i\times S}(s))$ is the integer defined before* Definition 1.6.

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Proof. Take any point s of S. Let $t_1,\dots,t_d \in H^0(X,\mathcal{O}_X(1))$ be general members such that $Z_1\cap\cdots\cap Z_d\cap Y=\emptyset$ and $E|_{(Z_1\cap\cdots\cap Z_{i-1})\times S}(s)\otimes\mathcal{O}_X(-1)\xrightarrow{t_i} E|_{(Z_1\cap\cdots\cap Z_{i-1})\times S}(s)$ are injective for $i=1,\dots,d$, where $Z_i := Z(t_i)$ is the zero scheme of t_i. Then each $E|_{(Z_1\cap\cdots\cap Z_i)\times S}$ is flat over a neighborhood of s. In particular $E|_{(Z_1\cap\cdots\cap Z_d)\times S}$ is finite and flat over a neighborhood of s. Put $W_1 := (Z_1\cap\cdots\cap Z_d\cap X_1)\times S$ and $W_2 := (Z_1\cap\cdots\cap Z_d\cap X_2)\times S$. Then $E|_{W_1}$ and $E|_{W_2}$ are both flat over a neighborhood of s. Hence $a_0(E|_{X_1\times S}(t))=\chi(E|_{W_1}(t))$ and $a_0(E|_{X_2\times S}(t))=\chi(E|_{W_2}(t))$ are constant on a neighborhood of s. □

§2. Direct construction of the moduli space of triples

In this section we will give another construction of the moduli space $M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ of triples. There exists an integer m_0 such that for any integer $m\geq m_0$ and for any geometric point $(E_1,\tilde{E}_2,f)\in\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}(k)$,

(i) $H^0(E_1(m))\otimes\mathcal{O}_{X_1}\to E_1(m)$ and $H^0(\tilde{E}_2(m))\otimes\mathcal{O}_{X_2}\to\tilde{E}_2(m)$ are surjective, and

(ii) $H^i(E_1(m))=0$, $H^i(\tilde{E}_2(m))=0$ and $H^i(\tilde{E}_2(-Y)(m))=0$ for $i>0$.

Let V_1 (resp. V_2) be a vector space over k of dimension $H_1(m_0)$ (resp. $H_2(m_0)$). Let us consider the open subschemes

$$Q_1 := \left\{ [V_1\otimes\mathcal{O}_{X_1}\to E_1]\in \mathrm{Quot}^{H_1(x+m_0)}_{V_1\otimes\mathcal{O}_{X_1}/X_1/k} \;\middle|\; \begin{array}{l} V_1\to H^0(E_1)\text{ is bijective,}\\ E_1\text{ is of pure dimension } d\\ \text{and } H^i(E_1)=0\text{ for all } i>0\end{array}\right\},$$

$$Q_2 := \left\{ [V_2\otimes\mathcal{O}_{X_2}\to \tilde{E}_2]\in \mathrm{Quot}^{H_2(x+m_0)}_{V_2\otimes\mathcal{O}_{X_2}/X_2/k} \;\middle|\; \begin{array}{l} V_2\to H^0(\tilde{E}_2)\text{ is bijective,}\\ \tilde{E}_2\text{ is of pure dimension } d\\ \text{and } H^i(\tilde{E}_2)=0\text{ for all } i>0\end{array}\right\}$$

of $\mathrm{Quot}^{H_1(x+m_0)}_{V_1\otimes\mathcal{O}_{X_1}/X_1/k}$ and $\mathrm{Quot}^{H_2(x+m_0)}_{V_2\otimes\mathcal{O}_{X_2}/X_2/k}$ respectively. Let $V_1\otimes\mathcal{O}_{X_1\times Q_1\times Q_2}\to\mathcal{E}_1$ and $V_2\otimes\mathcal{O}_{X_2\times Q_1\times Q_2}\to\tilde{\mathcal{E}}_2$ be the pull back of the universal quotient sheaves. Then $\mathcal{E}_1|_{Y\times Q_1\times Q_2}$ and $\tilde{\mathcal{E}}_2|_{Y\times Q_1\times Q_2}$ are flat over $Q_1\times Q_2$. From the result of base change theorem ([1], (1.1)), there exists a coherent sheaf $\mathcal{H}$ on $Q_1 \times Q_2$ such that

$$\mathcal{H}om(\mathcal{H} \otimes \mathcal{O}_T, M) \xrightarrow{\sim} (\pi_T)_*(\mathcal{H}om(\mathcal{E}_1 \otimes \mathcal{O}_{Y\times T}, \tilde{\mathcal{E}}_2 \otimes \mathcal{O}_{Y\times T} \otimes M))$$

for any morphism $T \to Q_1 \times Q_2$ and any quasi-coherent sheaf M on T, where $\pi_T : Y \times T \to T$ is the projection. Let us consider $\mathbf{V}(\mathcal{H}) = \mathrm{Spec}(S(\mathcal{H}))$. From the property of $\mathcal{H}$, the canonical homomorphism $\mathcal{H} \otimes \mathcal{O}_{\mathbf{V}(\mathcal{H})} \to \mathcal{O}_{\mathbf{V}(\mathcal{H})}$ corresponds to a homomorphism:

$$\tilde{f} : \mathcal{E}_1 \otimes \mathcal{O}_{Y\times \mathbf{V}(\mathcal{H})} \longrightarrow \tilde{\mathcal{E}}_2 \otimes \mathcal{O}_{Y\times \mathbf{V}(\mathcal{H})}.$$

Then we can define the following homomorphism;

$$\begin{aligned} \varphi_{\tilde{f}} : (\mathcal{E}_1)_{\mathbf{V}(\mathcal{H})} \oplus (\tilde{\mathcal{E}}_2)_{\mathbf{V}(\mathcal{H})} &\longrightarrow \tilde{\mathcal{E}}_2 \otimes \mathcal{O}_{Y\times \mathbf{V}(\mathcal{H})}; \\ (a, b) &\mapsto \tilde{f}\,(a|_{Y\times \mathbf{V}(\mathcal{H})}) - b|_{Y\times \mathbf{V}(\mathcal{H})}. \end{aligned}$$

If we put $\mathcal{E}' := \ker \varphi_{\tilde{f}} \otimes \mathcal{O}_X(-m_0)$, then $\mathcal{E}'$ is flat over $\mathbf{V}(\mathcal{H})$. We put

$$P := \left\{ s \in \mathbf{V}(\mathcal{H}) \;\middle|\; \begin{array}{l} \mathcal{E}'(s) \text{ is a stable sheaf on } X \times k(s) \\ \chi(\mathcal{E}_1(s)(-Y)(m)) = H'_1(m + m_0) \text{ and} \\ \chi(\tilde{\mathcal{E}}_2(s)(-Y)(m)) = H'_2(m + m_0) \end{array} \right\}.$$

If we put $\mathcal{E} := \mathcal{E}'|_{X\times P}$, then it induces a morphism

$$\Pi' : h_P \longrightarrow \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}.$$

Let $\Pi : P \to M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ be the morphism induced by the composition

$$h_P \xrightarrow{\Pi'} \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2} \to h_{M^{(1),H_1,H_2}_{X,H'_1,H'_2}}.$$

Put $G := (GL(V_1) \times GL(V_2))/\mathbf{G}_m$, where $\mathbf{G}_m \hookrightarrow GL(V_1) \times GL(V_2)$ is the diagonal embedding.

THEOREM 2.1. $\Pi : P \longrightarrow M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ *is a principal G-bundle.*

Proof. Let S be a locally noetherian scheme over k and take elements $g \in G$ and $x \in \mathbf{V}(\mathcal{H})(S)$. g is given by a line bundle $\mathcal{L}$ on S and two isomorphisms $g_i : V_i \otimes \mathcal{O}_S \xrightarrow{\sim} V_i \otimes \mathcal{O}_S \otimes \mathcal{L}$ $(i = 1, 2)$. x is determined by

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quotients $p_1 : V_1 \otimes \mathcal{O}_{X_1 \times S} \to E_1$, $p_2 : V_2 \otimes \mathcal{O}_{X_2 \times S} \to \tilde{E}_2$, and a homomorphism $f : E_1|_{Y \times S} \to \tilde{E}_2|_{Y \times S}$. Let $g \cdot x$ be the S-valued point of $\mathbf{V}(\mathcal{H})$ determined by the following data:

$$\begin{aligned}
&V_1 \otimes \mathcal{O}_{X_1 \times S} \xrightarrow{g_1} V_1 \otimes \mathcal{O}_{X_1 \times S} \otimes \mathcal{L} \xrightarrow{p_1} E_1 \otimes \mathcal{L}, \\
&V_2 \otimes \mathcal{O}_{X_2 \times S} \xrightarrow{g_2} V_2 \otimes \mathcal{O}_{X_1 \times S} \otimes \mathcal{L} \xrightarrow{p_2} \tilde{E}_2 \otimes \mathcal{L} \quad \text{and} \\
&E_1 \otimes \mathcal{L}|_{Y \times S} \xrightarrow{f \otimes 1} \tilde{E}_2 \otimes \mathcal{L}|_{Y \times S}.
\end{aligned}$$

Then we can define the action:

$$\sigma : G \times \mathbf{V}(\mathcal{H}) \to \mathbf{V}(\mathcal{H}); \quad (g, x) \mapsto g \cdot x.$$

σ induces the action of G on P and the morphism $\Pi' : h_P \to \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}$ is G-equivariant. So we obtain a morphism of functors $\psi : h_P/h_G \to \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}$, where h_P/h_G is the functor defined by

$$(h_P/h_G)\,(S) := P(S)/G(S)$$

for any locally noetherian scheme S over k. Let x and x' be S-valued points of P such that $\Pi'(x) = \Pi'(x')$. Let $p_1 : V_1 \otimes \mathcal{O}_{X_1 \times S} \to E_1$ and $p_2 : V_2 \otimes \mathcal{O}_{X_2 \times S} \to \tilde{E}_2$ be the quotient sheaves determined by x and $p'_1 : V_1 \otimes \mathcal{O}_{X_1 \times S} \to E'_1$ and $p'_2 : V_2 \otimes \mathcal{O}_{X_2 \times S} \to \tilde{E}'_2$ be the quotient sheaves determined by x'. Let $f : E_1|_{Y \times S} \to \tilde{E}_2|_{Y \times S}$ and $f' : E'_1|_{Y \times S} \to \tilde{E}'_2|_{Y \times S}$ be the homomorphisms determined by x and x' respectively. Then there exists a line bundle $\mathcal{L}$ on S such that $E_1 \cong E'_1 \otimes \mathcal{L}$, $\tilde{E}_2 \cong \tilde{E}'_2 \otimes \mathcal{L}$ and the following diagram commutes;

$$\begin{array}{ccc}
E_1|_{Y \times S} & \xrightarrow{f} & \tilde{E}_2|_{Y \times S} \\
\wr\downarrow & & \wr\downarrow \\
E'_1 \otimes \mathcal{L}|_{Y \times S} & \xrightarrow{f'} & \tilde{E}'_2 \otimes \mathcal{L}|_{Y \times S}.
\end{array}$$

So we have an isomorphism $g_1 : V_1 \otimes \mathcal{O}_S \xrightarrow{\sim} \pi_*(E_1) \xrightarrow{\sim} \pi_*(E'_1 \otimes \mathcal{L}) \xrightarrow{\sim} V_1 \otimes \mathcal{L}$, where $\pi : X_1 \times S \to S$ is the projection. The isomorphism $\tilde{E}_2 \cong \tilde{E}'_2 \otimes \mathcal{L}$ induces the isomorphism $g_2 : V_2 \otimes \mathcal{O}_S \xrightarrow{\sim} V_2 \otimes \mathcal{L}$. Then $[(g_1, g_2)] \cdot x' = x$ and so ψ is a monomorphism. It is easy to see that $\psi(R)$ is surjective for any local ring R. Hence the sheaves associated to the presheaves h_P/h_G , $\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}$ with respect to Zariski topology are isomorphic. From this fact one sees that the morphism $\Pi' : P \to \mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}$ is formally smooth and thus the morphism $\Pi : P \to M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ is smooth.

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Let us consider the morphism

$$\varphi : G \times P \longrightarrow P \times_{M^{(1),H_1,H_2}_{X,H'_1,H'_2}} P.$$

We must prove that φ is an isomorphism. Note that $P \times_{M^{(1),H_1,H_2}_{X,H'_1,H'_2}} P \cong P \times_{\mathcal{M}^{(1),H_1,H_2}_{X,H'_1,H'_2}} P$. First we will prove that $\varphi(R)$ is injective for any artinian local ring R over k. Let $(g,x),(g',x')$ be two elements of $G(R) \times P(R)$ such that $(g \cdot x, x) = (g' \cdot x', x')$. Then $x = x'$ and $g^{-1}g'x = x$. Let $(E_1, \tilde{E}_2, f)$ be the triple determined by x and E be the associated coherent sheaf on $X \times R$. Then we have $\mathrm{Hom}_{X \times R}(E, E) \cong R$. If we write $(g_1, g_2) := g^{-1}g'$, then there are isomorphisms $h_1 : E_1 \xrightarrow{\sim} E_1$, $h_2 : \tilde{E}_2 \xrightarrow{\sim} \tilde{E}_2$ such that the diagrams

$$\begin{array}{ccc} V_1 \otimes \mathcal{O}_{X_1} & \xrightarrow{p_1} & E_1 \\ g_1 \downarrow \wr & & h_1 \downarrow \wr \\ V_1 \otimes \mathcal{O}_{X_1} & \xrightarrow{p_1} & E_1 \end{array} \qquad \begin{array}{ccc} V_2 \otimes \mathcal{O}_{X_2} & \xrightarrow{p_2} & \tilde{E}_2 \\ g_2 \downarrow \wr & & h_2 \downarrow \wr \\ V_2 \otimes \mathcal{O}_{X_2} & \xrightarrow{p_2} & \tilde{E}_2 \end{array}$$

commute, where p_1 and p_2 are quotients determined by x, and the diagram

$$\begin{array}{ccc} E_1|_{Y \times R} & \xrightarrow{f} & \tilde{E}_2|_{Y \times R} \\ h_1 \downarrow & & h_2 \downarrow \\ E_1|_{Y \times R} & \xrightarrow{f} & \tilde{E}_2|_{Y \times R} \end{array}$$

commutes. (h_1, h_2) induces the automorphism $E \xrightarrow{\sim} E$. Since $\mathrm{Hom}(E,E) = R$, this isomorphism is a multiplication by an element c of R. So $h_1 = c \cdot 1$ and $h_2 = c \cdot 1$. Since $V_1 \otimes R \cong H^0(E_1)$ and $V_2 \otimes R \cong H^0(\tilde{E}_2)$, $g_1 = c \cdot 1$ and $g_2 = c \cdot 1$. Hence $g^{-1}g' = 1$ in $G(R)$ and $(g,x) = (g',x')$.

Hence $\varphi(R)$ is injective for all artinian local rings R over k and so φ is a monomorphism. Since ψ is a monomorphism, $\varphi(S)$ is surjective for any $S \in (\mathrm{Sch}/k)$. Hence φ is an isomorphism. □

Remark 2.2. If X_1, X_2 are non-singular curves, then $\mathcal{H}$ is a locally free sheaf. However, $\mathcal{H}$ is not necessarily locally free in higher dimensional case. (See Remark 5.2 for this example.)

Remark 2.3. By [[9], Proposition 4.10], there exists an integer l_0 such that for any $l \geq l_0$, for any $E \in \mathcal{M}^H_X(k)$ and for any coherent subsheaf F of E with $0 < a_0(F) < a_0(E)$, the inequality $h^0(F(l))/a_0(F) < h^0(E(l))/a_0(E)$

holds. We assume farther that $m_0 \geq l_0$. Then we can also prove that the quotient P/G exists by using geometric invariant theory. Namely one can show that all points of P are stable points with respect to the action of G and some polarization.

Remark 2.4. Assume that $H_1 \neq 0$ and $H_2 \neq 0$. Consider the projective bundle $\mathbf{P}(\mathcal{H})$ over $Q_1 \times Q_2$. Let $\bar{f} : \mathcal{E}_1 \otimes \mathcal{O}_{Y \times \mathbf{P}(\mathcal{H})} \longrightarrow \tilde{\mathcal{E}}_2 \otimes \mathcal{O}_{Y \times \mathbf{P}(\mathcal{H})} \otimes \mathcal{O}_{\mathbf{P}(\mathcal{H})}(1)$ be the homomorphism corresponding to the canonical surjection $\mathcal{H} \otimes \mathcal{O}_{\mathbf{P}(\mathcal{H})} \to \mathcal{O}_{\mathbf{P}(\mathcal{H})}(1)$. Let $\varphi_{\bar{f}} : (\mathcal{E}_1)_{\mathbf{P}(\mathcal{H})} \oplus \tilde{\mathcal{E}}_2 \otimes \mathcal{O}_{\mathbf{P}(\mathcal{H})}(1) \to \tilde{\mathcal{E}}_2 \otimes \mathcal{O}_{\mathbf{P}(\mathcal{H})}(1) \otimes \mathcal{O}_{Y \times \mathbf{V}(\mathcal{H})}$ be the induced homomorphism. Put

$$\bar{P} := \left\{ x \in \mathbf{P}(\mathcal{H}) \;\middle|\; \begin{array}{l} \ker \varphi_{\bar{f}}(x) \otimes \mathcal{O}_X(-m_0) \text{ is stable,} \\ \chi(\mathcal{E}_1(s)(-Y)(m)) = H'_1(m+m_0) \text{ and} \\ \chi(\tilde{\mathcal{E}}_2(s)(-Y)(m)) = H'_2(m+m_0) \end{array} \right\}.$$

Then $\bar{P}$ is an open subscheme of $\mathbf{P}(\mathcal{H})$. By the same arguments as Theorem 2.1, we can see that $\bar{P} \to M^{(1),H_1,H_2}_{X,H'_1,H'_2}$ is a principal $PGL(V_1) \times PGL(V_2)$-bundle.

§3. Parabolic triples

Let X be a projective scheme over k satisfying the condition (†) of section 1.

DEFINITION 3.1. A *parabolic triple* is a triple $((E_1)_*, (E_2)_*, \sigma)$, where $(E_i)_*$ is a filtration $E_i(-Y) \subset E'_i \subset E_i$ of coherent sheaves on X_i of pure dimension d for $i = 1, 2$ and σ an isomorphism $\sigma : E_1/E'_1 \xrightarrow{\sim} E_2/E'_2$ on Y.

DEFINITION 3.2. Two parabolic triples $((E_1)_*, (E_2)_*, \sigma)$ and $((F_1)_*, (F_2)_*, \tau)$ are said to be isomorphic if there exist isomorphisms $\theta_i : E_i \xrightarrow{\sim} F_i$ for $i = 1, 2$ such that $\theta_i(E'_i) = F'_i$ for $i = 1, 2$ and the diagram

$$\begin{array}{ccc} E_1/E'_1 & \xrightarrow{\sigma} & E_2/E'_2 \\ {\scriptstyle \bar{\theta}_1}\downarrow & & {\scriptstyle \bar{\theta}_2}\downarrow \\ F_1/F'_1 & \xrightarrow{\tau} & F_2/F'_2 \end{array}$$

commutes, where $\bar{\theta}_i : E_i/E'_i \xrightarrow{\sim} F_i/F'_i$ is the isomorphism induced by θ_i.

PROPOSITION 3.3. *There exists a bijective correspondence between the isomorphism classes of coherent sheaves of pure dimension d on X and the isomorphism classes of parabolic triples.*

Proof. Let E be a coherent sheaf of pure dimension d on X. As in the argument in section 1, there is an exact sequence

$$0 \longrightarrow E \longrightarrow E^{(1)} \oplus E^{(2)} \longrightarrow E^{(0)} \longrightarrow 0.$$

Let $E^{(i)}_*$ be the filtration $E^{(i)}(-Y) \subset \tilde{E}^{(i)}(-Y) \subset E^{(i)}$ for $i = 1, 2$. Let σ be the canonical isomorphism $E^{(1)}/\tilde{E}^{(1)}(-Y) \xrightarrow{\sim} E^{(0)} \xrightarrow{\sim} E^{(2)}/\tilde{E}^{(2)}(-Y)$. Then we obtain a parabolic triple $(E^{(1)}_*, E^{(2)}_*, \sigma)$ for E. It is easy to see that $E \mapsto (E^{(1)}_*, E^{(2)}_*, \sigma)$ is a bijective correspondence. □

A flat family of parabolic triples on X_T/T is a triple $((E_1)_*, (E_2)_*, \sigma)$ such that for each i, $(E_i)_*$ is a filtration $E_i(-Y) \subset E'_i \subset E_i$ of coherent sheaves on X_T such that E_i and E_i/E'_i are flat over T, $E_i(t)$ is of pure dimension d for any $t \in T$ and $\sigma : E_1/E'_1 \xrightarrow{\sim} E_2/E'_2$ is an isomorphism. For a flat family $((E_1)_*, (E_2)_*, \sigma)$ of parabolic triples, we define a homomorphism

$$\varphi_\sigma : E_1 \oplus E_2 \longrightarrow E_2/E'_2$$

by $\varphi_\sigma(a, b) = \sigma(\bar{a}) - \bar{b}$ where $\bar{a}$ is the image of a by $E_1 \to E_1/E'_1$. Let $(H_i)_*$ be sequences of numerical polynomials $\tilde{H}_i(m), H_i(m), \tilde{H}'_i(m), H'_i(m)$ for $i = 1, 2$ such that $H(m) = H_1(m) + H'_2(m) = H'_1(m) + H_2(m)$.

Definition 3.4. We define a functor

$$\text{par-}\mathcal{M}_X^{(H_1)_*, (H_2)_*} : (\mathrm{Sch}/k) \longrightarrow (\mathrm{Sets})$$

by

$$\text{par-}\mathcal{M}_X^{(H_1)_*, (H_2)_*}(T) := \left\{ \begin{array}{l} ((E_1)_*, (E_2)_*, \sigma)\,;\text{ a flat family of parabolic} \\ \text{triples on } X_T/T \text{ such that for any } t \in T, \\ \ ((E_1)_*, (E_2)_*, \sigma) \otimes k(t) \text{ satisfies the} \\ \text{following condition (a)} \end{array} \right\} \Big/ \sim$$

where $\sim$ is the equivalence relation defined by (b).

(a) $\chi(E'_i(t)(Y)(m)) = \tilde{H}_i(m)$, $\chi(E_i(t)(m)) = H_i(m)$, $\chi(E'_i(t)(m)) = \tilde{H}'_i(m)$ and $\chi(E_i(t)(-Y)(m)) = H'_i(m)$ for $i = 1, 2$ and $\ker \varphi_\sigma(t)$ is a stable sheaf on $X \times k(t)$.

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(b) $((E_1)_*, (E_2)_*, \sigma) \sim ((F_1)_*, (F_2)_*, \tau)$ if there are a line bundle L on T and isomorphisms $\theta_i : E_i \xrightarrow{\sim} F_i \otimes L$ with $\theta_i(E'_i) = F'_i \otimes L$ for $i = 1, 2$ such that the diagram

$$\begin{array}{ccc} E_1/E'_1 & \xrightarrow{\bar{\theta}_1} & F_1/F'_1 \otimes L \\ \sigma\downarrow & & \tau\downarrow \\ E_2/E'_2 & \xrightarrow{\bar{\theta}_2} & F_2/F'_2 \otimes L \end{array}$$

commutes, where $\bar{\theta}_i : E_i/E'_i \xrightarrow{\sim} F_i/F'_i \otimes L$ is the isomorphism induced by θ_i.

We can define a morphism of functors

$$\Psi : \text{par-}\mathcal{M}_X^{(H_1)_*,(H_2)_*} \longrightarrow \mathcal{M}_X^H$$

by $\Psi((E_1)_*, (E_2)_*, \sigma) := \ker \varphi_\sigma$.

PROPOSITION 3.5. *$\Psi : \text{par-}\mathcal{M}_X^{(H_1)_*,(H_2)_*} \to \mathcal{M}_X^H$ is a monomorphism. Moreover $\text{par-}\mathcal{M}_X^{(H_1)_*,(H_2)_*} = \mathcal{M}_{X,H'_1,\tilde{H}'_2}^{(1),H_1,\tilde{H}_2} \cap \mathcal{M}_{X,\tilde{H}'_1,H'_2}^{(2),\tilde{H}_1,H_2}$ as subfunctors of $\mathcal{M}_X^H$.*

Proof. Let $((E_1)_*, (E_2)_*, \sigma)$ and $((F_1)_*, (F_2)_*, \tau)$ be two T-valued points of $\text{par-}\mathcal{M}_X^{(H_1)_*,(H_2)_*}$ such that $\Psi((E_1)_*, (E_2)_*, \sigma) = \Psi((F_1)_*, (F_2)_*, \tau)$. If we put $E := \ker \varphi_\sigma$ and $F := \ker \varphi_\tau$, then there exists a line bundle L on T such that $E \cong F \otimes L$. From the similar arguments to the proof of Proposition 1.8, the homomorphism

$$\begin{array}{ccc} E_1(-Y) & \hookrightarrow & E_1 \oplus E_2 \\ a & \mapsto & (a, 0) \end{array}$$

factors through E and the induced sequence

$$0 \longrightarrow E_1(-Y) \longrightarrow E \longrightarrow E \otimes \mathcal{O}_{X_2} \longrightarrow 0$$

is exact. Similarly the exact sequence

$$0 \longrightarrow F_1(-Y) \longrightarrow F \longrightarrow F \otimes \mathcal{O}_{X_2} \longrightarrow 0$$

is obtained. Thus there exists an isomorphism $E_1(-Y) \xrightarrow{\sim} F_1(-Y) \otimes L$ such that the diagram

$$\begin{array}{ccccc} I_{X_2} \otimes E & \longrightarrow & E_1(-Y) & \hookrightarrow & E \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ I_{X_2} \otimes F \otimes L & \longrightarrow & F_1(-Y) \otimes L & \hookrightarrow & F \otimes L \end{array}$$

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commutes. Then for the induced isomorphism $E_1 \xrightarrow{\sim} F_1 \otimes L$, the diagram

$$\begin{array}{ccc} E & \longrightarrow & E_1 \\ \downarrow \wr & & \downarrow \wr \\ F \otimes L & \longrightarrow & F_1 \otimes L \end{array}$$

commutes. Similarly we obtain an isomorphism $E_2 \xrightarrow{\sim} F_2 \otimes L$ such that the diagram

$$\begin{array}{ccc} E & \longrightarrow & E_2 \\ \downarrow \wr & & \downarrow \wr \\ F \otimes L & \longrightarrow & F_2 \otimes L \end{array}$$

commutes. Thus we obtain an isomorphism $E_2/E_2' \xrightarrow{\sim} F_2/F_2' \otimes L$ and the exact commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & E & \longrightarrow & E_1 \oplus E_2 & \longrightarrow & E_2/E_2' & \longrightarrow 0 \\ & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\ 0 \longrightarrow & F \otimes L & \longrightarrow & F_1 \otimes L \oplus F_2 \otimes L & \longrightarrow & (F_2/F_2') \otimes L & \longrightarrow 0. \end{array}$$

This implies that $((E_1)_*, (E_2)_*, \sigma) \sim ((F_1)_*, (F_2)_*, \tau)$ and so Ψ is a monomorphism.

Take $((E_1)_*, (E_2)_*, \sigma) \in \text{par-}\mathcal{M}_X^{(H_1)_*,(H_2)_*}(T)$ and put $E := \ker \varphi_\sigma$. By definition, the canonical homomorphism $E_1(-Y)(t) \to E(t)$ is injective for all $t \in T$. So the exact sequence

$$0 \longrightarrow E_1(-Y) \longrightarrow E \longrightarrow E \otimes \mathcal{O}_{X_2} \longrightarrow 0$$

concludes that $E \otimes \mathcal{O}_{X_2}$ is flat over T ([4], IV Proposition (11.3.7)). Similarly $E \otimes \mathcal{O}_{X_1}$ is also flat over T. Hence the proof of Proposition 1.8 implies that $E \in \mathcal{M}^{(1),H_1,\tilde{H}_2}_{X,H_1',\tilde{H}_2'}(T) \cap \mathcal{M}^{(2),\tilde{H}_1,H_2}_{X,\tilde{H}_1',H_2'}(T)$.

Conversely take $E \in \mathcal{M}^{(1),H_1,\tilde{H}_2}_{X,H_1',\tilde{H}_2'}(T) \cap \mathcal{M}^{(2),\tilde{H}_1,H_2}_{X,\tilde{H}_1',H_2'}(T)$. Let $(E_1, \tilde{E}_2, f_1)$ and $(\tilde{E}_1, E_2, f_2)$ be the corresponding triples. Then we have the following two exact sequences:

$$\begin{array}{c} 0 \longrightarrow E \longrightarrow E_1 \oplus \tilde{E}_2 \longrightarrow \tilde{E}_2|_{Y \times T} \longrightarrow 0 \\ 0 \longrightarrow E \longrightarrow \tilde{E}_1 \oplus E_2 \longrightarrow \tilde{E}_1|_{Y \times T} \longrightarrow 0. \end{array}$$

Consider the composition $\tilde{E}_1(-Y) \hookrightarrow E \to E_1$. Since $\tilde{E}_1(-Y)(t) \to E_1(t)$ is injective for all $t \in T$, $\tilde{E}_1(-Y) \to E_1$ is injective and $E_1/\tilde{E}_1(-Y)$ is flat over T ([4], IV Proposition (11.3.7)). Moreover we have a factorization

$E_1(-Y) \hookrightarrow \tilde{E}_1(-Y) \hookrightarrow E_1$. Similarly we have a factorization $E_2(-Y) \hookrightarrow \tilde{E}_2(-Y) \hookrightarrow E_2$. Since $E(t) \to E_1(t) \oplus E_2(t)$ is injective for any $t \in T$, $E \to E_1 \oplus E_2$ is injective and the cokernel E_0 is flat over T ([4], IV Proposition (11.3.7)). The exact sequence

$$0 \longrightarrow \tilde{E}_1(-Y) \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

implies that $E_1/\tilde{E}_1(-Y) \cong E_0$. Similarly $E_2/\tilde{E}_2(-Y) \cong E_0$. Hence E is contained in par-$\mathcal{M}_X^{(H_1)_*,(H_2)_*}(T)$. ∎

THEOREM 3.6. *A coarse moduli scheme* par-$M_X^{(H_1)_*,(H_2)_*}$ *of* par-$\mathcal{M}_X^{(H_1)_*,(H_2)_*}$ *exists. Moreover* par-$M_X^{(H_1)_*,(H_2)_*}$ *is a subscheme of* M_X^H *and is the scheme theoretic intersection* $M^{(1),H_1,\tilde{H}_2}_{X,H_1',\tilde{H}_2'} \cap M^{(2),\tilde{H}_1,H_2}_{X,\tilde{H}_1',H_2'}$.

Proof. The proof is similar to Theorem 1.10. ∎

We will give another construction of the moduli space of parabolic triples. There exists an integer m_0 such that for any integer $m \geq m_0$ and for any $((E_1)_*, (E_2)_*, \sigma) \in$ par-$M_X^{(H_1)_*,(H_2)_*}(k)$,

(i) $E_i(m), (E_i/E_i')(m)$ are globally generated for $i = 1, 2$,

(ii) $H^j(E_i(m)) = 0, H^j((E_i/E_i')(m)) = 0$ for $i = 1, 2$ and for any $j > 0$.

For $i = 1, 2$, put $V_i := k^{\oplus H_i(m_0)}$ and

$$Q_i := \left\{ [V_i \otimes \mathcal{O}_{X_i} \to E_i] \in \mathrm{Quot}^{H_i[m_0]}_{V_i \otimes \mathcal{O}_{X_i}/X_i/k} \;\middle|\; \begin{array}{l} E_i \text{ is of pure dimension } d, \\ V_i \to H^0(E_i) \text{ is bijective} \\ \text{and } H^j(E_i) = 0 \text{ for } j > 0 \end{array} \right\}.$$

Let $V_i \otimes \mathcal{O}_{X_i \times Q_i} \to \mathcal{E}_i$ be the universal quotient sheaf. Let R_i' be the Quot-scheme $\mathrm{Quot}^{H_i[m_0] - \tilde{H}_i'[m_0]}_{\mathcal{E}_i/X_i \times Q_i/Q_i}$ and $(\mathcal{E}_i)_{R_i'} \to \mathcal{G}_i$ be the universal quotient sheaf. From [[19], Corollary 2.3] there exists a subscheme R_i of R_i' such that for any $T \in (\mathrm{Sch}/k)$,

$$R_i(T) := \left\{ T \to R_i' \;\middle|\; \begin{array}{l} \mathcal{E}_i(-Y)_T \hookrightarrow (\mathcal{E}_i)_T \to (\mathcal{G}_i)_T \text{ is zero and} \\ H^j(\mathcal{G}_i(t)) = 0 \text{ for } j > 0 \text{ and for any } t \in T \end{array} \right\}.$$

Then $(\mathcal{G}_i)_{R_i}$ is an $\mathcal{O}_{Y \times R_i}$-module. Put $\tilde{\mathcal{E}}_i := \ker g \otimes \mathcal{O}_{X_i}(Y)$, where $g : (\mathcal{E}_i)_{R_i} \to (\mathcal{G}_i)_{R_i}$ is the canonical surjection. There exists a coherent sheaf $\mathcal{H}$ on $R_1 \times R_2$ such that

$$\mathrm{Hom}_T(\mathcal{H}_T, \mathcal{M}) \cong \mathrm{Hom}_{X_T}((\mathcal{G}_1)_T, \mathcal{G}_2 \otimes \mathcal{M})$$

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for any $T \to R_1 \times R_2$ and any quasi-coherent sheaf $\mathcal{M}$ on T ([1], (1.1)). Let $\sigma : (\mathcal{G}_1)_{\mathbf{V}(\mathcal{H})} \to (\mathcal{G}_2)_{\mathbf{V}(\mathcal{H})}$ be the homomorphism corresponding to the canonical homomorphism $\mathcal{H} \otimes \mathcal{O}_{\mathbf{V}(\mathcal{H})} \to \mathcal{O}_{\mathbf{V}(\mathcal{H})}$. Put

$$\varphi_\sigma : (\mathcal{E}_1)_{\mathbf{V}(\mathcal{H})} \oplus (\mathcal{E}_2)_{\mathbf{V}(\mathcal{H})} \longrightarrow (\mathcal{G}_2)_{\mathbf{V}(\mathcal{H})};\ (a, b) \mapsto \sigma(\bar{a}) - \bar{b}.$$

Set

$$P := \left\{ s \in \mathbf{V}(\mathcal{H}) \;\middle|\; \begin{array}{l} \sigma(s) \text{ is isomorphic, } \ker \varphi_\sigma(s) \otimes \mathcal{O}_X(-m_0) \text{ is stable,} \\ \chi(\tilde{\mathcal{E}}_i(s)(n)) = \tilde{H}_i(n + m_0) \text{ and} \\ \chi(\mathcal{E}_i(-Y)(s)(n)) = H'_i(n + m_0) \text{ for } i = 1, 2 \end{array} \right\}.$$

Then $\ker \varphi_\sigma \otimes \mathcal{O}_X(-m_0)|_{X \times P}$ induces a morphism

$$\pi : P \longrightarrow \text{par-}M_X^{(H_1)_*, (H_2)_*}.$$

Put $G := (GL(V_1) \times GL(V_2))/\mathbf{G}_m$.

THEOREM 3.7. *$\pi : P \to \text{par-}M_X^{(H_1)_*, (H_2)_*}$ is a principal G-bundle.*

Proof. The same arguments as proof of Theorem 2.1 shows that π is a smooth morphism. Moreover the surjectivity of π is obvious.

Let us consider the morphism

$$\psi : G \times P \longrightarrow P \times_{\text{par-}M_X^{(H_1)_*, (H_2)_*}} P.$$

We have only to prove that ψ is an isomorphism. Note that there is a canonical isomorphism $P \times_{\text{par-}\mathcal{M}_X^{(H_1)_*, (H_2)_*}} P \cong P \times_{\text{par-}M_X^{(H_1)_*, (H_2)_*}} P$. In order to prove that ψ is a monomorphism, it is sufficient to show that $\psi(R)$ is injective for all artinian local rings R over k. Take $p \in P(R)$ and $[(g_1, g_2)] \in G(R)$ such that $[(g_1, g_2)] \cdot p = p$. Let

$$\begin{array}{ccccc} V_1 \otimes \mathcal{O}_{X_1 \times_k R} & \longrightarrow & E_1 & \longrightarrow & G_1 \\ & & & & \wr \downarrow \sigma \\ V_2 \otimes \mathcal{O}_{X_2 \times_k R} & \longrightarrow & E_2 & \longrightarrow & G_2 \end{array}$$

be the diagram corresponding to p. Since $(g_1, g_2) \cdot p = p$, there are isomorphisms $\theta_i : E_i \xrightarrow{\sim} E_i$, $\bar{\theta}_i : G_i \xrightarrow{\sim} G_i$ for $i = 1, 2$ such that the diagrams

$$\begin{array}{ccc} V_i \otimes \mathcal{O}_{X_1 \times_k R} & \xrightarrow{g_i} & V_i \otimes \mathcal{O}_{X_i \times_k R} \\ \downarrow & & \downarrow \\ E_i & \xrightarrow{\theta_i} & E_i \\ \downarrow & & \downarrow \\ G_i & \xrightarrow{\bar{\theta}_i} & G_i. \end{array}$$

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commute for $i = 1, 2$ and the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\sigma} & G_2 \\ {\scriptstyle \bar{\theta}_1}\downarrow & & {\scriptstyle \bar{\theta}_2}\downarrow \\ G_1 & \xrightarrow{\sigma} & G_2 \end{array}$$

commutes. θ_1 and θ_2 induce an automorphism of $\ker \varphi_\sigma$. However, this automorphism is a scalar multiplication by $c \in R$, since $\mathrm{End}(\ker \varphi_\sigma) = R$. Hence $\theta_1 = c$ and $\theta_2 = c$, and so $[(g_1, g_2)] = 1$ in $G(R)$, which proves that $\varphi(R)$ is injective.

Take $(p, q) \in (P \times_{\mathrm{par}-\mathcal{M}_X^{(H_1)_*,(H_2)_*}} P)(S)$, where $S \in (\mathrm{Sch}/k)$. Let

$$\begin{array}{ccccc} V_1 \otimes \mathcal{O}_{X_1 \times_k S} & \longrightarrow & E_1 & \longrightarrow & G_1 \\ & & & & \wr \downarrow \sigma \\ V_2 \otimes \mathcal{O}_{X_2 \times_k S} & \longrightarrow & E_2 & \longrightarrow & G_2 \end{array}$$

be the diagram corresponding to p and

$$\begin{array}{ccccc} V_1 \otimes \mathcal{O}_{X_1 \times_k S} & \longrightarrow & \bar{E}_1 & \longrightarrow & \bar{G}_1 \\ & & & & \wr \downarrow \tau \\ V_2 \otimes \mathcal{O}_{X_2 \times_k S} & \longrightarrow & \bar{E}_2 & \longrightarrow & \bar{G}_2 \end{array}$$

be the diagram corresponding to q. From the choice of p, q, there exist a line bundle $\mathcal{L}$ on S and an isomorphism $\theta : \ker \varphi_\sigma \xrightarrow{\sim} \ker \varphi_\tau \otimes \mathcal{L}$. θ induces isomorphisms $h_1 : E_1 \xrightarrow{\sim} \bar{E}_1 \otimes \mathcal{L}$, $\bar{h}_1 : G_1 \xrightarrow{\sim} \bar{G}_1 \otimes \mathcal{L}$, $h_2 : E_2 \xrightarrow{\sim} \bar{E}_2 \otimes \mathcal{L}$ and $\bar{h}_2 : G_2 \xrightarrow{\sim} \bar{G}_2 \otimes \mathcal{L}$ such that the diagram

$$\begin{array}{ccccccc} E_1 & \longrightarrow & G_1 & \xrightarrow{\sigma} & G_2 & \longleftarrow & E_2 \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle \bar{h}_1}\downarrow & & {\scriptstyle \bar{h}_2}\downarrow & & {\scriptstyle h_2}\downarrow \\ \bar{E}_1 \otimes \mathcal{L} & \longrightarrow & \bar{G}_1 \otimes \mathcal{L} & \xrightarrow{\tau} & \bar{G}_2 \otimes \mathcal{L} & \longleftarrow & \bar{E}_2 \otimes \mathcal{L} \end{array}$$

commutes. The isomorphisms h_1, h_2 induce isomorphisms $g_1 : V_1 \otimes \mathcal{O}_S \xrightarrow{\sim} V_1 \otimes \mathcal{O}_S \otimes \mathcal{L}$ and $g_2 : V_2 \otimes \mathcal{O}_S \xrightarrow{\sim} V_2 \otimes \mathcal{O}_S \otimes \mathcal{L}$ such that the diagrams

$$\begin{array}{ccc} V_1 \otimes \mathcal{O}_{X_1 \times S} & \longrightarrow & E_1 \\ {\scriptstyle g_1 \otimes 1}\downarrow & & {\scriptstyle h_1}\downarrow \\ V_1 \otimes \mathcal{O}_{X_1 \times S} \otimes \mathcal{L} & \longrightarrow & \bar{E}_1 \otimes \mathcal{L} \end{array}$$

and

$$\begin{array}{ccc} V_2 \otimes \mathcal{O}_{X_2 \times S} & \longrightarrow & E_2 \\ {\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow \\ V_2 \otimes \mathcal{O}_{X_2 \times S} \otimes \mathcal{L} & \longrightarrow & \bar{E}_2 \otimes \mathcal{L} \end{array}$$

both commute. Then $[(g_1, g_2)] \cdot q = p$, which proves that $\psi(S)$ is surjective. Hence ψ is an isomorphism. $\square$

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§4. Local deformations of sheaves on reducible surfaces

Lemma 4.1. *Let (A, m) be an artinian local ring with residue field $k = A/m$ and I an ideal of A such that $Im = 0$. Let X be a projective scheme flat over A and E a coherent sheaf on $X \times_A A/I$ flat over A/I. Assume that there exist finite points $p_1, \dots, p_n$ of $X \times_A k$ such that $E \otimes_A k$ is locally free over $X \times_A k \setminus \{p_1, \dots, p_n\}$. Moreover assume that each stalk E_{p_i} can be lifted to an $\mathcal{O}_{X,p_i}$-module $\tilde{M}_i$ flat over A. Then there exists an element $\omega(E) \in H^2(X \times k, \mathcal{E}nd(E \otimes k)) \otimes I$ whose vanishing is equivalent to the liftability of E to an A-flat coherent sheaf $\tilde{E}$ on X such that $\tilde{E}_{p_i} \cong M_i$ for each i.*

Proof. There exists a finite covering $\{U_i\}_{i=1}^r$ of X by affine open sets such that $E|_{(U_i \cap U_j)\otimes A/I}$ is locally free for each i, j with $U_i \neq U_j$ and each $E|_{U_i}$ can be lifted to a coherent $\mathcal{O}_{U_i}$-module $\tilde{E}_i$ which is flat over U_i. Let η_i be the element of $\mathrm{Ext}^1_{U_i}(E|_{U_i}, I \otimes_A E|_{U_i})$ corresponding to the extension

$$0 \longrightarrow I \otimes_A E|_{U_i} \longrightarrow \tilde{E}_i \longrightarrow E|_{U_i} \longrightarrow 0.$$

Since $\tilde{E}_i|_{U_i \cap U_j}$ and $\tilde{E}_j|_{U_i \cap U_j}$ are locally free sheaves for $U_i \neq U_j$, there exists an isomorphism of extensions:

$$\begin{array}{ccccccc}
0 \longrightarrow & I \otimes_A E|_{U_i \cap U_j} & \longrightarrow & \tilde{E}_i|_{U_i \cap U_j} & \longrightarrow & E|_{U_i \cap U_j} & \longrightarrow 0 \\
& \| & & \wr\downarrow & & \| & \\
0 \longrightarrow & I \otimes_A E|_{U_i \cap U_j} & \longrightarrow & \tilde{E}_j|_{U_i \cap U_j} & \longrightarrow & E|_{U_i \cap U_j} & \longrightarrow 0.
\end{array}$$

Hence we have $\eta_i|_{U_i \cap U_j} = \eta_j|_{U_i \cap U_j}$ and so $\{\eta_i\}_{i=1}^r$ determines an element η of $H^0(X, \mathcal{E}xt^1_X(I \otimes_A E, E))$. From the spectral sequence $E_2^{p,q} = H^p(X, \mathcal{E}xt^q_X(I \otimes_A E, E)) \Rightarrow E^{p+q} = \mathrm{Ext}^{p+q}_X(I \otimes_A E, E)$, the following exact sequence is obtained:

$$0 \longrightarrow H^1(X, \mathcal{H}om_X(I \otimes_A E, E)) \longrightarrow \mathrm{Ext}^1_X(I \otimes_A E, E)$$

$$\longrightarrow H^0(X, \mathcal{E}xt^1_X(I \otimes_A E, E)) \longrightarrow H^2(X, \mathcal{H}om_X(I \otimes_A E, E)).$$

Let ω be the image of η by $H^0(X, \mathcal{E}xt^1_X(I \otimes_A E, E)) \to H^2(X, \mathcal{H}om_X(I \otimes_A E, E)) = H^2(X \times_A k, \mathcal{E}nd_X(E \otimes_A k) \otimes_k I)$. Then $\omega = 0$ if and only if η comes from an element of $\mathrm{Ext}^1_X(I \otimes_A E, E)$, that is an extension $0 \to I \otimes_A E \to \tilde{E} \to E \to 0$ such that

$$\begin{array}{ccccccc}
0 \longrightarrow & I \otimes_A E|_{U_i \cap U_j} & \longrightarrow & \tilde{E}_i|_{U_i \cap U_j} & \longrightarrow & E|_{U_i \cap U_j} & \longrightarrow 0 \\
& \| & & \wr\downarrow & & \| & \\
0 \longrightarrow & I \otimes_A E|_{U_i \cap U_j} & \longrightarrow & \tilde{E}|_{U_i \cap U_j} & \longrightarrow & E|_{U_i \cap U_j} & \longrightarrow 0
\end{array}$$

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for each i. Hence ω is the desired obstruction. □

Let R be a discrete valuation ring and $k = R/m_R$ be the residue field. (We assume that $k = \bar{k}$.) Let t_0 be the closed point of $\operatorname{Spec} R$ and η be the generic point of $\operatorname{Spec} R$. Let $\tilde{X}$ be a projective scheme flat over R such that $\tilde{X}_\eta$ is a smooth surface and $X := \tilde{X}_{t_0} = X_1 \cup X_2$, where X_1 and X_2 are smooth surfaces and $Y := X_1 \cap X_2$ is a smooth curve. We will investigate which sheaves on X can be lifted to sheaves on $\tilde{X}$.

Proposition 4.2. *Let E be a coherent sheaf of pure dimension 2 on X such that* $\operatorname{rank} E|_{X_1} \neq \operatorname{rank} E|_{X_2}$. *Then E can not be lifted to a coherent sheaf on $\tilde{X}$ flat over R.*

Proof. Take a very ample line bundle $\mathcal{O}_{\tilde{X}}(1)$. Since $\tilde{X}$ is flat over R, we may assume that $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(1))$ is a projective module over R and that $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(1)) \otimes k(t_0) \cong H^0(X, \mathcal{O}_X(1))$, where $\mathcal{O}_X(1) := \mathcal{O}_{\tilde{X}}(1)|_X$. Put $r_i := \operatorname{rank} E|_{X_i}$ for $i = 1, 2$. Assume that E is lifted to a coherent sheaf $\tilde{E}$ on $\tilde{X}$ flat over R. Take general sections $s_1, s_2 \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(1))$. Let $Z(s_i) \subset \tilde{X}$ be the zero scheme of s_i. Then we may assume that $\tilde{E}|_{Z(s_1)\cap Z(s_2)}$ is flat over R, $\tilde{E}|_{Z(s_1)\cap Z(s_2)\cap \tilde{X}_\eta}$ is a locally free sheaf of rank $r := \operatorname{rank} \tilde{E}|_{\tilde{X}_\eta}$ and $E|_{Z(s_1)\cap Z(s_2)\cap X_i}$ are locally free sheaves of rank r_i on $Z(s_1) \cap Z(s_2) \cap X_i$ for $i = 1, 2$. Then $\tilde{E}|_{Z(s_1)\cap Z(s_2)}$ is a locally free sheaf on $Z(s_1) \cap Z(s_2)$ and its rank is $r_1 = \operatorname{rank} \tilde{E}|_{Z(s_1)\cap Z(s_2)\cap X_1} = \operatorname{rank} \tilde{E}|_{Z(s_1)\cap Z(s_2)\cap \tilde{X}_\eta} = \operatorname{rank} \tilde{E}|_{Z(s_1)\cap Z(s_2)\cap X_2} = r_2$, which is a contradiction. □

Let E be a coherent sheaf of pure dimension 2 on X and $(E^{(1)}, \tilde{E}^{(2)}, f)$ be the corresponding triple. Assume that $E^{(1)}$ and $\tilde{E}^{(2)}$ are of rank r, locally free along Y and $f : E^{(1)}|_Y \to \tilde{E}^{(2)}|_Y$ is injective. Let $\{p_1, \dots, p_n\}$ be the set of points of Y where f is not an isomorphism. Assume that $\operatorname{coker} f_{p_i} \cong k(p_i)$ for each i. Let M_i be the stalk of E at p_i.

Let $\mathcal{C}_R$ be the category of artinian local rings A over R with $A/m_A = k$. We define a functor

$$D_E : \mathcal{C}_R \longrightarrow (\text{Sets})$$

by

$$D_E(A) := \left\{ \tilde{E} \;\middle|\; \begin{array}{l} \tilde{E} \text{ is a coherent sheaf on } \tilde{X}_A \text{ flat over } A \\ \text{such that } \tilde{E} \otimes_A A/m_A \cong E \otimes_k A/m_A \end{array} \right\}.$$

We also define a functor

$$D_{M_i} : \mathcal{C}_R \longrightarrow (\mathrm{Sets})$$

by

$$D_{M_i}(A) := \left\{ \tilde{M} \;\middle|\; \begin{array}{l} \tilde{M} \text{ is an } \mathcal{O}_{\tilde{X}_A}\text{-module flat over } A \\ \text{such that } \tilde{M} \otimes_A A/m_A \cong M_i \otimes_k A/m_A \end{array} \right\}.$$

Then we can define a morphism of functors

$$\Phi_1 : D_E \longrightarrow D_{M_1} \times \cdots \times D_{M_n}$$

by $\Phi_1(\tilde{E}) := (\tilde{E}_{p_1}, \ldots, \tilde{E}_{p_n})$. If $H^2(X, \mathcal{E}nd(E, E)) = 0$, then from Lemma 4.1, Φ_1 is formally smooth. So we will study D_{M_i}. Put $M := M_i$ and $p := p_i$. There is an exact sequence

$$0 \longrightarrow M \longrightarrow \mathcal{O}_{X_1,p}^{\oplus r} \oplus \mathcal{O}_{X_2,p}^{\oplus r} \xrightarrow{\varphi_{f_p}} \mathcal{O}_{Y,p}^{\oplus r} \longrightarrow 0.$$

Let $\tilde{f} : \mathcal{O}_{X_1,p}^{\oplus r} \to \mathcal{O}_{X_1,p}^{\oplus r}$ be a lift of $f_p : \mathcal{O}_{Y,p}^{\oplus r} \to \mathcal{O}_{Y,p}^{\oplus r}$. Then the following exact commutative diagram is obtained:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & \mathcal{O}_{X_1,p}^{\oplus r} \oplus \mathcal{O}_{X_2,p}^{\oplus r} & \xrightarrow{\varphi_{f_p}} & \mathcal{O}_{Y,p}^{\oplus r} & \longrightarrow & 0 \\
 & & \downarrow & & \tilde{f} \oplus 1 \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{X,p}^{\oplus r} & \longrightarrow & \mathcal{O}_{X_1,p}^{\oplus r} \oplus \mathcal{O}_{X_2,p}^{\oplus r} & \longrightarrow & \mathcal{O}_{Y,p}^{\oplus r} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{coker} \tilde{f} & = & \operatorname{coker} \tilde{f} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

We consider M as a submodule of $\mathcal{O}_{X,p}^{\oplus r}$ with respect to this diagram. We define a functor

$$D_{M \subset \mathcal{O}_{X,p}^{\oplus r}} : \mathcal{C}_R \longrightarrow (\mathrm{Sets})$$

by

$$D_{M \subset \mathcal{O}_{X,p}^{\oplus r}}(A) := \left\{ \tilde{M} \subset \mathcal{O}_{\tilde{X}_A,p}^{\oplus r} \;\middle|\; \begin{array}{l} \tilde{M} \otimes_A A/m \to \mathcal{O}_{X,p}^{\oplus r} \otimes A/m \text{ is just the} \\ \text{inclusion } M \otimes_k A/m \hookrightarrow \mathcal{O}_{X,p}^{\oplus r} \otimes_k A/m \end{array} \right\}.$$

Let I_{X_i} be the ideal of $\mathcal{O}_X$ corresponding to the subscheme $X_i \subset X$ for $i = 1, 2$. Then $I_{X_1} \cong \mathcal{O}_{X_2}(-Y)$ and so $I_{X_1,p}$ is a principal ideal. So we can write $I_{X_1,p} = (f_1)$ and $I_{X_2,p} = (f_2)$.

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LEMMA 4.3. $\mathrm{Ext}^i_{\mathcal{O}_{X,p}}(M, \mathcal{O}_{X,p}) = 0$ *for* $i \geq 1$.

Proof. We have the following free resolution of $\mathcal{O}_{X_1,p}$:

$$\cdots \longrightarrow \mathcal{O}_{X,p} \xrightarrow{f_1} \mathcal{O}_{X,p} \xrightarrow{f_2} \mathcal{O}_{X,p} \xrightarrow{f_1} \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{X_1,p} \longrightarrow 0.$$

Taking the dual of this complex, we have the following exact sequence:

$$\mathcal{O}^\vee_{X,p} \xrightarrow{f_1} \mathcal{O}^\vee_{X,p} \xrightarrow{f_2} \mathcal{O}^\vee_{X,p} \xrightarrow{f_1} \cdots.$$

Thus $\mathrm{Ext}^i_{\mathcal{O}_{X,p}}(\mathcal{O}_{X_1,p}, \mathcal{O}_{X,p}) = 0$ for $i \geq 1$. Similarly $\mathrm{Ext}^i_{\mathcal{O}_{X,p}}(\mathcal{O}_{X_2,p}, \mathcal{O}_{X,p}) = 0$ for $i \geq 1$. There is an isomorphism $I_{Y,p} \cong I_{X_1,p} \oplus I_{X_2,p}$ and an exact sequence

$$\mathrm{Ext}^{i-1}_{\mathcal{O}_{X,p}}(I_{Y,p}, \mathcal{O}_{X,p}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_{X,p}}(\mathcal{O}_{Y,p}, \mathcal{O}_{X,p}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}, \mathcal{O}_{X,p}).$$

Thus $\mathrm{Ext}^i_{\mathcal{O}_{X,p}}(\mathcal{O}_{Y,p}, \mathcal{O}_{X,p}) = 0$ for $i \geq 2$. From the exact sequence

$$0 \longrightarrow M \longrightarrow \mathcal{O}^{\oplus r}_{X_1,p} \oplus \mathcal{O}^{\oplus r}_{X_2,p} \xrightarrow{\varphi_{f_p}} \mathcal{O}^{\oplus r}_{Y,p} \longrightarrow 0,$$

the following exact sequence is obtained:

$$\mathrm{Ext}^i_{\mathcal{O}_{X,p}}(\mathcal{O}^{\oplus r}_{X_1,p} \oplus \mathcal{O}^{\oplus r}_{X_2,p}, \mathcal{O}_{X,p}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_{X,p}}(M, \mathcal{O}_{X,p}) \longrightarrow \mathrm{Ext}^{i+1}_{\mathcal{O}_{X,p}}(\mathcal{O}^{\oplus r}_{Y,p}, \mathcal{O}_{X,p}).$$

Hence we have $\mathrm{Ext}^i_{\mathcal{O}_{X,p}}(M, \mathcal{O}_{X,p}) = 0$ for $i \geq 1$. □

We will use the following lemma.

LEMMA 4.4. *Let $A \to B$ be a local homomorphism of noetherian local rings and $k = A/m$ be the residue field. Let M be a B-module of finite type which is flat over A and satisfies $\mathrm{Ext}^1_{B\otimes_A k}(M \otimes_A k, B \otimes_A k) = 0$. Then $\mathrm{Hom}_B(M, B)$ is flat over A and $\mathrm{Hom}_B(M, B) \otimes_A k = \mathrm{Hom}_{B\otimes_A k}(M \otimes_A k, B \otimes_A k)$.*

Proof. See [12], Appendix. □

We define a morphism of functors

$$\Phi_2 : D_{M\subset \mathcal{O}^{\oplus r}_{X,p}} \longrightarrow D_M$$

by putting $\Phi_2(\tilde{M} \subset \mathcal{O}^{\oplus r}_{\tilde{X}_A,p}) := \tilde{M}$.

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Take any $A \in \mathcal{C}_R$ and an ideal $I \subset A$ such that $Im = 0$. If $\tilde{M} \in D_E(A)$ and an injection $\tilde{M} \otimes_A A/I \subset \mathcal{O}^{\oplus r}_{\tilde{X}_{A/I},p}$ is given, then from Lemma 4.3 and Lemma 4.4, $\mathrm{Hom}(\tilde{M}, \mathcal{O}^{\oplus r}_{\tilde{X}_A,p}) \to \mathrm{Hom}(\tilde{M} \otimes_A A/I, \mathcal{O}^{\oplus r}_{\tilde{X}_{A/I},p})$ is surjective. So the injection $\tilde{M} \otimes_A A/I \subset \mathcal{O}^{\oplus r}_{\tilde{X}_{A/I},p}$ can be lifted to an injection $\tilde{M} \subset \mathcal{O}^{\oplus r}_{\tilde{X}_A,p}$. Hence Φ_2 is formally smooth.

Let N be the cokernel of the injection $M \hookrightarrow \mathcal{O}^{\oplus r}_{X,p}$. We define a functor

$$D_N : \mathcal{C}_R \longrightarrow (\mathrm{Sets})$$

by

$$D_N(A) := \left\{ \tilde{N} \;\middle|\; \begin{array}{l} \tilde{N} \text{ is an } \mathcal{O}_{\tilde{X}_A,p}\text{-module flat over } A \\ \text{such that } \tilde{N} \otimes_A A/m \cong N \otimes_k A/m \end{array} \right\}.$$

Then we obtain the following morphism of functors;

$$\Phi_3 : D_{M \subset \mathcal{O}^{\oplus r}_{X,p}} \longrightarrow D_N; \quad [\tilde{M} \subset \mathcal{O}^{\oplus r}_{\tilde{X}_A,p}] \mapsto \mathcal{O}^{\oplus r}_{\tilde{X}_A,p}/\tilde{M}.$$

It is obvious that Φ_3 is formally smooth.

By the assumption on $f : E^{(1)}|_Y \to \tilde{E}^{(2)}|_Y$, N is generated by one element. So there is an exact sequence

$$0 \longrightarrow I_N \longrightarrow \mathcal{O}_{X,p} \longrightarrow N \longrightarrow 0.$$

We define a functor $D_{I_N \subset \mathcal{O}_{X,p}}$ in the same way as $D_{M \subset \mathcal{O}^{\oplus r}_{X,p}}$ and a morphism of functors $\Phi_4 : D_{I_N \subset \mathcal{O}_{X,p}} \to D_N$ in the same way as Φ_3.

Now let $\tilde{f}_1 \in \mathcal{O}_{\tilde{X},p}$ and $\tilde{f}_2 \in \mathcal{O}_{\tilde{X},p}$ be lifts of f_1 and f_2 respectively. Then there exists an element $g \in \mathcal{O}_{\tilde{X},p}$ such that $\tilde{f}_1 \tilde{f}_2 = tg$ where $t \in R$ is a local parameter.

Proposition 4.5. *Let $A \in \mathcal{C}_R$ and $\tilde{M}$ be an element of $D_M(A)$. If g is a unit in $\mathcal{O}_{\tilde{X},p}$, then $tA = 0$.*

Proof. Since Φ_2, Φ_4 are formally smooth, $\tilde{M}$ induces an A-valued point $[\tilde{I} \subset \mathcal{O}_{\tilde{X}_A,p}]$ of $D_{I_N \subset \mathcal{O}_{X,p}}$. I_N is generated by f_1 and another element $h \in \mathcal{O}_{X,p}$. Then the induced homomorphism $\mathcal{O}_{Y,p} \xrightarrow{\bar{h}} \mathcal{O}_{Y,p}$ is injective.

From the exact commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathcal{O}_{X_1,p} & \xrightarrow{f_2} & \mathcal{O}_{X_1,p} & \longrightarrow & \mathcal{O}_{Y,p} & \longrightarrow 0 \\
 & & h\downarrow & & h\downarrow & & \bar{h}\downarrow & \\
0 & \longrightarrow & \mathcal{O}_{X_1,p} & \xrightarrow{f_2} & \mathcal{O}_{X_1,p} & \longrightarrow & \mathcal{O}_{Y,p} & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & N & \xrightarrow{f_2} & N & \longrightarrow & N/f_2N & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0, &
\end{array}$$

one sees that $N \overset{f_2}{\to} N$ is injective. $\tilde{N} := \mathcal{O}_{\tilde{X}_A,p}/\tilde{I}$ is flat over A by definition. Thus $\tilde{N} \overset{\tilde{f}_2}{\to} \tilde{N}$ is injective and $\tilde{N}/\tilde{f}_2\tilde{N}$ is flat over A ([4] IV Proposition (11.3.7)). On the other hand, since $g \in \mathcal{O}_{\tilde{X}}^{\times}$, $t = -g^{-1}\tilde{f}_1\tilde{f}_2$. Thus $t(\tilde{N}/\tilde{f}_2\tilde{N}) = 0$. Since $\tilde{N}/\tilde{f}_2\tilde{N}$ is faithfully flat over A, we have $tA = 0$. ∎

From Proposition 4.5, we can see that M can never be lifted to a sheaf on $\tilde{X}$ if g is a unit in $\mathcal{O}_{\tilde{X},p}$.

The following proposition is useful for the lifting problem of sheaves on a degenerate quadric surface in the next section (Theorem 5.4).

Proposition 4.6. *Assume that the image $\bar{g} \in \mathcal{O}_{Y,p}$ of g is a regular parameter. Then M can be lifted to an $\mathcal{O}_{\tilde{X}_R,p}$-module flat over R.*

Proof. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_{X_1,p} \xrightarrow{h} \mathcal{O}_{X_1,p} \longrightarrow N \longrightarrow 0$$

for some $h \in \mathcal{O}_{X_1,p}$. h can be written as $h = ug + f_2\varphi$ where $u \in \mathcal{O}_{X_1,p}^{\times}$ and $\varphi \in \mathcal{O}_{X_1,p}$. Let $\tilde{u}$ and $\tilde{\varphi}$ be lifts of u and φ to $\mathcal{O}_{\tilde{X},p}$ respectively. Let $\tilde{I}$ be the ideal of $\mathcal{O}_{\tilde{X},p}$ generated by $\tilde{u}g + \tilde{f}_2\tilde{\varphi}, \tilde{f}_1 + t\tilde{u}^{-1}\tilde{\varphi}$. Since there is an equality $(\tilde{f}_1 + t\tilde{u}^{-1}\tilde{\varphi})\tilde{f}_2 - t\tilde{u}^{-1}(\tilde{u}g + \tilde{f}_2\tilde{\varphi}) = 0$, we can define a homomorphism

$$\mathcal{O}_{\tilde{X},p}/(\tilde{f}_2, \tilde{u}g + \tilde{f}_2\tilde{\varphi}) \xrightarrow{\tilde{f}_1 + t\tilde{u}^{-1}\tilde{\varphi}} \mathcal{O}_{\tilde{X},p}/(\tilde{u}g + \tilde{f}_2\tilde{\varphi}).$$

Since $\mathcal{O}_{X,p} \xrightarrow{\tilde{u}g + f_2\tilde{\varphi}} \mathcal{O}_{X,p}$ is injective, $\mathcal{O}_{\tilde{X},p} \xrightarrow{\tilde{u}g + \tilde{f}_2\tilde{\varphi}} \mathcal{O}_{\tilde{X},p}$ is injective and $\mathcal{O}_{\tilde{X},p}/(\tilde{u}g + \tilde{f}_2\tilde{\varphi})$ is flat over R ([4], IV Proposition (11.3.7)). Let us consider

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the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X_2,p} & \xrightarrow{f_1} & \mathcal{O}_{X_2,p} & \longrightarrow & \mathcal{O}_{Y,p} & \longrightarrow & 0 \\
 & & {\scriptstyle \tilde{u}g+\tilde{f}_2\tilde{\varphi}}\downarrow & & {\scriptstyle \tilde{u}g+\tilde{f}_2\tilde{\varphi}}\downarrow & & {\scriptstyle ug}\downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X_2,p} & \xrightarrow{f_1} & \mathcal{O}_{X_2,p} & \longrightarrow & \mathcal{O}_{Y,p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{X_2,p}/(\tilde{u}g+\tilde{f}_2\tilde{\varphi}) & \xrightarrow{f_1} & \mathcal{O}_{X_2,p}/(\tilde{u}g+\tilde{f}_2\tilde{\varphi}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Then the injectivity of $\mathcal{O}_{Y,p} \xrightarrow{ug} \mathcal{O}_{Y,p}$ implies that $\mathcal{O}_{X_2,p}/(\tilde{u}g + \tilde{f}_2\tilde{\varphi}) \xrightarrow{f_1} \mathcal{O}_{X_2,p}/(\tilde{u}g + \tilde{f}_2\tilde{\varphi})$ is injective, and so $\mathcal{O}_{X_2,p}/(\tilde{u}g + \tilde{f}_2\tilde{\varphi}) \xrightarrow{f_1} \mathcal{O}_{X,p}/(\tilde{u}g + \tilde{f}_2\tilde{\varphi})$ is injective. Hence the homomorphism $\mathcal{O}_{\tilde{X},p}/(\tilde{f}_2, \tilde{u}g + \tilde{f}_2\tilde{\varphi}) \xrightarrow{\tilde{f}_1+t\tilde{u}^{-1}\tilde{\varphi}} \mathcal{O}_{\tilde{X},p}/(\tilde{u}g + \tilde{f}_2\tilde{\varphi})$ is injective and $\mathcal{O}_{\tilde{X},p}/(\tilde{f}_1 + t\tilde{u}^{-1}\tilde{\varphi}, \tilde{u}g + \tilde{f}_2\tilde{\varphi}) = \mathcal{O}_{\tilde{X},p}/\tilde{I}$ is flat over R ([4], IV Proposition (11.3.7)). Hence $[\tilde{I} \subset \mathcal{O}_{\tilde{X},p}]$ is an element of $D_{I_N \subset \mathcal{O}_{X,p}}(R)$. Since $\Phi_3(R)$ is surjective, there exists an element $[\tilde{M} \subset \mathcal{O}_{\tilde{X},p}^{\oplus r}]$ of $D_{M \subset \mathcal{O}_{X,p}^{\oplus r}}(R)$ such that $\mathcal{O}_{\tilde{X},p}^{\oplus r}/\tilde{M} \cong \mathcal{O}_{\tilde{X},p}/\tilde{I}$. $\tilde{M}$ is the desired lift of M. □

§5. Moduli space of rank 2 stable sheaves on a reducible quadric surface

Let H_1, H_2 be two distinct planes in $\mathbf{P}^3$ and put $Q_0 := H_1 \cup H_2$. We consider the reduced structure on Q_0. We put $L := H_1 \cap H_2$. Then Q_0 satisfies the hypothesis (†) in section 1. We consider the polarization with respect to $\mathcal{O}_{Q_0}(1) := \mathcal{O}_{\mathbf{P}^3}(1)|_{Q_0}$. Fix a positive integer n_0 and put $P^{(0,0)}_{n_0}(m) := 4\binom{m+2}{2} - 2(m+1) - n_0$. For an integer n, put $P^{(0)}_n(m) := 2\binom{m+2}{2} - n$, $P^{(-1)}_n(m) := 2\binom{m+2}{2} - (m+1) - n$ and $P^{(1)}_n(m) := 2\binom{m+2}{2} + (m+1) + 1 - n$. We denote $\mathcal{M}^{(1),P_1(m),P_2(m)}_{Q_0,P_1(m-1),P_2(m-1)}$ simply by $\mathcal{M}^{P_1,P_2}_{(1),Q_0}$. We write $M^{P_1,P_2}_{(1),Q_0}$, $\mathcal{M}^{P_1,P_2}_{(2),Q_0}$ and $M^{P_1,P_2}_{(2),Q_0}$ similarly. If we put

$$M^{\circ P^{(0,0)}_{n_0}}_{Q_0} := \left\{ E \in M^{P^{(0,0)}_{n_0}}_{Q_0} \,\middle|\, \operatorname{rank} E|_{H_1} = \operatorname{rank} E|_{H_2} = 2 \right\},$$

then by Proposition 1.13 it is an open subscheme of $M^{P^{(0,0)}_{n_0}}_{Q_0}$.

THEOREM 5.1. *Let n_0 be an integer with $n_0 \geq 4$. Then*

$$M^{\circ P^{(0,0)}_{n_0}}_{Q_0} = \bigcup_{\substack{n_1+n_2=n_0 \\ n_1 \geq 0, n_2 \geq 0}} M^{P^{(0)}_{n_1}, P^{(0)}_{n_2}}_{(1),Q_0} \cup \bigcup_{\substack{n_1+n_2=n_0 \\ 0<n_1<n_2}} M^{P^{(-1)}_{n_1}, P^{(1)}_{n_2}}_{(1),Q_0} \cup \bigcup_{\substack{n_1+n_2=n_0 \\ n_1>n_2>0}} M^{P^{(1)}_{n_1}, P^{(-1)}_{n_2}}_{(2),Q_0}.$$

Moreover each component of the right hand side is non-empty.

Proof. Let E be an element of $M^{\circ P^{(0,0)}_{n_0}}_{Q_0}(k)$. We have the following exact sequence:

$$0 \longrightarrow E \longrightarrow E^{(1)} \oplus \tilde{E}^{(2)} \longrightarrow \tilde{E}^{(2)}|_L \longrightarrow 0.$$

Since $\operatorname{rank} E^{(1)} = \operatorname{rank} \tilde{E}^{(2)} = 2$, we have

$$\begin{aligned}
\chi(E(m)) &= \chi(E^{(1)}(m)) + \chi(\tilde{E}^{(2)}(m)) - \chi(\tilde{E}^{(2)}|_L(m)) \\
&= 4\binom{m+2}{2} + \left(c_1(E^{(1)}) + c_1(\tilde{E}^{(2)}) - 2\right)(m+1) - c_2(E^{(1)}) \\
&\qquad -c_2(\tilde{E}^{(2)}) + \frac{c_1(E^{(1)})^2 + c_1(E^{(1)}) + c_1(\tilde{E}^{(2)})^2 - c_1(\tilde{E}^{(2)})}{2}.
\end{aligned}$$

If we put $a := c_1(E^{(1)})$, then we have $c_1(\tilde{E}^{(2)}) = -a$ since $\chi(E(m)) = P^{(0,0)}_{n_0}(m)$. If we assume $a \leq -2$, then $\mu^S(E^{(1)}) \leq -1 < -1/2 = \mu^S(E)$ which contradicts the stability of E. (Recall that we defined $\mu^S(E) = a_1(E)/a_0(E)$.) Assume that $a \geq 2$. For the subsheaf $E^{(1)}(-L) \subset E$, we have $\mu^S(E^{(1)}(-L)) = a/2 - 1 \geq 0 > -1/2 = \mu^S(E)$ which also contradicts the stability of E. Hence we have $-1 \leq a \leq 1$.

Case 1. $a = -1$

In this case

$$\begin{aligned}
\chi(E^{(1)}(m)) &= 2\binom{m+2}{2} - (m+1) - c_2(E^{(1)}), \\
\chi(\tilde{E}^{(2)}(m)) &= 2\binom{m+2}{2} + (m+1) + 1 - c_2(\tilde{E}^{(2)}) \quad \text{and} \\
\chi(E(m)) &= 4\binom{m+2}{2} - 2(m+1) - c_2(E^{(1)}) - c_2(\tilde{E}^{(2)}).
\end{aligned}$$

Thus $E \in M_1^{P_{n_1}^{(-1)}, P_{n_2}^{(1)}}(k)$, where $n_1 = c_2(E^{(1)})$ and $n_2 = c_2(\tilde{E}^{(2)})$. The equality $\chi(E(m)) = P_{n_0}^{(0,0)}(m)$ implies that $n_1 + n_2 = n_0$. Since E is stable, $\chi(E(m))/4 < \chi(E^{(1)}(m))/2$ for all sufficiently large integers m. Hence we have the inequality $n_1 < n_2$. If $E^{(1)}$ is not μ-semi-stable, there exists a quotient coherent sheaf F of $E^{(1)}$ of rank 1, such that $\mu^S(E^{(1)}) = -1/2 > -1 = \mu^S(F)$. Since F is a quotient sheaf of E and $\mu^S(E) = -1/2 > \mu^S(F)$, this contradicts the stability of E. Hence $E^{(1)}$ is μ-semi-stable. From Schwarzenberger's inequality, we have $n_1 = c_2(E^{(1)}) \geq c_1(E^{(1)})^2/4 > 0$.
Case 2. $a = 1$

Assume that the homomorphism $E^{(1)}|_L \to \tilde{E}^{(2)}|_L$ is not isomorphic at the generic point of L. If F is the kernel of the homomorphism $E^{(1)} \to \tilde{E}^{(2)}|_L$, then F is a subsheaf of E and $c_1(F) \geq c_1(E^{(1)}) - 1 = 0$ which contradicts the stability of E. Hence the homomorphism $E^{(1)}|_L \to \tilde{E}^{(2)}|_L$ is isomorphic at the generic point of L. So we have $c_1(\tilde{E}^{(1)}) = 1$ and $c_1(E^{(2)}) = -1$. From the same argument as Case 1, $E \in M_{(2),Q_0}^{P_{n_1}^{(1)}, P_{n_2}^{(-1)}}(k)$ for integers n_1, n_2 with $n_1 + n_2 = n_0$ and $n_1 > n_2 > 0$.
Case 3. $a = 0$

If F is a rank 1 quotient coherent sheaf of $E^{(1)}$, $\mu^S(F) \geq -1/2 = \mu^S(E)$. Hence we have $\mu^S(F) \geq 0 = \mu^S(E^{(1)})$ and so $E^{(1)}$ is μ-semi-stable. From Schwarzenberger's inequality, we have $c_2(E^{(1)}) \geq 0$. Take any rank 1 quotient coherent sheaf F of $\tilde{E}^{(2)}$. If we put F' the image of the homomorphism $E \to F$, we have $\mu^S(F) \geq \mu^S(F') > \mu^S(E) = -1/2$. Hence we have $\mu^S(F) \geq 0 = \mu^S(\tilde{E}^{(2)})$ and so $\tilde{E}^{(2)}$ is μ-semi-stable. So we have $c_2(\tilde{E}^{(2)}) \geq 0$. Hence $E \in M_{(1),Q_0}^{P_{n_1}^{(0)}, P_{n_2}^{(0)}}(k)$ for integers n_1, n_2 with $n_1 \geq 0, n_2 \geq 0$ and $n_1 + n_2 = n_0$. Thus we have proved the first part of the theorem.

Take integers n_1, n_2 with $n_1 + n_2 = n_0$ and $0 < n_1 < n_2$. We will show that $M_{(1),Q_0}^{P_{n_1}^{(-1)}, P_{n_2}^{(1)}} \neq \emptyset$. There exist rank 2 stable bundles E_1 on H_1 and $\tilde{E}_2$ on H_2 such that $\chi(E_1(m)) = P_{n_1}^{(-1)}(m)$, $\chi(\tilde{E}_2(m)) = P_{n_2}^{(1)}(m)$, $E_1|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L$ and $\tilde{E}_2|_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L$. There is a homomorphism $f : E_1|_L \to \tilde{E}_2|_L$ which is isomorphic at the generic point of L. Let E be the coherent sheaf of pure dimension 2 on Q_0 associated to the triple $(E_1, \tilde{E}_2, f)$.
Claim 1. E is stable.

Let F be a coherent subsheaf of E with $0 < a_0(F) < a_0(E)$. We have the following exact commutative diagram with $F^{(1)} \to E_1$ and $\tilde{F}^{(2)} \to \tilde{E}_2$

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injective:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & F^{(1)} \oplus \tilde{F}^{(2)} & \longrightarrow & \tilde{F}^{(2)}|_L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & E_1 \oplus \tilde{E}_2 & \longrightarrow & \tilde{E}_2|_L & \longrightarrow & 0.
\end{array}
$$

Assume that $\operatorname{rank} F^{(1)} = 0$. Then $F \cong \tilde{F}^{(2)}(-L)$. If $\operatorname{rank} \tilde{F}^{(2)} = 1$, then we have $c_1(\tilde{F}^{(2)}(-L)) \leq -1 < -1/2 = \mu(\tilde{E}_2(-L))$ by the stability of $\tilde{E}_2(-L)$. Hence we have $\mu^S(F) \leq -1 < \mu^S(E)$. If $\operatorname{rank} \tilde{F}^{(2)} = 2$, F is contained in $\tilde{E}_2(-L)$ and

$$
\begin{aligned}
\frac{\chi(F(m))}{2} \leq \frac{\chi(\tilde{E}_2(-L)(m))}{2}
&= \binom{m+2}{2} - \frac{m+1}{2} - \frac{c_2(\tilde{E}_2)}{2} \\
&< \binom{m+2}{2} - \frac{m+1}{2} - \frac{c_2(E_1) + c_2(\tilde{E}_2)}{4} = \frac{\chi(E(m))}{4}
\end{aligned}
$$

for all sufficiently large integers m.

Assume that $\operatorname{rank} F^{(1)} = 1$. If $\operatorname{rank} \tilde{F}^{(2)} = 0$, then we have $\mu^S(F) = \mu^S(F^{(1)}) \leq -1 < -1/2 = \mu^S(E_1)$ by the stability of E_1. Hence we have $\mu^S(F) < -1/2 = \mu^S(E)$. If $\operatorname{rank} \tilde{F}^{(2)} = 1$, we have $\mu^S(F^{(1)}) \leq -1 < -1/2 = \mu^S(E_1)$ and $\mu^S(\tilde{F}^{(2)}) \leq 0 < 1/2 = \mu^S(\tilde{E}_2)$ by the stability of E_1 and $\tilde{E}_2$. Hence we have

$$
\mu^S(F) = \frac{\mu^S(F^{(1)}) + \mu^S(\tilde{F}^{(2)}) - 1}{2} \leq -1 < -\frac{1}{2} = \mu^S(E).
$$

If $\operatorname{rank} \tilde{F}^{(2)} = 2$, then

$$
\mu^S(F) \leq \frac{\mu^S(F^{(1)}) + 2\mu^S(\tilde{E}_2) - 2}{3} \leq -2/3 < -1/2 = \mu^S(E).
$$

Assume that $\operatorname{rank} F^{(1)} = 2$. If $\operatorname{rank} \tilde{F}^{(2)} = 0$, F is contained in the kernel F' of the homomorphism $E_1 \to \tilde{E}_2|_L$. Since $E_1 \to \tilde{E}_2|_L$ is not zero at the generic point of L, we have $\mu^S(F) \leq (2\mu^S(E_1) - 1)/2 = -1 < \mu^S(E)$. If $\operatorname{rank} \tilde{F}^{(2)} = 1$, we have $\mu^S(\tilde{F}^{(2)}) \leq 0 < 1/2 = \mu^S(\tilde{E}_2)$ by the stability of $\tilde{E}_2$. Hence we have

$$
\mu^S(F) = \frac{2\mu^S(F^{(1)}) + \mu^S(\tilde{F}^{(2)}) - 1}{3} \leq -\frac{2}{3} < \mu^S(E).
$$

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This completes the proof of the stability of E. Hence $E \in M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(1)}}(k) \neq \emptyset$. Similarly for integers n_1, n_2 with $n_1 + n_2 = n_0$ and $0 < n_2 < n_1$, $M_{(2),Q_0}^{P_{n_1}^{(1)},P_{n_2}^{(-1)}} \neq \emptyset$.

Let n_1, n_2 be integers with $n_1 + n_2 = n_0$ and $n_1, n_2 \geq 0$. We will show that $M_{(1),Q_0}^{P_{n_1}^{(0)},P_{n_2}^{(0)}}(k) \neq \emptyset$. There exist rank 2 μ-semi-stable sheaves E_1 on H_1, $\tilde{E}_2$ on H_2 and a generically injective homomorphism $f : E_1|_L \xrightarrow{\sim} \tilde{E}_2|_L$ such that $\chi(E_1(m)) = P_{n_1}^{(0)}(m)$, $\chi(\tilde{E}_2(m)) = P_{n_2}^{(0)}(m)$ and either E_1 or $\tilde{E}_2$ is μ-stable. Let E be the coherent sheaf on Q_0 corresponding to the triple $(E_1, \tilde{E}_2, f)$.

Claim 2. E is stable.

Let F be a coherent subsheaf of E with $0 < a_0(F) < a_0(E)$. The following exact commutative diagram is obtained;

$$\begin{array}{ccccccc}
0 \longrightarrow & F & \longrightarrow & F^{(1)} \oplus \tilde{F}^{(2)} & \longrightarrow & \tilde{F}^{(2)}|_L & \longrightarrow 0 \\
& \cap & & \cap & & \downarrow & \\
0 \longrightarrow & E & \longrightarrow & E_1 \oplus \tilde{E}_2 & \longrightarrow & \tilde{E}_2|_L & \longrightarrow 0.
\end{array}$$

Assume that $\operatorname{rank} F^{(1)} = 0$. Then F is contained in $\tilde{E}_2(-L)$ and so $\mu^S(F) \leq \mu^S(\tilde{E}_2(-L)) = -1 < -1/2 = \mu^S(E)$.

Assume that $\operatorname{rank} F^{(1)} = 1$. If $\operatorname{rank} \tilde{F}^{(2)} = 0$, then $F = F^{(1)}$ and $F^{(1)} \subset \ker(E_1 \to \tilde{E}_2|_L)$. Thus $\mu^S(F) = \mu^S(F^{(1)}) \leq \mu^S(E_1(-L)) = -1 < \mu^S(E)$. If $\operatorname{rank} \tilde{F}^{(2)} = 1$, then $\mu^S(F^{(1)}) \leq 0$ and $\mu^S(\tilde{F}^{(2)}) \leq 0$. Moreover $\mu^S(F^{(1)}) < 0$ or $\mu^S(\tilde{F}^{(2)}) < 0$. Hence we have

$$\mu^S(F) = \frac{\mu^S(F^{(1)}) + \mu^S(\tilde{F}^{(2)}) - 1}{2} \leq -1 < \mu^S(E).$$

If $\operatorname{rank} \tilde{F}^{(2)} = 2$, then

$$\mu^S(F) = \frac{\mu^S(F^{(1)}) + 2\mu^S(\tilde{F}^{(2)}) - 2}{3} \leq -2/3 < \mu^S(E).$$

Assume that $\operatorname{rank} F^{(1)} = 2$. If $\operatorname{rank} \tilde{F}^{(2)} = 0$, then $F = F^{(1)} \subset \ker(E_1 \to \tilde{E}_2|_L)$, and so

$$\mu^S(F) = \mu^S(F^{(1)}) \leq \mu^S(E_1(-L)) = -1 < \mu^S(E).$$

If $\operatorname{rank} \tilde{F}^{(2)} = 1$, then $F^{(1)}$ is contained in the kernel of $E_1 \to (\tilde{E}_2/\tilde{F}^{(2)})|_L$. Thus $\mu^S(F^{(1)}) \leq -1/2$ and

$$\mu^S(F) = \frac{2\mu^S(F^{(1)}) + \mu^S(\tilde{F}^{(2)}) - 1}{3} \leq -\frac{2}{3} < \mu^S(E).$$

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Hence E is a stable sheaf on Q_0. This implies that $M_{(1),Q_0}^{P_{n_1}^{(0)},P_{n_2}^{(0)}}(k) \neq \emptyset$. This completes the proof of the theorem. ∎

Remark 5.2. We consider structures of the moduli spaces $M_{(1),Q_0}^{P_{n_1}^{(0)},P_{n_2}^{(0)}}$, $M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(1)}}$ and $M_{(2),Q_0}^{P_{n_1}^{(1)},P_{n_2}^{(-1)}}$ appeared in Theorem 5.1. Let $M_{H_i}^P$ be the moduli scheme of stable sheaves on H_i with Hilbert polynomial P for $i = 1, 2$.

Assume that $n_1 \geq 2$ and $n_2 \geq 2$. From Theorem 2.1, there exist an open subscheme U_0 of $M_{(1),Q_0}^{P_{n_1}^{(0)},P_{n_2}^{(0)}}$ and a dominant morphism $\pi_0 : U_0 \to M_{H_1}^{P_{n_1}^{(0)}} \times M_{H_2}^{P_{n_2}^{(0)}}$ such that for a point x which corresponds to a triple $(E_1, \tilde{E}_2, f)$, $\pi_0(x)$ is the point which corresponds to $(E_1, \tilde{E}_2)$. Let η be a general point of a general fiber of π_0. Then we have

$$\begin{aligned}\dim_\eta M_{(1),Q_0}^{P_{n_1}^{(0)},P_{n_2}^{(0)}} &= \dim(M_{H_1}^{P_{n_1}^{(0)}} \times M_{H_2}^{P_{n_2}^{(0)}}) + 3 \\ &= 4n_1 - 3 + 4n_2 - 3 + 3 = 4n_0 - 3\end{aligned}$$

Similarly for $1 \leq n_1 < n_2$ (resp. $1 \leq n_2 < n_1$), there exist an open subscheme $U_1 \subset M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(1)}}$ (resp. $U_2 \subset M_{(2),Q_0}^{P_{n_1}^{(1)},P_{n_2}^{(-1)}}$) and a dominant morphism $\pi_1 : U_1 \to M_{H_1}^{P_{n_1}^{(-1)}} \times M_{H_2}^{P_{n_2}^{(1)}}$ (resp. $\pi_2 : U_2 \to M_{H_1}^{P_{n_1}^{(1)}} \times M_{H_2}^{P_{n_2}^{(-1)}}$). If η_1 is a general point of a general fiber of π_1, then

$$\begin{aligned}\dim_{\eta_1} M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(1)}} &= \dim(M_{H_1}^{P_{n_1}^{(-1)}} \times M_{H_2}^{P_{n_2}^{(1)}}) + 7 \\ &= 4n_1 - 4 + 4n_2 - 4 + 7 = 4n_0 - 1.\end{aligned}$$

Similarly for a general point η_2 of the general fiber of π_2, we have

$$\dim_{\eta_2} M_{(2),Q_0}^{P_{n_1}^{(1)},P_{n_2}^{(-1)}} = 4n_0 - 1.$$

Take $p \in M_{H_1}^{P_{n_1}^{(0)}} \times M_{H_2}^{P_{n_2}^{(0)}}$ such that the corresponding sheaves E_1,$\tilde{E}_2$ are locally free. Then the dimension of the fiber $\pi_0^{-1}(p)$ jumps if L is a jumping line of E_1 and $\tilde{E}_2$. See the definition of jumping line for [[13], 2.2]. In particular the coherent sheaf $\mathcal{H}$ mentioned in Remark 2.2 is not locally free in this case.

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Next we will consider relationships with the relative moduli space of stable sheaves on quadric surfaces. Put $\Sigma := \mathbf{P}(H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))^\vee)$. Let $\tilde{Q} \subset \mathbf{P}^3 \times \Sigma$ be the universal family of quadric surfaces. We will consider the relative moduli space $M_{\tilde{Q}/\Sigma}^{P_{n_0}^{(0,0)}}$ of stable sheaves on $\tilde{Q}/\Sigma$ with respect to the polarization $\mathcal{O}_{\tilde{Q}}(1) = \mathcal{O}_{\mathbf{P}^3_\Sigma}(1)|_{\tilde{Q}}$. Take a point $\xi \in \Sigma$ such that $\tilde{Q}_\xi = Q_0$ is a reducible quadric surface. The following proposition means that $M_{\tilde{Q}/\Sigma}^{P_{n_0}^{(0,0)}}$ is smooth over Σ at "general points" of $M_{(1),Q_0}^{P_{n_1}^{(0)},P_{n_2}^{(0)}}$.

Proposition 5.3. *If $n_1 \geq 2$ and $n_2 \geq 2$, then there exists a locally free stable sheaf $E \in M_{(1),Q_0}^{P_{n_1}^{(0)},P_{n_2}^{(0)}}$ such that $H^2(X, \mathcal{E}nd(E)) = 0$.*

Proof. Take stable bundles E_1 on H_1 with $c_1(E_1) = 0, c_2(E_1) = n_1$ and $\tilde{E}_2$ on H_2 with $c_1(\tilde{E}_1) = 0, c_2(\tilde{E}_2) = n_2$ such that $E_1|_L \cong \mathcal{O}_L^{\oplus 2}$ and $\tilde{E}_2|_L \cong \mathcal{O}_L^{\oplus 2}$. Take an isomorphism $f : E_1|_L \xrightarrow{\sim} \tilde{E}_2|_L$. Let E be the coherent sheaf on Q_0 corresponding to the triple $(E_1, \tilde{E}_2, f)$. Then E is locally free and stable by the proof of Theorem 5.1. The exact sequence

$$0 \longrightarrow \mathcal{E}nd(E) \longrightarrow \mathcal{E}nd(E_1) \oplus \mathcal{E}nd(\tilde{E}_2) \longrightarrow \mathcal{E}nd(\tilde{E}_2)|_L \longrightarrow 0$$

induces the following exact sequence;

$$0 = H^1(\mathcal{O}_L^{\oplus 4}) \longrightarrow H^2(\mathcal{E}nd(E)) \longrightarrow H^2(\mathcal{E}nd(E_1) \oplus \mathcal{E}nd(\tilde{E}_2)).$$

Since $E_1, \tilde{E}_2$ are stable bundles on $\mathbf{P}^2$, $H^2(\mathcal{E}nd(E_1)) = 0$ and $H^2(\mathcal{E}nd(\tilde{E}_2)) = 0$. Hence $H^2(\mathcal{E}nd(E)) = 0$. ∎

Theorem 5.4. *Assume that $1 \leq n_1 < n_2$. Take a general point p of a general fiber of π_1 in* Remark 5.2 *and let $E \in M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(1)}}$ be the corresponding sheaf. Put $R := k[[t]]$. Let t_0 be the closed point of* $\operatorname{Spec} R$ *and t_1 the generic point of* $\operatorname{Spec} R$. *Then there exists a morphism* $\operatorname{Spec} R \to \Sigma$ *such that $\tilde{Q} \otimes k(t_1)$ is a smooth quadric surface, $\tilde{Q} \otimes k(t_0) = Q_0$ and E can be lifted to a coherent sheaf on $\tilde{Q}_R$ flat over R.*

Proof. Take $E \in M_1^{P_{n_1}^{(-1)},P_{n_2}^{(1)}}$ which is a general point of a general fiber of π_1 and let $(E_1, \tilde{E}_2, f)$ be the corresponding triple. We may assume that E_1 and $\tilde{E}_2$ are locally free, $E_1|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$ and $\tilde{E}_2|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(1)$. For the homomorphism $f : \mathcal{O}_L \oplus \mathcal{O}_L(-1) \to \mathcal{O}_L \oplus \mathcal{O}_L(1)$, $\det f$ can be

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considered as a section of $\mathcal{O}_L(2)$. So we may assume that $f(p)$ is isomorphic except at distinct two points p_1, p_2 of L. Then coker $f_{p_i} \cong k(p_i)$ for $i = 1, 2$.

On the other hand, there is an exact sequence

$$\mathcal{E}nd(E) \longrightarrow \mathcal{E}nd(E)|_{H_1} \oplus \mathcal{E}nd(E)|_{H_2} \longrightarrow \mathcal{E}nd(E)|_L \longrightarrow 0.$$

Canonical homomorphisms $\mathcal{E}nd(E)|_{H_1} \to \mathcal{E}nd(E_1)$, $\mathcal{E}nd(E)|_{H_2} \to \mathcal{E}nd(\tilde{E}_2)$ are induced and they are isomorphic on $H_1 \setminus \{p_1, p_2\}$, $H_2 \setminus \{p_1, p_2\}$ respectively. Since E_1 and $\tilde{E}_2$ are stable bundles on $\mathbf{P}^2$, $H^2(\mathcal{E}nd(E_1)) = H^2(\mathcal{E}nd(\tilde{E}_2)) = 0$. Therefore we have $H^2(\mathcal{E}nd(E)|_{H_1}) = H^2(\mathcal{E}nd(E)|_{H_2}) = 0$. The composition $g : \mathcal{E}nd(E)|_{H_1} \to \mathcal{E}nd(E)|_L \to \mathcal{E}nd(\tilde{E}_2)|_L$ induces the following homomorphism:

$$\psi : \mathcal{E}nd(E)|_{H_1} \oplus \mathcal{E}nd(\tilde{E}_2) \longrightarrow \mathcal{E}nd(\tilde{E}_2)|_L; \quad \psi(a, b) := g(a) - b|_L.$$

Then we obtain the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & \mathcal{E}nd(E) & \longrightarrow & \mathcal{E}nd(E)|_{H_1} \oplus \mathcal{E}nd(E)|_{H_2} & \longrightarrow & \mathcal{E}nd(E)|_L & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \ker \psi & \longrightarrow & \mathcal{E}nd(E)|_{H_1} \oplus \mathcal{E}nd(\tilde{E}_2) & \xrightarrow{\psi} & \mathcal{E}nd(\tilde{E}_2)|_L & \longrightarrow 0. \end{array}$$

Since $\mathcal{E}nd(\tilde{E}_2)|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L^{\oplus 2} \oplus \mathcal{O}_L(1)$, we have $H^1(\mathcal{E}nd(\tilde{E}_2)|_L) = 0$. Thus by the exact sequence

$$H^1(\mathcal{E}nd(\tilde{E}_2)|_L) \longrightarrow H^2(\ker \psi) \longrightarrow H^2(\mathcal{E}nd(E)|_{H_1} \oplus \mathcal{E}nd(\tilde{E}_2)),$$

we have $H^2(\ker \psi) = 0$. From the construction, $\mathcal{E}nd(E) \to \ker \psi$ is isomorphic on $Q_0 \setminus \{p_1, p_2\}$. Hence we have $H^2(\mathcal{E}nd(E)) = 0$.

After a suitable projective linear transformation, we may assume that Q_0 is given by the equation $\{xy = 0\}$ in $\mathbf{P}^3 = \operatorname{Proj} k[x, y, z, w]$. Take $a_1 := \alpha z + \beta w \neq 0, a_2 := \gamma z + \delta w \neq 0$ with $\alpha, \beta, \gamma, \delta \in k$ such that $a_1(p_1) = 0$ and $a_2(p_2) = 0$. Consider $\mathcal{Q} := \operatorname{Proj} R[x, y, z, w]/(xy - ta_1a_2)$. Then $\mathcal{Q}_{t_0} = Q_0$ and $\mathcal{Q}_{t_1}$ is a smooth quadric surface. From Lemma 4.1 and Proposition 4.6, E can be lifted to a coherent sheaf $\tilde{E}$ on $\mathcal{Q}$ flat over R. ∎

Conclusion 5.5. Let $\eta \in \Sigma$ be the scheme theoretic generic point. Then we have

$$M_{\tilde{Q}_\eta}^{P_{n_0}^{(0,0)}} = \coprod_{a \geq 0} M_a,$$

where $M_a \otimes_{k(\eta)} \overline{k(\eta)} = M(2, (a, -a), n_0 - a^2) \coprod M(2, (-a, a), n_0 - a^2)$ for $a > 0$, $M_0 \otimes_{k(\eta)} \overline{k(\eta)} = M(2, (0, 0), n_0)$ and $M(2, (a, b), c)$ is the moduli scheme of rank 2 stable sheaves E on $\tilde{Q}\otimes_{k(\eta)}\overline{k(\eta)} \cong \mathbf{P}^1\times\mathbf{P}^1$ with $c_1(E)=(a,b)$ and $c_2(E)=c$. Note that $M(2,(a,-a),n_0-a^2)$ and $M(2,(-a,a),n_0-a^2)$ are contained in the same irreducible component of $M^{P^{(0,0)}_{n_0}}_{\tilde{Q}/\Sigma}$, which is the closure of M_a in $M^{P^{(0,0)}_{n_0}}_{\tilde{Q}/\Sigma}$. If $M(2,(a,-a),n_0-a^2)\neq\emptyset$, then $\dim M(2,(a,-a),n_0-a^2)=4n_0-3-2a^2$. Thus we can see from the dimension calculation in Remark 5.2 that $M^{P^{(0,0)}_{n_0}}_{\tilde{Q}/\Sigma}$ is not flat over Σ at "general points" of $M_1^{P^{(-1)}_{n_1},P^{(1)}_{n_2}}$. On the other hand, Theorem 5.4 means that "general points" in $M_1^{P^{(-1)}_{n_1},P^{(1)}_{n_2}}$ are contained in the closure of $M^{P^{(0,0)}_{n_0}}_{\tilde{Q}_\eta}$. Such a non-flat relative moduli space appears because $\dim\Sigma\geq 2$. (Compare it with the argument in [[3], section 2].) We can see more explicit properties of degeneration of sheaves. Let us consider a degeneration $\mathcal{Q}=\operatorname{Proj}R[x,y,z,w]/(xy-ta_1a_2)$ as in Theorem 5.4 such that a_1,a_2 are linearly independent. Let p_i be the zero point of a_i for $i=1,2$. Take $E\in M_1^{P^{(-1)}_{n_1},P^{(1)}_{n_2}}$ such that E_1 and $\tilde{E}_2$ are stable bundles, $E_1|_L\cong\mathcal{O}_L\oplus\mathcal{O}_L(-1)$, $\tilde{E}_2|_L\cong\mathcal{O}_L\oplus\mathcal{O}_L(1)$ and f is injective, where $(E_1,\tilde{E}_2,f)$ is the corresponding triple. The arguments of the proof of Theorem 5.4 conclude that E can be lifted to $\mathcal{Q}$ if and only if $\det f(p_i)=0$ for $i=1,2$. The dimension of

$$\left\{(E_1,\tilde{E}_2,f)\in M^{P^{(-1)}_{n_1},P^{(1)}_{n_2}}_{(1),Q_0}\;\middle|\;\begin{array}{l}E_1 \text{ and } \tilde{E}_2 \text{ are stable bundles,}\\ E_1|_L\cong\mathcal{O}_L\oplus\mathcal{O}_L(-1),\tilde{E}_2|_L\cong\mathcal{O}_L\oplus\mathcal{O}_L(1),\\ f \text{ is injective and } \det f(p_i)=0 \text{ for } i=1,2\end{array}\right\}$$

is $4n_0-3$. So it is contained in the closure of $M_0\subset M^{P^{(0,0)}_{n_0}}_{\tilde{Q}_\eta}$.

Next we will consider stable sheaves on Q_0 with another Hilbert polynomial. We fix a positive integer n_0 and put $P^{(-1,-1)}_{n_0}(m):=4\binom{m+2}{2}-4(m+1)+1-n_0$. For an integer n, put $P^{(-2)}_n(m):=2\binom{m+2}{2}-2(m+1)+1-n$. Put

$$M^{\circ P^{(-1,-1)}_{n_0}}_{Q_0}:=\left\{E\in M^{P^{(-1,-1)}_{n_0}}_{Q_0}\;\middle|\;\operatorname{rank}E|_{H_1}=\operatorname{rank}E|_{H_2}=2\right\}.$$

THEOREM 5.6. *Let n_0 be an integer with $n_0\geq 6$. Then*

$$M^{\circ P^{(-1,-1)}_{n_0}}_{Q_0}=\bigcup_{\substack{n_1+n_2=n_0\\ n_1>0,n_2>0}}M^{P^{(-1)}_{n_1},P^{(-1)}_{n_2}}_{(1),Q_0}\cup\bigcup_{\substack{n_1+n_2=n_0\\ 0<n_1<n_2+1}}M^{P^{(-2)}_{n_1},P^{(0)}_{n_2}}_{(1),Q_0}\cup\bigcup_{\substack{n_1+n_2=n_0\\ n_1+1>n_2>0}}M^{P^{(0)}_{n_1},P^{(-2)}_{n_2}}_{(2),Q_0}.$$

Moreover each component of the right hand side is non-empty.

Proof. Let E be an element of $M^{\circ P^{(-1,-1)}_{n_0}}_{Q_0}(k)$. Then

$$\chi(E(m)) = 4\binom{m+2}{2} + \left(c_1(E^{(1)}) + c_1(\tilde{E}^{(2)}) - 2\right)(m+1) - c_2(E^{(1)})$$
$$-c_2(\tilde{E}^{(2)}) + \frac{c_1(E^{(1)})^2 + c_1(E^{(1)}) + c_1(\tilde{E}^{(2)})^2 - c_1(\tilde{E}^{(2)})}{2}.$$

If we put $a := c_1(E^{(1)})$, then $c_1(\tilde{E}^{(2)}) = -a-2$. Assume that $a < -2$. Then $c_1(\tilde{E}^{(2)}(-L)) = -a-4 > -2$ and so $\mu^S(\tilde{E}^{(2)}(-L)) > -1 = \mu^S(E)$. Since $\tilde{E}^{(2)}(-L)$ is a subsheaf of E, this contradicts the stability of E. Assume that $a > 0$. Then $c_1(E^{(1)}(-L)) = a-2 > -2$. So $E^{(1)}(-L)$ is a subsheaf of E with $\mu^S(E^{(1)}(-L)) > -1 = \mu^S(E)$ which contradicts the stability of E. Hence we have $-2 \le a \le 0$.

Case 1. $a = -2$.

If we put $n_1 = c_2(\tilde{E}^{(1)})$ and $n_2 = c_2(E^{(2)})$, then $n_1 + n_2 = n_0$ and $E \in M^{P^{(-2)}_{n_1}, P^{(0)}_{n_2}}_{(1),Q_0}$. Since $E^{(1)}$ is a quotient sheaf of E, the stability of E implies that

$$\frac{\chi(E(m))}{4} < \frac{\chi(E^{(1)}(m))}{2} \quad \text{for } m \gg 0.$$

Hence we have $n_1 < n_2 + 1$. Since $E^{(1)}$ is a quotient sheaf of E with $\mu^S(E^{(1)}) = -1 = \mu^S(E)$, $E^{(1)}$ is μ-semi-stable. Thus $n_1 \ge 1$.

Case 2. $a = 0$.

Assume that the homomorphism $E^{(1)}|_L \to \tilde{E}^{(2)}|_L$ is not isomorphic at the generic point of L. Then the kernel F of the homomorphism $E^{(1)} \to \tilde{E}^{(2)}|_L$ is a subsheaf of E and $c_1(F) \ge -1$. So $\mu^S(F) \ge -1/2 > \mu^S(E)$ which contradicts the stability of E. Thus $E^{(1)}|_L \to \tilde{E}^{(2)}|_L$ is generically isomorphic. Consider the triple $(\tilde{E}^{(1)}, E^{(2)}, f)$ $(f : E^{(2)}|_L \to \tilde{E}^{(1)}|_L)$ corresponding to E. Then $c_1(\tilde{E}^{(1)}) = 0$ and $c_1(E^{(2)}) = -2$. Hence $E \in M_2^{P^{(0)}_{n_1}, P^{(-2)}_{n_2}}$ where $n_1 = c_2(\tilde{E}^{(1)})$ and $n_2 = c_2(E^{(2)})$. The proof of Case 1 implies that $n_1 + 1 > n_2 > 0$ and $n_1 + n_2 = n_0$.

Case 3. $a = -1$.

If we put $n_1 = c_2(E^{(1)})$ and $n_2 = c_2(\tilde{E}^{(2)})$, then $n_1 + n_2 = n_0$ and $E \in M^{P^{(-1)}_{n_1}, P^{(-1)}_{n_2}}_{(1),Q_0}$. Take any rank 1 coherent quotient sheaf F of $E^{(1)}$. The stability of E implies that $c_1(F) = \mu^S(F) > \mu^S(E) = -1$. Thus $\mu^S(F) = c_1(F) \ge 0 > \mu^S(E^{(1)})$. Hence $E^{(1)}$ is stable and so $n_1 > 0$. Take any rank 1 coherent quotient sheaf F of $\tilde{E}^{(2)}$. Let F' be the image of $E \to F$. Then $\mu^S(E) = -1 < \mu^S(F') \leq \mu^S(F)$ and so $\mu^S(F) \geq 0 > \mu^S(\tilde{E}^{(2)})$. Hence $\tilde{E}^{(2)}$ is stable and $n_2 > 0$. This completes the proof of the first part of the theorem.

Take integers n_1, n_2 with $n_1 + n_2 = n_0$ and $0 < n_1 < n_2 + 1$. We will show that $M_{(1),Q_0}^{P_{n_1}^{(-2)}, P_{n_2}^{(0)}} \neq \emptyset$. If $n_1 \neq 1, 2$, then there exist rank 2 μ-stable bundles E_1 on H_1 with $\chi(E_1(m)) = P_{n_1}^{(-2)}(m)$, $\tilde{E}_2$ on H_2 with $\chi(\tilde{E}_2(m)) = P_{n_2}^{(0)}(m)$ and an injective homomorphism $f : E_1|_L \to \tilde{E}_2|_L$. If $n_1 = 1$ or 2, then there exist a rank 2 μ-semi-stable sheaf E_1 on H_1 with $\chi(E_1(m)) = P_{n_1}^{(-2)}(m)$, a rank 2 μ-stable sheaf $\tilde{E}_2$ on H_2 with $\chi(\tilde{E}_2(m)) = P_{n_2}^{(0)}(m)$ and a generically injective homomorphism $f : E_1|_L \to \tilde{E}_2|_L$. Let E be the coherent sheaf of pure dimension 2 on Q_0 corresponding to the triple $(E_1, \tilde{E}_2, f)$.

Claim 1. E is stable

Take any coherent subsheaf F of E with $0 < a_0(F) < a_0(E)$. The following exact commutative diagram is obtained:

$$\begin{array}{ccccccc}
0 \longrightarrow & F & \longrightarrow & F^{(1)} \oplus \tilde{F}^{(2)} & \longrightarrow & \tilde{F}^{(2)}|_L & \longrightarrow 0 \\
 & \cap & & \cap & & \downarrow & \\
0 \longrightarrow & E & \longrightarrow & E_1 \oplus \tilde{E}_2 & \longrightarrow & \tilde{E}_2|_L & \longrightarrow 0.
\end{array}$$

Assume that $F^{(1)} = 0$. Then $F = \tilde{F}^{(2)}(-L)$. If $\operatorname{rank} \tilde{F}^{(2)} = 1$, then $\mu^S(F) < \mu^S(\tilde{E}_2(-L)) = -1 = \mu^S(E)$. If $\operatorname{rank} \tilde{F}^{(2)} = 2$, then for all sufficiently large integers m,

$$\begin{aligned}
\frac{\chi(F(m))}{2} &\leq \frac{\chi(\tilde{E}_2(-L)(m))}{2} \\
&= \binom{m+2}{2} - (m+1) - \frac{n_2}{2} \\
&< \binom{m+2}{2} - (m+1) + \frac{1-n_0}{4} = \frac{\chi(E(m))}{4}.
\end{aligned}$$

Assume that $\operatorname{rank} F^{(1)} = 1$. If $\operatorname{rank} \tilde{F}^{(2)} = 0$, then $F = F^{(1)} \subset \ker(E_1 \to \tilde{E}_2|_L)$. Since $E_1|_L \to \tilde{E}_2|_L$ is generically injective, $\mu^S(E_1(-L)) = \mu^S(\ker(E_1 \to \tilde{E}_2|_L))$. Thus $\mu^S(F) = \mu^S(F^{(1)}) \leq \mu^S(E_1(-L)) = -2 < \mu^S(E)$. If $\operatorname{rank} \tilde{F}^{(2)} = 1$, then $\mu^S(\tilde{F}^{(2)}) < \mu^S(\tilde{E}_2) = 0$. Thus

$$\mu^S(F) = \frac{\mu^S(F^{(1)}) + \mu^S(\tilde{F}^{(2)}) - 1}{2}$$

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$$\leq \frac{-1-1-1}{2} < -1 = \mu^S(E).$$

Assume that $\operatorname{rank} \tilde{F}^{(2)} = 2$. If $n_1 \neq 1, 2$, then E_1 is μ-stable and so $\mu^S(F^{(1)}) < \mu^S(E_1) = -1$. Hence

$$\mu^S(F) = \frac{\mu^S(F^{(1)}) + 2\mu^S(\tilde{F}^{(2)}) - 2}{3} \leq \frac{-2-2}{3} < -1 = \mu^S(E).$$

If $n_1 = 1$, then $E_1 \cong \mathcal{O}_{H_1}(-1)^{\oplus 2}$ and so

$$\chi(F^{(1)}(m)) \leq \chi(\mathcal{O}_{H_1}(m-1)) = \binom{m+2}{2} - (m+1)$$

for all sufficiently large integers m. If $n_1 = 2$, then $E_1^{\vee\vee} \cong \mathcal{O}_{H_1}(-1)^{\oplus 2}$ and $E_1^{\vee\vee}/E_1 \cong k(p)$ for some $p \in H_1$. Thus $\chi(F^{(1)}(m)) \leq \chi(\mathcal{O}_{H_1}(m-1))$ for $m \gg 0$. Moreover $n_1 = 1$ implies $n_2 \geq 5$ and $n_1 = 2$ does $n_2 \geq 4$. Hence for $n_1 = 1, 2$,

$$\begin{aligned}
\frac{\chi(F(m))}{3} &\leq \frac{\chi(F^{(1)}(m)) + 2\chi(\tilde{E}_2(m)) - 2\chi(\tilde{E}_2|_L(m))}{3} \\
&\leq \frac{\binom{m+2}{2} - (m+1) + 2\binom{m+2}{2} - 2(m+1) - n_2}{3} \\
&= \binom{m+2}{2} - (m+1) - \frac{n_2}{3} \\
&< \binom{m+2}{2} - (m+1) + \frac{1-n_1-n_2}{4} = \frac{\chi(E(m))}{4}
\end{aligned}$$

for all sufficiently large integers m.

Assume that $\operatorname{rank} F^{(1)} = 2$. If $\operatorname{rank} \tilde{F}_2 = 0$, then $F = F^{(1)} \subset \ker(E_1 \to \tilde{E}_2|_L)$. Since $E_1(-L) \to \ker(E_1 \to \tilde{E}_2|_L)$ is isomorphic in codimension 1, $\mu^S(F) \leq \mu^S(E_1(-L)) = -2 < \mu^S(E)$. If $\operatorname{rank} \tilde{F}_2 = 1$, then $\mu^S(\tilde{F}_2) < \mu^S(\tilde{E}_2) = 0$. Hence

$$\mu^S(F) \leq \frac{2\mu^S(E_1) + \mu^S(\tilde{F}_2) - 1}{3} \leq -\frac{4}{3} < -1 = \mu^S(E).$$

Hence E is a stable sheaf on Q_0.

We have proved Claim 1 and so we have $M_{(1),Q_0}^{P_{n_1}^{(-2)}, P_{n_2}^{(0)}} \neq \emptyset$. Similarly for $n_1, n_2 \in \mathbf{Z}$ with $0 < n_2 < n_1 + 1$, we have $M_{(2),Q_0}^{P_{n_1}^{(0)}, P_{n_2}^{(-2)}} \neq \emptyset$.

Let n_1, n_2 be integers with $n_1 > 0$, $n_2 > 0$ and $n_1 + n_2 = n_0$. We will prove that $M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(-1)}} \neq \emptyset$. There are rank 2 stable bundles E_1 on H_1 with $\chi(E_1(m)) = 2\binom{m+2}{2} - (m+1) - n_1$, $\tilde{E}_2$ on H_2 with $\chi(\tilde{E}_2(m)) = 2\binom{m+2}{2} - (m+1) - n_2$, and an injective homomorphism $f : E_1|_L \to \tilde{E}_2|_L$. Let E be the coherent sheaf on Q_0 corresponding to the triple $(E_1, \tilde{E}_2, f)$.
Claim 2. E is stable.

Let F be a coherent subsheaf of E with $0 < a_0(F) < a_0(E)$. Assume that $\operatorname{rank} F^{(1)} = 0$. Then $F = \tilde{F}^{(2)}(-L) \subset \tilde{E}_2(-L)$. Thus $\mu^S(F) \leq \mu^S(\tilde{E}_2(-L)) = -3/2 < -1 = \mu^S(E)$. Assume that $\operatorname{rank} F^{(1)} = 1$. If $\operatorname{rank} \tilde{F}^{(2)} = 0$, then $F = F^{(1)} \subset \ker(E_1 \to \tilde{E}_2|_L) = E_1(-L)$. Thus $\mu^S(F) \leq \mu^S(E_1(-L)) = -3/2 < -1 = \mu^S(E)$. If $\operatorname{rank} \tilde{F}^{(2)} \geq 1$, then

$$\begin{aligned}\mu^S(F) &= \frac{\mu^S(F^{(1)}) + \operatorname{rank} \tilde{F}^{(2)}(\mu^S(\tilde{F}^{(2)}) - 1)}{1 + \operatorname{rank} \tilde{F}^{(2)}} \\ &\leq \frac{-1 - 3\operatorname{rank} \tilde{F}^{(2)}/2}{1 + \operatorname{rank} \tilde{F}^{(2)}} < -1 = \mu^S(E).\end{aligned}$$

Assume that $\operatorname{rank} F^{(1)} = 2$. If $\operatorname{rank} \tilde{F}^{(2)} = 0$, then $F = F^{(1)} \subset \ker(E_1 \to \tilde{E}_2|_L) = E_1(-L)$ Thus $\mu^S(F) \leq \mu^S(E_1(-L)) = -3/2 < -1 = \mu^S(E)$. If $\operatorname{rank} \tilde{F}^{(2)} = 1$, then the commutative diagram

$$\begin{array}{ccc} F^{(1)} & \longrightarrow & \tilde{F}^{(2)}|_L \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & \tilde{E}_2|_L \end{array}$$

implies that $\mu^S(F^{(1)}) \leq \mu^S(E_1) - 1/2 = -1$. Hence

$$\mu^S(F) = \frac{2\mu^S(F^{(1)}) + \mu^S(\tilde{F}^{(2)}) - 1}{3} \leq \frac{-2 - 1 - 1}{3} < -1 = \mu^S(E).$$

Thus E is a stable sheaf on Q_0. Hence $M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(-1)}} \neq \emptyset$. □

Remark 5.7. Similar calculations to Remark 5.2 show that at a "general" point p of $M_1^{P_{n_1}^{(-1)},P_{n_2}^{(-1)}}$, $\dim_p M_1^{P_{n_1}^{(-1)},P_{n_2}^{(-1)}} = 4n_1 - 4 + 4n_2 - 4 + 3 = 4n_0 - 5$. Similarly at a "general" point p of $M_1^{P_{n_1}^{(-2)},P_{n_2}^{(0)}}$, $\dim_p M_1^{P_{n_1}^{(-2)},P_{n_2}^{(0)}} = 4n_1 - 7 + 4n_2 - 3 + 7 = 4n_0 - 3$.

Similar arguments to Proposition 5.3 conclude the following proposition.

Proposition 5.8. *For $n_1 \geq 1$ and $n_2 \geq 1$, there exists a locally free stable sheaf $E \in M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(-1)}}$ such that $H^2(X, \mathcal{E}nd(E)) = 0$.*

Remark 5.9. Consider a degeneration $\mathcal{Q} = \operatorname{Proj}(R[x,y,z,w]/(xy - ta_1a_2))$ as in the proof of Theorem 5.4. We assume that a_1, a_2 are linearly independent. Let $p_i \in L$ be the zero point of a_i. Let $\bar{t}_1$ be a generic geometric point of $\operatorname{Spec} R$. There is a decomposition of a moduli space of stable sheaves on a smooth quadric surface;

$$M_{\mathcal{Q}_{\bar{t}_1}}^{P_{n_0}^{(-1,-1)}} = \coprod_{a \in \mathbf{Z}} M(2, (a-1, -a-1), n_0 - a^2).$$

If $M(2,(a-1,-a-1), n_0 - a^2) \neq \emptyset$, then $\dim M(2,(a-1,-a-1), n_0 - a^2) = 4n_0 - 5 - 2a^2$. From Proposition 5.8, there exists an open subscheme $U \subset M_{\mathcal{Q}/R}^{P_{n_0}^{(-1,-1)}}$ smooth over R such that $U_{\bar{\eta}} \subset M(2,(-1,-1),n_0)$ and $U_{t_0} \subset M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(-1)}}$. We may assume that E is locally free for any $E \in U_{t_0}$ and that $E|_{H_1}, E|_{H_2}$ are stable. There exist a scheme V over R and a morphism of functors $\theta : V \to \mathcal{M}_{\mathcal{Q}/R}^{P_{n_0}^{(-1,-1)}}$ such that the induced morphism $\theta' : V \to M_{\mathcal{Q}/R}^{P_{n_0}^{(-1,-1)}}$ is étale and that the image is U. On the other hand, let $\tilde{I}_{x,a_1}$ be the image of the homomorphism $\mathcal{O}_{\mathcal{Q}}(-1)^{\oplus 2} \to \mathcal{O}_{\mathcal{Q}}$ defined by the sections x, a_1. Then $\mathcal{O}_{\mathcal{Q}}/\tilde{I}_{x,a_1}$ is flat over R. Let $\tilde{E}$ be a flat family of stable bundles on $\mathcal{Q} \times_R V/V$ corresponding to θ. Then $\tilde{E} \otimes \tilde{I}_{x,a_1}(1)$ is a flat family of stable sheaves whose fiber over $V_{\bar{\eta}}$ is contained in $M(2,(-1,1),n_0-1)$ and whose fiber over V_{t_0} is contained in $M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(1)}}$.

Let us consider an open subscheme

$$W := \left\{ (E_1, \tilde{E}_2, f) \in M_1^{P_{n_1}^{(-1)},P_{n_2}^{(1)}} \;\middle|\; \begin{array}{l} E_1 \text{ and } \tilde{E}_2 \text{ are stable bundles,} \\ E_1|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1), \\ \tilde{E}_2|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(1), \\ f : E_1|_L \to \tilde{E}_2|_L \text{ is injective} \end{array} \right\}$$

of $M_{(1),Q_0}^{P_{n_1}^{(-1)},P_{n_2}^{(1)}}$. Then the set $W' := \{\tilde{E} \otimes \tilde{I}_{x,a_1}(1) \otimes k(s)\}_{s \in V_{t_0}}$ is contained in W and

$$W' = \left\{ (E_1, \tilde{E}_2, f) \in W \;\middle|\; \begin{array}{l} f = \begin{pmatrix} a_1 & (\alpha z + \beta w)a_1 \\ 0 & \gamma a_1 \end{pmatrix} \\ \alpha, \beta, \gamma \in k, \ \alpha \neq 0 \end{array} \right\},$$

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where $f : \mathcal{O}_L \oplus \mathcal{O}_L(-1) = E_1|_L \to \tilde{E}_2|_L = \mathcal{O}_L \oplus \mathcal{O}_L(1)$ is regarded as a matrix $\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_1 \in H^0(\mathcal{O}_L(1)), b_2 \in H^0(\mathcal{O}_L(2)), b_3 \in k$ and $b_4 \in H^0(\mathcal{O}_L(1))$. Moreover $\dim M(2,(-1,1),n_0-1) = \dim W' = 4n_0 - 5$.

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Department of Mathematics
Faculty of Science
Kyoto University
Kyoto 606-8502, Japan
inaba@kusm.kyoto-u.ac.jp