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# NORM CONVERGENCE IN ERGODIC THEORY AND THE BEHAVIOR OF FOURIER TRANSFORMS

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ABSTRACT. The $L_p$-norm convergence of weighted averages $\mu_n f$ in ergodic theory is equivalent to the pointwise convergence of the Fourier transforms $\hat{\mu}_n$. If $h(\gamma) = \lim_{n\to\infty} \hat{\mu}_n(\gamma)$, then the behavior of $h$ determines when the $L_p$-norm limit of $\mu_n f$ is $\int f\,dm$. The nature of such limit functions $h$ is the focus of this article.

Let $(X, \beta, m)$ be a probability space and let $\tau$ be an invertible measure-preserving transformation of $(X, \beta, m)$. Let $(\mu_n)$ be a sequence of probability measures on $Z$ and define the operators $\mu_n f$ by

$$\mu_n f(x) = \sum_{k\in Z} \mu_n(k) f(\tau^k x)$$

for all $x \in X$, and $f: X \longrightarrow \mathbb{C}$. This gives well-defined linear operators from $L_p(X)$ to $L_p(X)$, $1 \le p \le \infty$; moreover, $\|\mu_n\|_p = 1$. The questions treated here are 1) when does $(\mu_n f)$ converge in $L_p$-norm for $f \in L_p(X)$, $1 \le p < \infty$, and 2) what is the limit function $f^* = \lim_{n\to\infty} \mu_n f$, when it exists?

The first very well-known principle is the following one which characterizes general norm convergence in terms of the behavior of the Fourier transforms. The Fourier transform of the measure $\mu_n$ is given by

$$\hat{\mu}_n(\gamma) = \sum_{k\in Z} \mu_n(k)\bar{\gamma}^k, \quad \text{for } \gamma \in T = \{z : |z| = 1\}.$$

The proof of Theorem 1 uses only a small part of the harmonic analysis which is inherent in the usual arguments for the Spectral Theorem on normal operators.

THEOREM 1. *For $(\mu_n f)$ to converge in $L_p$-norm for all $f \in L_p(X, \beta, m)$, $1 \le p < \infty$, and all $(X, \beta, m, \tau)$, it is necessary and sufficient that* $\lim_{n\to\infty} \hat{\mu}_n(\gamma)$ *exists for all* $\gamma \in T$.

To prove this theorem, we first need a lemma.

LEMMA 2. *The sequence $(\mu_n f)$ converges in $L_p$-norm for all $f \in L_p(X, \beta, m)$, $1 \le p < \infty$, if and only if $(\mu_n f)$ converges in $L_2$-norm for all $f \in L_2(X, \beta, m)$.*

PROOF. It suffices to prove only the non-trivial direction of implication above. Since $\|\mu_n\|_p \le 1$ for all $p$, $1 \le p \le \infty$, to prove $L_p$-norm convergence, it suffices to prove $L_p$-norm convergence on the dense subspace $L_\infty(X, \beta, m)$ of $L_p(X, \beta, m)$. Let $f \in L_\infty(X, \beta, m)$, and assume $(\mu_n f)$ converges to $f^*$ in $L_2$-norm. Some subsequence $(\mu_{n_m} f)$ converges a.e. to

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$f^*$. Since $\|\mu_n f\|_\infty \leq \|f\|_\infty$, this shows $f^* \in L_\infty(X, \beta, m)$. Let $(\mu_{n_m} f)$ be any subsequence. There is a further subsequence $(\mu_{n_{m_k}} f)$ converging a.e. to $f^*$. Then by Lebesgue's bounded convergence theorem, $(\mu_{n_{m_k}} f)$ converges in $L_p$-norm to $f^*$ for any $p$, $1 \leq p < \infty$. Hence, since the $L_p$-norm topology is a metric topology, the full sequence $(\mu_n f)$ converges in $L_p$-norm to $f^*$. ∎

PROOF OF THEOREM 1. Assume $(\hat{\mu}_n)$ converges pointwise on $T$. Let $f \in L_2(X, \beta, m)$ and consider the positive definite function $\rho$ on $Z$ given by $\rho_f(k) = \langle f \circ \tau^k, f\rangle$ with $\langle\, ,\rangle$ being the inner-product in $L_2(X, \beta, m)$.

By the Herglotz Theorem, there is a positive regular Borel measure $\nu_f$ on $T$ with $\hat{\nu}_f(k) = \rho_f(k)$ for all $k \in Z$. But then for all finite measures $\mu$ on $Z$,

$$\begin{aligned}
\|\mu f\|_2^2 &= \langle \mu f, \mu f\rangle \\
&= \sum_{k\in Z}\sum_{\ell\in Z} \mu(k)\overline{\mu(\ell)}\langle f\circ\tau^{k-\ell}, f\rangle \\
&= \sum_{k\in Z}\sum_{\ell\in Z} \mu(k)\overline{\mu(\ell)}\hat{\nu}_f(k-\ell) \\
&= \sum_{k\in Z}\sum_{\ell\in Z} \mu(k)\overline{\mu(\ell)} \int_T \bar{\gamma}^{k-\ell}\, d\nu_f(\gamma) \\
&= \int_T |\hat{\mu}(\gamma)|^2\, d\nu_f(\gamma).
\end{aligned}$$

Hence, for $m, n \geq 1$,

$$\|\mu_m f - \mu_n f\|_2^2 = \int_T |\hat{\mu}_m(\gamma) - \hat{\mu}_n(\gamma)|^2\, d\nu_f(\gamma).$$

But $(\hat{\mu}_n)$ is a uniformly bounded sequence of continuous functions on $T$ which converges pointwise on $T$. Hence, by Lebesgue's bounded convergence theorem $(\hat{\mu}_n)$ converges in $L_2(T, \nu)$. Thus, $(\mu_n f)$ is $L_2$-norm Cauchy and must converge in $L_2$-norm.

Conversely, let $(X, \beta, m, \tau)$ be the dynamical system with $X = T$ and $m = \lambda_T$, the usual Lebesgue measure on $T$. Let $\tau(\gamma) = \overline{\gamma_0}\gamma$ for some $\gamma_0 \in T$. Then if $f \in L_2(T, \lambda_T)$ is given by $f(\gamma) = \gamma$ for $\gamma \in T$, we see that $\mu_n f(\gamma) = \hat{\mu}_n(\gamma_0) f(\gamma)$. Thus, the convergence of $(\mu_n f)$ in $L_2$-norm implies the convergence of $\big(\hat{\mu}_n(\gamma_0)\big)$. Since we can vary the choice of $\gamma_0$, the Fourier transforms $(\hat{\mu}_n)$ converge pointwise on $T$. ∎

REMARK 3. a) Because of the consequence, if $(\mu_n)$ is as in Theorem 1, we will say it is *universally $L_2$-norm good.*

b) See Feller [5], XIXb, for an elementary proof of the Herglotz Theorem.

c) The von Neumann Mean Ergodic Theorem for a unitary operator $U$ on $L_2(X)$ says that for all $f \in L_2(X)$, $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N U^k f$ exists in $L_2$-norm and is the value $P_U f$ of the normal projection $P_U$ of $L_2(X)$ on the $U$-invariant functions $I_U$. The intrinsic proof that is usually given for this is to observe that $L_2(X) = I_U \oplus \mathrm{cl}_{\|\cdot\|_2} \operatorname{span}\{f - Uf : f \in L_2(X)\}$ where $I_U = \{f \in L_2(X) : Uf = f\}$. Then the behavior of the averages on the whole space is obvious. But also the previous argument can be used, only one takes $U$ in place of $f \longmapsto f \circ \tau$, and each $\mu_n f = \frac{1}{n}\sum_{k=1}^n U^k f$. Since each function $k \longmapsto \langle U^k f, f\rangle$ is positive definite, the norm convergence of the averages $\frac{1}{N}\sum_{k=1}^{N} U^k$ is immediate. But then also, since the limit function $f^* = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} U^k f$ is clearly $U$-invariant, it follows that $f^* = P_U f$ for all $f \in L_2(X)$. Indeed, when $f \perp I_U$, any $U^k f \perp I_U$ and so $f^* \perp I_U$ too. That is, $f^* = 0$ if $f \perp I_U$ while $f^* = f$ if $f \in I_U$.

The first part of Theorem 1 can of course be proved by directly appealing to the Spectral Theorem. This has some advantages. For example, let $E$ be the projection-valued measure associated with the unitary operator $Uf = f \circ \tau$, for $f \in L_2(X, \beta, m)$. If $h(\gamma) = \lim_{n\to\infty} \hat{\mu}_n(\gamma)$ exists for all $\gamma$, then $h$ is a Borel measurable function and $(\mu_n)$ converges to the bounded operator $\int_T h(\gamma)E(d\gamma)$ in the strong operator topology.

So the Spectral Theorem shows that if $h(\gamma) = 0$ except for $\gamma = 1$, $\gamma \in T$, then $(\mu_n)$ converges to the normal projection $P_I = E(\{1\})$ onto the $\tau$-invariant functions $I$ in $L_2(X, \beta, m)$. However, this can be seen easily also in the style of the proof above because if $I = \{f \in L_2(X, \beta, m) : f \text{ is } \tau\text{-invariant}\}$, then for any $f \in I^\perp$, the measure $\nu_f$ has no point mass at 1. Indeed, in this case, by Remark 3.c, $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f \circ \tau^k = P_I f = 0$ in $L_2$-norm. So $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \hat{\nu}_f(k) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \langle f \circ \tau^k, f\rangle = 0$. But

$$\lim_{N\to\infty} \frac{1}{N}\sum_{K=1}^{N} \hat{\nu}_f(k) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \int_T \bar{\gamma}^k \, d\nu_f(\gamma) = \int_T \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \bar{\gamma}^k \, d\nu_f(\gamma) = \nu_f(\{1\}).$$

So $\nu_f(\{1\}) = 0$. Actually, this argument shows $\nu_f(\{1\}) = 0$ if and only if $f \in I^\perp$. Hence, if $f \in I^\perp$, we have

$$\begin{aligned} \lim_{n\to\infty} \|\mu_n f\|_2^2 &= \lim_{n\to\infty} \int_T |\hat{\mu}_n(\gamma)|^2 \, d\nu_f(\gamma) \\ &= \nu_f(\{1\}) = 0. \end{aligned}$$

See also Jones, Rosenblatt, Tempelman [6] for extensions of the above arguments to general (abelian) groups.

More generally, if $h(\gamma)$ exists and is 0 except for at most a countable number of values $\gamma$, then the same argument will work to identify the limit $f^*$ as long as $\nu_f$ has no discrete part. This is the reason for observing the following.

PROPOSITION 4. *The transformation $\tau$ is weakly mixing if and only if whenever $f \in L_2(X, \beta, m)$, $f \perp 1$, then $\nu_f$ has no point masses.*

While this is also a well-known principle, there are serveral aspects of this which are worth pointing out, not the least of which is that the arguments to prove it are typical of the ones being used here.

First, observe that

LEMMA 5. *The measure $\nu_f$ has no point mass at $\lambda$ if and only if* $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \hat{\nu}_f(k)\lambda^k = 0$.

PROOF. Let $\delta_{\gamma,\lambda} = \begin{cases} 1, & \text{if } \gamma = \lambda \\ 0, & \text{otherwise.} \end{cases}$ Then $\frac{1}{N}\sum_{k=1}^{N} \hat{\nu}_f(k)\lambda^k = \int_T \frac{1}{N}\sum_{k=1}^{N} \bar{\gamma}^k \lambda^k \, d\nu_f(\gamma)$. So $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \hat{\nu}_f(k)\lambda^k$ exists and is $\int_T \delta_{\gamma,\lambda} \, d\nu_f(\gamma) = \nu_f(\{\lambda\})$. ■

REMARK 6. The proof is showing that $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \hat{\nu}_f(k)\lambda^k$ exists for all $\lambda \in T$.

Now $\tau$ is *weakly mixing* if and only if for all $f \in L_2(X)$, $\int f\,dm = 0$, $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N |\hat{\nu}_f(k)| = 0$. Indeed, the usual definition is that for $g, h \in L_2(X)$, $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N |\langle g\circ\tau^k, h\rangle - \int g\,dm \int h\,dm| = 0$. But functions of the form $k \longmapsto \langle g\circ\tau^k, h\rangle$ are linear combinations over $\mathbb{C}$ of functions $k \longmapsto \langle f\circ\tau^k, f\rangle = \hat{\nu}_f(k)$ for suitable $f$. So the equivalence of these two definitions is obvious.

However, it is also a fact about sequences $(\alpha_k)$ which are bounded that $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N |\alpha_k| = 0$ if and only if $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N |\alpha_k|^2 = 0$. So we have

LEMMA 7. *The transformation $\tau$ is weakly mixing if and only if $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N |\hat{\nu}_f(k)|^2 = 0$ for all $f \in L_2(X)$, $\int f\,dm = 0$.*

These remarks give

PROOF OF PROPOSITION 4. Clearly, if $\tau$ is weakly mixing, then $\limsup_{N\to\infty} \frac{1}{N}|\sum_{k=1}^N \hat{\nu}_f(\lambda)\lambda^k| \leq \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^N |\hat{\nu}_f(k)| = 0$ for all $f \perp 1$. So by Lemma 5, $\nu_f$ has no point masses for $f \perp 1$.

Conversely, suppose $\nu_f$ has no point masses for $f \perp 1$. Then for all $f \in L_2(X)$, $\int f\,dm = 0$, $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k)\lambda^k = 0$ for all $\lambda \in T$. But the reflected measure $\tilde{\nu}_f$ given by $\tilde{\nu}_f(E) = \nu_f(E^{-1})$ has $\widehat{\tilde{\nu}_f}(k) = \overline{\nu_f(k)}$ for all $k \in Z$. Hence, for $f \perp 1$,

$$
\begin{aligned}
\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N |\hat{\nu}_f(k)|^2 &= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k)\widehat{\tilde{\nu}_f}(k) \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k) \int_T \bar{\gamma}^k\, d\tilde{\nu}_f(\gamma) \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k) \int_T \gamma^k\, d\nu_f(\gamma) \\
&= \int_T \Big(\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k)\gamma^k\Big)\, d\nu_f(\gamma) \\
&= 0
\end{aligned}
$$

because $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k)\gamma^k = 0$ for all $\gamma \in T$. Thus, if $\nu_f$ has no point masses for $f \perp 1$, $\tau$ is weakly mixing by Lemma 7. ∎

It is worth noting here that this elementary argument also easily gives another characterization of weakly mixing transformation; namely, they are transformations with no non-trivial eigenvalues. First, we need a little more notation. Let $\lambda \in T$ and define $\mathcal{E}_\lambda = \{f \in L_2(X) : f\circ\tau = \lambda f\}$. This is the *$\lambda$-eigenspace* of $\tau$. Let $E_\lambda : L_2(X) \longrightarrow L_2(X)$ be the unitary operator $E_\lambda f = \bar{\lambda} f\circ\tau$. Then let $P_\lambda : L_2(X) \longrightarrow L_2(X)$ be the normal projection onto the eigenspace $\mathcal{E}_\lambda$. The von Neumann Mean Ergodic Theorem as in Remark 3.c says

PROPOSITION 8. *For all $f \in L_2(X)$, $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N E_\lambda^k f = P_\lambda f$ in $L_2$-norm.*

But this easily gives this localized version of Lemma 5.

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PROPOSITION 9. *For $\lambda \in T$ and $f \in L_2(X)$, $f \perp \mathcal{E}_\lambda$ if and only if $\lim_{N\to\infty} \frac{1}{N}\sum_{K=1}^N \hat{\nu}_f(k)\bar{\lambda}^k = 0$.*

PROOF. Here $\frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k)\bar{\lambda}^k = \frac{1}{N}\sum_{k=1}^N \langle E_\lambda^k f, f\rangle$. So $\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k)\bar{\lambda}^k$ exists and is $\langle P_\lambda f, f\rangle$. Hence, this limit is 0 if and only if $f \perp E_\lambda$. ■

These corollaries are all immediate consequences of the previous discussion.

COROLLARY 10. *If $\lambda \in T$ and $f \in L_2(X)$, then $f \perp \mathcal{E}_\lambda$ if and only if $\nu_f$ has no point mass at $\bar{\lambda}$.*

PROOF. See the proof of Lemma 5. ■

COROLLARY 11. *If $\lambda \in T$, then $\mathcal{E}_\lambda = \{0\}$ if and only if $\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k)\bar{\lambda}^k = 0$ for all $f \in L_2(X)$.*

COROLLARY 12. *If $\lambda \in T$, then $\mathcal{E}_\lambda = \{0\}$ if and only if for all $f \in L_2(X)$, $\nu_f$ has no point mass at $\bar{\lambda}$.*

THEOREM 13. *Given a transformation $\tau$, $\tau$ is weakly mixing if and only if $\mathcal{E}_\lambda = \{0\}$ for all $\lambda \neq 1$ and $\mathcal{E}_1$ consists only of constants.*

PROOF. Assume $\mathcal{E}_\lambda = \{0\}$ for all $\lambda \neq 1$. Then $\nu_f$ has no point mass at $\lambda \neq 1$. But if also $f \perp 1$, then $\nu_f$ has no point mass at 1 either, by the remarks after the proof of Theorem 3. So $\nu_f$ has no point mass at all for $f \perp 1$. Thus, Proposition 4 shows $\tau$ is weakly mixing. Conversely, assume $\tau$ is weakly mixing. If $\lambda \neq 1$ and $f \in \mathcal{E}_\lambda$, then $f \perp 1$. So by Proposition 4, $\nu_f$ has no point masses and $\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N \hat{\nu}_f(k)\bar{\lambda}^k = 0$ by Lemma 5. But $f \in \mathcal{E}_\lambda$ means $\hat{\nu}_f(k) = \langle f\circ\tau^k, f\rangle = \lambda^k \|f\|_2^2$. So $0 = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N \lambda^k\|f\|_2^2\bar{\lambda}^k = \|f\|_2^2$. Thus, $\mathcal{E}_\lambda = \{0\}$ if $\lambda \neq 1$. ■

REMARK 14. a) The differences between this development of Theorem 13, and the one in Parry [10], Petersen [11], or Walters [14] is that this approach is almost entirely Fourier analytic, with the Herglotz theorem being the only important tool. The argument in this form does not even require the machinery of the Spectral Theorem.

b) A more sophisticated version of the previous results would center around the fact that $\hat{\nu}_f$ is weakly almost periodic and has the form $a+n$ where $a$ is almost periodic and $n$ is weakly almost periodic with mean value for $|n|$ being 0. Then for at most a countable number of $\lambda \in T$, $\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N a(k)\lambda^k \neq 0$, and if $(\lambda_s : s \geq 1)$ are these values, then there exists $f_s \in \mathcal{E}_{\lambda_s}$ such that $a = \sum_{s=1}^\infty f_s$. Hence, the eigenspaces $\mathcal{E}_\lambda = \{0\}$ for $\lambda \neq 1$ if and only if $a = 0$ in every case. The previous arguments are preferred here because they avoid any discussion of the structure of AP(Z) and the behavior of the unique invariant mean on the space of weakly almost periodic functions.

The above discussion gave Proposition 4 and Theorem 13 and should make the following two theorems quite clear. But it is also easy to see how this same discussion can be used to state a more local form of the next proposition.

PROPOSITION 15. *If the* $\lim_{n\to\infty}\hat{\mu}_n(\gamma) = h(\gamma)$ *exists and is* 0 *for all but a countable number of values and* $\nu_f$ *has no point masses, then* $\lim_{n\to\infty}\|\mu_n f - \int f\,dm\|_2 = 0$.

COROLLARY 16. *If the* $\lim_{n\to\infty}\hat{\mu}_n(\gamma) = h(\gamma)$ *exists and is* 0 *for all but a countable number of values, then for any weakly mixing system* $(X,\tau)$*, and* $f \in L_2(X)$, $\lim_{n\to\infty}\|\mu_n f - \int f\,dm\|_2 = 0$.

Such a result as in Corollary 16 is what can be called a *norm theorem with the classical limit*. To what degree are such theorems generally available? For example, if we have a sequence $(\mu_n)$ for which $(\mu_n f)$ converges in $L_2$-norm for all $f \in L_2(X)$ and all $\tau$, can one assert anything about the limit $f^*$? There are plenty of examples to show why $f^*$ is not the classical limit in general. For instance, if $\mu_n = \frac{1}{n}\sum_{k=1}^{n}\delta_{k^2}$, then $\big(\hat{\mu}_n(\gamma)\big)$ converges for all $\gamma$. However, the limit is not zero for roots of unity. So for some dynamical systems, namely ones with $\mathcal{E}_\lambda \neq \{0\}$ for $\lambda \neq 1$ and a root of unity, there will be $f \in L_2(X)$ for which $f^* \neq \int f\,dm$.

Such examples lead to the following question. First, we need this

DEFINITION 17. The sequence $(\mu_n)$ is *dissipative* if $\lim_{n\to\infty}\mu_n(k) = 0$ for all $k$. The sequence is *uniformly dissipative* if $\lim_{n\to\infty}\sup_{k\in Z}\mu_n(k) = 0$.

It is clear that only averaging sequences $(\mu_n)$ which are dissipative are really measuring some aspects of the long term behavior of $(\tau_k)$. For this reason, this technical assumption is added here. Note that a characterization of uniform dissipation in terms of $(\hat{\mu}_n)$ is given later in Example 23.b. See also Bellow, Jones, and Rosenblatt [1] where the concept of uniform dissipation is used.

QUESTION 18. If $(\mu_n)$ is dissipative and $(\hat{\mu}_n)$ converges pointwise on $T$, is the limit $h$ necessarily 0 except for countably many values?

If the answer to Question 18 were affirmative, then for weakly mixing systems, the limit $f^*$ would be the classical limit. On the other hand, if not, then for some $\alpha > 0$, the set $\{\gamma \in T : |h(\gamma)| \geq \alpha\}$ must be uncountable. Since this set is a Borel set, it must contain a closed perfect set $K$. But then there is a positive measure $0 \neq \nu \in M(K)$ with no point masses. Such a measure $\nu$ is $\nu_f$ for some $f \in L_2(X)$ and some weakly mixing system. This is seen by using the Gaussian measure space construction. See Schmidt [13] or Cornfeld et al. [4], 8.2 and 14. Hence, the answer to Question 18 is exactly resolving whether all dissipative $L_2$-norm good averaging mathods must necessarily converge to the classical limit. An example given later in this paper in Proposition 33 shows that the answer to Question 18 is negative.

There are some extremes when the norm limit $f^*$ is always the classical limit. Here is an extension of the result in Blum and Hanson [2].

THEOREM 19. *If* $(\mu_n)$ *is uniformly dissipative and* $(X,\tau)$ *is strongly mixing, then for all* $f \in L_p(X)$, $1 \leq p < \infty$, $\lim_{n\to\infty}\|\mu_n f - \int f\,dm\|_p = 0$.

PROOF. As in Lemma 1, it suffices to prove $\lim_{n\to\infty}\|\mu_n f\|_2 = 0$ for $f \in L_2(X)$, $f \perp 1$. For such $f$, the assumption that $\tau$ is strongly mixing says that $\hat{\nu}_f$ is vanishing at

$\infty$. Hence, for $C \geq 1$ fixed,

$$\|\mu_n f\|_2^2 = \sum_{k,\ell \in Z} \mu_n(k)\mu_n(\ell)\hat{\nu}_f(k-\ell)$$
$$= \sum\nolimits_1 + \sum\nolimits_2$$

where $\sum_1$ is the sum over $\{(k,\ell) \in Z^2 : |k-\ell| \leq C\}$ and $\sum_2$ is the sum over $\{(k,\ell) \in Z^2 : |k-\ell| > C\}$. But

$$\left|\sum\nolimits_2\right| \leq \sum_{\substack{k,\ell \in Z \\ |k-\ell|>C}} \mu_n(k)\mu_n(\ell) \sup_{|s|>C} |\hat{\nu}_f(s)|$$
$$\leq \sup_{|s|\geq C} |\hat{\nu}_f(s)|.$$

Also,

$$\left|\sum\nolimits_1\right| \leq \|f\|_2^2 \sum_{k \in Z} \mu_n(k) \sum_{\substack{\ell \in Z \\ |k-\ell| \leq C}} \mu_n(\ell)$$
$$\leq (2C+1)\|f\|_2^2 \sum_{k \in Z} \mu_n(k) \sup_s \mu_n(s)$$
$$= (2C+1)\|f\|_2^2 \sup_s \mu_n(s).$$

Hence, we can choose $C \geq 1$ such that $|\sum_2|$ is as small as we like, and then let $n \longrightarrow \infty$ to get $|\sum_1|$ as small as we like. Thus, $\lim_{n\to\infty} \|\mu_n f\|_2^2 = 0$. ■

An immediate corollary of the proof above is that if $\nu_f$ is absolutely continuous with respect to $m_T$, then, for $(\mu_n)$ which is uniformly dissipative, $\lim_{n\to\infty} \mu_n f = 0$ in $L_2$-norm. In fact, the following was shown.

PROPOSITION 20. *If $(\mu_n)$ is uniformly dissipative and $f \in L_2(X)$ has $\hat{\nu}_f$ vanishing at $\infty$, then* $\lim_{n\to\infty} \mu_n f = 0$ *in $L_2$-norm.*

REMARK 21. a) Because there exist singular measures $\nu$ with $c_0$ Fourier coefficients, the above applies to a wider class of functions than those for which $\nu_f$ is of Lebesgue type.

b) Notice also that the uniform dissipation assumption is needed above. For example, if $\mu_n = \delta_n$, then $(\mu_n)$ is dissipative but $\|\mu_n f\|_2 = \|f\|_2$ for all $f \in L_2(X)$.

This result suggests some Fourier analytic properties of the levels sets of $|h|$. To confirm this, we first need this lemma.

LEMMA 22. *If $(\mu_n)$ is uniformly dissipative and $h(\gamma) = \lim_{n\to\infty} \hat{\mu}_n(\gamma)$ exists for all $\gamma \in T$, then $h(\gamma) = 0$ for a.e. $\gamma$ with respect to Lebesgue measure $m_T$.*

PROOF. Choose a subsequence $(n_m)$ such that $\sum_{m=1}^{\infty} \big(\sup_k \mu_{n_m}(k)\big) < \infty$. If $\lim_{m\to\infty} \hat{\mu}_{n_m}(\gamma) = 0$ for $m_T$ a.e. $\gamma$, then $h(\gamma) = 0$ for $m_T$ a.e. $\gamma$ too. But

$$\int_T |\hat{\mu}_{n_m}(\gamma)|^2 \, dm_T(\gamma) = \sum_{k,\ell \in Z} \mu_{n_m}(k)\mu_{n_m}(\ell) \int_T \bar{\gamma}^k \gamma^\ell \, dm_T(\gamma)$$
$$= \sum_{k \in Z} \mu_{n_m}(k)^2$$

So $\int_T |\hat{\mu}_{n_m}(\gamma)|^2\, dm_T(\gamma) \leq \sup_k \mu_{n_m}(k)$. This means $\sum_{m=1}^\infty |\hat{\mu}_{n_m}(\gamma)|^2$ is integrable with respect to $m_T$ and so the terms $\hat{\mu}_{n_m}(\gamma) \longrightarrow 0$ as $m \longrightarrow \infty$ for a.e. $\gamma$. ∎

EXAMPLE 23. a) If $\mu_n = \frac{1}{n}\sum_{k=1}^n \delta_{m_k}$, for some fixed sequence $(m_k)$ in $Z^+$, then $(\mu_n)$ is uniformly dissipative. Also, $\hat{\mu}_n(\gamma) \longrightarrow 0$ as $n \longrightarrow \infty$ for $m_T$ a.e. $\gamma$. This is a well known fact about sequences $(m_k)$: for a.e. real number $x$, $(m_k x)$ is uniformly distributed modulo 1; see Kuipers and Niederreiter [8] for a reference. But even though $(\mu_n)$ is uniformly dissipative and $\lim_{n\to\infty} \hat{\mu}_n(\gamma) = 0$ for $m_T$ a.e. $\gamma$, $h$ can still fail to exist. An example of this would be with $m_k = 2^k$, a lacunary sequence. However, in general for a uniformly dissipative sequence $(\mu_n)$, if $h$ fails to exist, then $(\mu_n)$ may not converge a.e. to 0. An example of this is given in Example 35.

b) It also should be noticed that $(\mu_n)$ is uniformly dissipative if and only if every subsequence $(\mu_{n_m})$ has a further subsequence $(\mu_{n_{m_s}})$ for which $\lim_{s\to\infty} \hat{\mu}_{n_{m_s}}(\gamma)$ exists for $m_T$ a.e. $\gamma$. Indeed, if $(\mu_n)$ is uniformly dissipative, then this property holds by the proof of Lemma 22 above. Conversely, if $\lim_{n\to\infty} \hat{\mu}_n(\gamma) = 0$ $m_T$ a.e. $\gamma$ then $|\mu_n(k)| = |\int_T \hat{\mu}_n(\gamma)\gamma^k\, dm_T(\gamma)| \leq \int_T |\hat{\mu}_n(\gamma)| dm_T(\gamma)$ shows $\sup_{k\in Z} \mu_n(k) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, if the subsequence property holds, then $(\mu_n)$ is uniformly dissipative. Indeed, otherwise there is a subsequence $(\mu_{n_m})$ and $\delta > 0$ with $\sup_{k\in Z} \mu_{n_m}(k) \geq \delta$ for all $m \geq 1$; and then some subsequence of this, $(\mu_{n_{m_s}})$, is uniformly dissipative by the integration argument. This would be a contradiction.

So, let us assume that $h = \lim \hat{\mu}_n$ exists pointwise and that $(\mu_n)$ is uniformly dissipative. Then $h(\gamma) = 0$, $m_T$ a.e. $\gamma$. But also, $h$ is a first Baire class function, the pointwise limit of continuous functions $(\hat{\mu}_n)$. So $h$ is continuous at a dense $G_\delta$ set of points in $T$. Thus, $h$ cannot be non-zero at a point of continuity because $h = 0$ $m_T$ a.e. and, hence, on a dense set. Thus, $h = 0$ $m_T$ a.e. and on a dense $G_\delta$ set simultaneously. This means that the level sets $\{\gamma \in T : |h(\gamma)| \geq \alpha\}$ are small sets in two different senses. This is certainly satisfied by $h$ if $h \neq 0$ only countably often, but it also can hold for more general $h$, as Proposition 33 shows.

Also, Proposition 20 suggests another way in which the level sets are small.

PROPOSITION 24. *Suppose $(\mu_n)$ is uniformly dissipative and $\nu \in M(T)$, $\nu \geq 0$, $\nu \neq 0$, with* $\operatorname{supp}\nu \subset \{\gamma \in T : \liminf_{n\to\infty} |\hat{\mu}_n(\gamma)| \geq \alpha\}$, $\alpha > 0$. *Then $\hat{\nu}$ does not vanish at $\infty$.*

PROOF. In the proof of Proposition 20, it was shown that if $\hat{\nu}$ is vanishing at infinity, then $\lim_{n\to\infty} \int |\hat{\mu}_n(\gamma)|^2\, d\nu(\gamma) = 0$. Indeed,

$$\int |\hat{\mu}_n(\gamma)|^2\, d\nu(\gamma) = \sum_{h,\ell\in Z} \mu_n(k)\mu_n(\ell)\hat{\nu}(k-\ell)$$

so the proof there applies with $\nu$ in place of $\nu_f$. But now, if $\operatorname{supp}\nu \subset \{\gamma \in T : \liminf_{n\to\infty} |\hat{\mu}_n(\gamma)| \geq \alpha\}$, then

$$\begin{aligned}\liminf_{n\to\infty} \int_T |\mu_n(\gamma)|^2\, d\nu(\gamma) &\geq \int_T \liminf_{n\to\infty} |\hat{\mu}_n(\gamma)|^2\, d\nu(\gamma)\\ &\geq \alpha^2 \|\nu\|_1.\end{aligned}$$

Since $\nu \neq 0$, this is a contradiction. ∎

COROLLARY 25. *If $(\mu_n)$ is uniformly dissipative and $E \subset \{\gamma : \liminf_{n\to\infty} |\mu_n(\gamma)| \geq \alpha\}$ is compact, then any $\nu \in M(E)$, $\nu \neq 0$, has $\hat{\nu}$ not vanishing at $\infty$.*

PROOF. It is observed by Körmer, in Lindahl & Poulsen [9], p. 153, that if there is $\nu \in M(E)$, $\nu \neq 0$, with $\hat{\nu}$ vanishing at $\infty$, then there is $\nu \in M(E)$, $\nu \neq 0$, $\nu \geq 0$, with $\hat{\nu}$ vanishing at $\infty$. ∎

The property of the level sets above is also a property of Helson sets. However, in Lindahl and Poulson [9] Körmer points out that this property is not equivalent to a set being a Helson set. It is possible that the level sets above are Helson sets.

EXAMPLE 26. Here is a construction of a closed perfect set $E$ as above. Let $\omega_n = \frac{1}{n}\sum_{k=1}^n \delta_{2^k}$. Then $\hat{\omega}_n(\gamma) \longrightarrow 1$ for all $\gamma$ which are $2^m$ roots of unity for some $m \geq 0$. Choose $\mu_1 = \omega_{N_1}$ and two points $\gamma_{11} \neq \gamma_{12}$ with $\hat{\mu}_1(\gamma_{1i}) > \frac{1}{2}, i = 1, 2$. Then choose two disjoint open arcs $I_{11}, I_{12}$ with $\gamma_{1i} \in I_{1i}$ such that $|\hat{\mu}_1(\gamma)| > \frac{1}{2}$ for all $\gamma \in I_{11} \cup I_{12}$. Inductively assume $\mu_1, \ldots, \mu_n$ have been chosen so that for each $i$, there are $2^i$ points $\gamma_{ij}, j = 1, \ldots, 2^i$, and $2^i$ pairwise disjoint arcs, $I_{ij}, j = 1, \ldots, 2^i$, with $\gamma_{ij} \in I_{ij}$. Assume also the usual Cantor type nesting that $I_{i+1,2k-1} \cup I_{i+1,2k} \subset I_{i,k}$ for $k = 1, \ldots, 2^i$. Now choose $\mu_{i+1}$ from $(\omega_n)$ and $\gamma_{n+1,1}, \ldots, \gamma_{n+1,2^{n+1}}$ such that for each $k = 1, \ldots, 2^n$, $\gamma_{n+1,2k-1}$, $\gamma_{n+1,2k} \in I_{nk}$, and $|\hat{\mu}_{n+1}(\gamma_{n+1,i})| > \frac{1}{2}$ for $i = 2k-1, 2k$. Then choose $I_{n+1,j}, j = 1, \ldots, 2^{n+1}$, pairwise disjoint open arcs with $\gamma_{n+1,i} \in I_{n+1,i}$ and $I_{n+1,2k-1} \cup I_{n+1,2k} \subset I_{n,k}, k = 1, \ldots, 2^n$. If these arcs are small enough, then $|\hat{\mu}_{n+1}(\gamma)| > \frac{1}{2}$ for all $\gamma \in \bigcup_{i=1}^{2^{n+1}} I_{n+1,j}$. This completes the induction. Let $E = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{2^i} I_{ij}$. Then $E$ is a closed perfect set and $|\hat{\mu}_n(\gamma)| \geq \frac{1}{2}$ for all $\gamma \in E$ and $n \geq 1$. This provides a construction of $(\mu_n)$ and $E$ as in Proposition 24.

It might be appropriate now to point out that the essential ingredient of the above construction was that the measures $(\omega_n)$ have $\liminf_{n\to\infty} |\hat{\omega}_n(\gamma)| = 1$ for a dense set of $\gamma \in T$. It follows that $\big(\hat{\omega}_n(\gamma)\big)$ does not converge for some $\gamma$.

PROPOSITION 27. *Suppose $\varphi_n : T \longrightarrow \mathbb{C}$ are continuous and $(\varphi_n)$ is pointwise convergent to 0 on a dense set $D_0$, and $\limsup_{n\to\infty} |\varphi_n(\gamma)| \geq \alpha > 0$ for all $\gamma$ on a dense set $D_1$. Then there is a dense $G_\delta$ set on which $(\varphi_n)$ does not converge.*

PROOF. Consider $L_1 = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\gamma \in T : |\varphi_m(\gamma)| \leq \frac{\alpha}{2}\}$. Each $\{\gamma \in T : |\varphi_m(\gamma)| \leq \frac{\alpha}{2}\}$ is closed and so is $\bigcap_{m=n}^{\infty} \{\gamma \in T : |\varphi_m(\gamma)| \leq \frac{\alpha}{2}\}$. This intersection has no interior since $\limsup |\varphi_m(\gamma)| \geq \alpha$ on some dense set $D_1$. Hence, $L_1$ is first category and $L_1^c$ is a dense $G_\delta$ set. For any $\gamma \in L_1^c$, $|\varphi_m(\gamma)| > \frac{\alpha}{2}$ for infinitely many $m$.

By a similar argument the set $L_2 = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\gamma \in T : |\varphi_m(\gamma)| \geq \frac{\alpha}{4}\}$ is a first category set because $D_0$ is disjoint from each $\bigcap_{n=m}^{\infty} \{\gamma \in T : |\varphi_m(\gamma)| \geq \frac{\alpha}{4}\}$. Hence, $L_2^c$ is also a dense $G_\delta$ set and for any $\gamma \in L_2^c$, $|\varphi_m(\gamma)| < \frac{\alpha}{4}$ for infinitely many $m$. But then on the dense $G_\delta$ set, $L_1^c \cap L_2^c$, $(|\varphi_m|)$ and $(\varphi_m)$ do not converge. ∎

COROLLARY 28. *Suppose $(\mu_n)$ is uniformly dissipative and $\liminf_{n\to\infty} |\hat{\mu}_n(\gamma)| \geq \alpha$, for some $\alpha > 0$, for all $\gamma \in D$, where $D$ is dense in a non-trivial arc $I \subset T$. Then on a dense $G_\delta$ subset of $I$, $\big(\hat{\mu}_n(\gamma)\big)$ does not converge.*

PROOF. Without loss of generality, we can replace $(\mu_n)$ by a subsequence as in the proof of Lemma 21 where $\big(\hat{\mu}_n(\gamma)\big)$ converges to 0 for $m_T$ a.e. $\gamma$. Then let $\varphi_n = \hat{\mu}_n$ and use the proof of Proposition 27 on $I$, instead of on all of $T$. ∎

REMARK 29. This proposition shows that if $(\hat{\mu}_n)$ does converge everywhere to $h$ and $(\mu_n)$ is uniformly dissipative, then the level sets of $|h|$ cannot be dense in any non-trivial arc. This is further information on how small these level sets must be.

Unfortunately, it does not seem to be clear whether Corollary 28 holds for the usual Cesaro averages of a lacunary sequence $(n_k)$. By numerical properties, it does for any $(n_k)$ of the form $b^k$, $b = 2, 3, \ldots$. It also does if $n_{k+1}/n_k \geq \lambda > 1$ and $\lambda$ is sufficiently large, depending on $\alpha$. See Rosenblatt [12]. But it is not clear in general. For this reason, we would like to point out

PROPOSITION 30. *For any Sidon set $E \subset Z$ and any dissipative sequence $(\mu_n)$ of probabilty measures on E, there exists $\gamma \in T$ such that* $\lim_{n\to\infty} \hat{\mu}_n(\gamma)$ *does not exist.*

PROOF. Without loss of generality, replace $(\mu_n)$ by a subsequence which is essentially disjointly supported. That is, if $\delta > 0$ is fixed, then there are pairwise disjoint set $(E_n)$ in $E$ with $\mu_n(E_n) \geq 1 - \delta$. Choose a measure $\nu \in M(T)$ with $\hat{\nu}(k) = a_k$, for all $k \in \bigcup_{n=1}^{\infty} E_n$. If we assume $|a_k| \leq 1$ for all $k$, then we can take $\|\nu\|_1 \leq C$, the Sidon constant of $E$. Assume $\lim_{n\to\infty} \hat{\mu}_n(\gamma) = h(\gamma)$ exists for all $\gamma \in T$. Then

$$\begin{aligned}\int_T \hat{\mu}_n(\gamma)\, d\nu(\gamma) &= \sum \mu_n(k) \int \bar{\gamma}^k\, d\nu(\gamma)\\ &= \sum_{k\in E_n} \mu_n(k) a_k + e_n\end{aligned}$$

where $|e_n| \leq \mu_n(Z \setminus E_n)\, \|\nu\|_1$.

If $a_k$ is constantly $b_n$ for $k \in E_n$, $|b_n| \leq 1$, this gives $\left| \int_T \hat{\mu}_n(\gamma)\, d\nu(\gamma) - b_n \mu_n(E_n) \right| \leq \delta C$. But then if $b_n = \begin{cases} 0, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$, this gives $\left| \int_T \hat{\mu}_n(\gamma)\, d\nu(\gamma) \right| \leq \delta C$ for $n$ even and $\left| \int_T \hat{\mu}_n(\gamma)\, d\nu(\gamma) \right| \geq 1 - \delta - \delta C$ for $n$ odd. Hence, for small enough $\delta$, we see that $\lim_{n\to\infty} \int_T \hat{\mu}_n(\gamma)\, d\nu(\gamma)$ fails to exist. But if $\lim_{n\to\infty} \hat{\mu}_n(\gamma) = h(\gamma)$ exists for all $\gamma \in T$, then also $\lim_{n\to\infty} \int_T \hat{\mu}_n(\gamma)\, d\nu(\gamma)$ must exist by Lebesgue's Bounded Convergence Theorem. ∎

COROLLARY 31. *If $(n_k)$ is lacunary, then there exists $\gamma$ such that* $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \gamma^{n_k}$ *fails to exist.*

REMARK 32

. A good question is whether there is a constant $\alpha > 0$, independent of $\lambda = \inf_{k\geq 1} n_{k+1}/n_k$, such that for all $\gamma$ in a dense set in $T$, $\liminf_{n\to\infty} \frac{1}{n} \left| \sum_{k=1}^{n} \gamma^{n_k} \right| \geq \alpha$. The previous corollary does not come close to answering this. See Rosenblatt [12] for an ergodic theoretic application of the above, when it holds. It is perhaps now clearer what are the properties of the level sets of $\liminf_{n\to\infty} |\mu_n|$. Question 18 is asking whether these properties are consistent with the existence of $\lim_{n\to\infty} \mu_n(\gamma)$ for all $\gamma$. The following answers this basic question.

PROPOSITION 33. *There exists a uniformly dissipative* $(\mu_n)$ *of symmetric probability measures on* $Z$ *with* $\hat{\mu}_n(\gamma) \geq 0$ *and decreasing everywhere to a limit* $h(\gamma) \geq 0$ *such that* $h(\gamma) \geq \frac{1}{2}$ *on some closed perfect set.*

COROLLARY 34. *There is a uniformly dissipative sequence* $(\mu_n)$ *which gives operators* $\mu_n f$ *converging in* $L_2$*-norm for all* $f \in L_2(X)$ *and all* $(X, \beta, m, \tau)$*, but for which there also exists a weakly mixing* $(X, \beta, m, \tau)$ *such that for some* $f \in L_2(X)$*,* $\lim_{n\to\infty} \mu_n f \neq \int f\, dm$.

PROOF OF PROPOSITION 33. Let $(p_n)$ be an increasing sequence of whole numbers. Let $w_n = \frac{1}{4}(\delta_0 + \delta_{p_n})^* * (\delta_0 + \delta_{p_n}) = \frac{1}{2}\delta_0 + \frac{1}{4}\delta_{p_n} + \frac{1}{4}\delta_{-p_n}$. Let $\mu_n = w_1 * \cdots * w_n$. We claim that for any rapidly increasing $(p_n)$, this sequence $(\mu_n)$ is the one that we want.

First, $w_n$ is symmetric and $\hat{w}_n(\gamma) = |\hat{\nu}_n(\gamma)|^2$ where $\nu_n = \frac{1}{2}(\delta_0 + \delta_{p_n})$. Also, $|\nu_n(\gamma)| \leq 1$ since $\nu_n$ is a probability measure. So, $\mu_n$ is symmetric for each $n \geq 1$, and $0 \leq \hat{\mu}_{n+1} \leq \hat{\mu}_n \leq 1$. Hence, $\lim_{n\to\infty} \hat{\mu}_n(\gamma)$ exists for all $\gamma \in T$.

Second, we can choose $p_{n+1}$ inductively so that $\sup_{k\in Z} \mu_{n+1}(k) \leq \frac{1}{2} \sup_{k\in Z} \mu_n(k)$. Hence, $\sup_{k\in Z} \mu_n(k)$ decreases to 0 geometrically. This only requires $p_{n+1} > 2\sum_{k=1}^n p_k$ so that the supports of $\mu_n$, $\delta_{p_{n+1}} * \mu_n$, and $\delta_{-p_{n+1}} * \mu_n$ are pairwise disjoint.

To keep the limit $h$ large on a Cantor set, we inductively choose $p_n$ as follows. Notice that $\hat{\nu}_n(\gamma) = 1$ if $\gamma$ is a $p_n$th root of unity. Choose $p_1$ so that $\hat{\mu}_1(\gamma) > \frac{1}{2}$ for all $\gamma \in I_{11} \cup I_{12}$, where $I_{1j}$ are two pairwise disjoint open arcs. Inductively assume $p_1, \ldots, p_n$ have been chosen so that $\hat{\mu}_n(\gamma) > \frac{1}{2}$ for all $\gamma \in I_{nj}, j = 1, \ldots, 2^n$, where the open arcs $\{I_{kj} : k = 1, \ldots, n, j = 1, \ldots, 2^k\}$ are nested as in Example 26. Then for any sufficiently large $p_{n+1}$, the appropriate new level of arcs $I_{n+1\,j}$ exist. Consequently, the limit $h$ will have $h(\gamma) \geq \frac{1}{2}$ for all $\gamma \in \bigcap_{i=1}^\infty \bigcup_{j=1}^{2^i} I_{ij}$, which is a closed perfect set. ∎

REMARK 35. a) The analogy with a Riesz product construction of the above is interesting. The idea of making $(\hat{\mu}_n)$ non-increasing is very useful because it is hard to get pointwise convergence otherwise. Indeed, it remains an open problem to construct a sequence $(m_k)$ such that the measures $\mu_n = \frac{1}{n}\sum_{k=1}^n \delta_{m_k}$ has $\lim_{n\to\infty} \hat{\mu}_n(\gamma) = h(\gamma)$ existing everywhere, but $h(\gamma) \neq 0$ more than countably often.

b) In the construction of Proposition 33, one can arrange for $h(\gamma) \geq 1 - \varepsilon$ for all $\gamma$ in a closed perfect set by the same argument, no matter what $\varepsilon > 0$ is given.

c) The construction also can easily give $\lim_{n\to\infty} \gamma^{p_n} = 1$ for all $\gamma \in \bigcap_{i=1}^\infty \bigcup_{j=1}^{2^i} I_{ij}$. But then the weakly mixing $\tau$ constructed in Corollary 34 will be rigid because for all $f \in L_2(X)$, $\lim_{n\to\infty} \langle f \circ \tau^{p_n}, f\rangle$ will be $\|f\|_2^2$.

PROOF OF COROLLARY 34. Take the construction of $(\mu_n)$ as in Proposition 33 and let $E$ be a closed perfect set on which $h(\gamma) \geq \frac{1}{2}$ for $\gamma \in E$. Let $\nu \in \mathcal{M}(E)$, $\nu \geq 0$, $\nu \neq 0$, be any regular Borel measure with no point masses. Since $\rho(k) = \hat{\nu}(k)$ is positive definite, there exists a unitary operator $V$ on a Hilbert space $H$ and a cyclic vector $w$ for $V$ in it such that $\langle V^k w, w\rangle = \hat{\nu}(k)$ for all $k \in Z$. As in Schmidt [13] or Cornfeld et al. [4], the Gaussian action of $Z$ on a Gaussian probability space $(X, \beta, m)$ arising from $V$ is a weakly mixing action by the ergodic invertible transformation $\tau$ corresponding to $1 \in Z$.

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But if $f \in L_2(X,\beta,m)$, such that $\langle f \circ \tau^k, f\rangle = \langle V^k w, w\rangle = \hat{\nu}(k)$, then $\int f\,dm = 0$ and $f^* = \lim_{n\to\infty} \mu_n f$ satisfies $\|f^*\|^2 = \lim_{n\to\infty} \int_T |\hat{\mu}_n(\gamma)|^2\,d\nu(\gamma) = \int_T h^2\,d\nu(\gamma) \geq \frac{1}{2}\nu(T) > 0$, so $f^* \neq \int f\,dm$. ∎

It should be remarked that the existence of the above example says something interesting about probability measures on the Bohr compactification $\beta Z$ of $Z$. Generally, if $(\mu_n)$ is a sequence of probability measures on $Z$, such that $\lim_{n\to\infty} \hat{\mu}_n(\gamma) = h(\gamma)$ exists for all $\gamma \in T$, then $(\mu_n)$ converges in the $\omega^*$-topology of $M(\beta Z)$. Let $B$ be the unit ball of $M(\beta Z)$. Then this $\omega^*$-topology is not metric on $B$, and there is no point $\mu \in B$ which has *a* (relative to B) local basis which is countable. Nonetheless, there are measures $\mu \in B$, $\mu \geq 0$, with $\hat{\mu} = h$ which is not zero uncountably often such that $\mu$ is a $\omega^*$-limit of a sequence $(\mu_n)$. This is despite the highly non-metric nature of the local topology for such a $\mu$.

Hopefully, the example above exhibits how complex the limit operator $f^* = \lim_{n\to\infty} \mu_n f$ can be even for weakly mixing systems. This construction is actually easier to achieve if one only wants the local convergence of $(\mu_n f)$ in some weakly mixing system.

One way to see this is as follows. Let $\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{2^k}$. As we have observed in Example 26, there is a closed perfect set $E$ such that $|\hat{\mu}_n(\gamma)| \geq \frac{1}{2}$ for all $\gamma \in E$. In general though, there is no subsequence of $(\mu_n)$ such that $\big(\hat{\mu}_n(\gamma)\big)$ converges everywhere; see Proposition 27 and Rosenblatt [12]. But we can localize to a metric space and fix this. Indeed, some subnet $(\mu_{n_\alpha})$ of $(\mu_n)$ does have $\lim_\alpha \hat{\mu}_{n_\alpha}(\gamma) = h(\gamma)$ exists for all $\gamma \in T$. Take a weakly mixing system $(X,\beta,m,\tau)$ given by some continuous non-zero positive measure $\nu$ on $E$. Then the associated space $L_2(X)$ is necessarily a separable Banach space in such constructions. The operators $(\mu_{n_\alpha})$ converge in the strong operator topology on $L_2(X)$.

But this is a metric topology. Hence, there is actually a sequence $(\omega_s)$, given by $(\mu_{n_\alpha})$, such that $(\omega_s f)$ converges in $L_2$-norm for all $f \in L_2(X)$. The choice of $\nu$ guarantees $f^* \neq \int f\,dm$ for some $f \in L_2(X)$. The sequence here $(\omega_s)$ is a subsequence of $(\mu_n)$. Its defect is that it cannot be universally $L_2$-norm good, but is only $L_2$-norm good, with a non-classical limit, in this particular weakly mixing system.

Moreover, the behavior of some $(\mu_n)$ on a particular $L_2(X)$ does not predict its global behavior unless $L_2(X)$ contains all possible eigenspaces $\mathcal{E}_\lambda$, $\lambda \in T$. Indeed, for separable spaces $L_2(X)$, this would never be enough as the following shows.

PROPOSITION 36. *Given a separable probability space $(X,\beta,m)$, and an ergodic transformation $\tau$, there exists a uniformly dissipative sequence $(\mu_n)$ such that $\lim_{n\to\infty} \mu_n f = \int f\,dm$ in $L_2$-norm for all $f \in L_2(X)$, but $(\mu_n)$ is not good for $L_2$-norm convergence for some other ergodic dynamical system.*

PROOF. Suppose $(f_s)$ is a sequence which is dense in $\{1\}^\perp \subset L_2(X)$. This exists by the separability hypothesis. Write each $\nu_s = \nu_{f_s}$ as $\nu_s^1 + \nu_s^2$ where $\nu_s^1 << m_T$, and the measure-theoretic support $E_S$ of $\nu_s^2$ is $m_T$ measure zero. Let $E = \bigcup_{s=1}^\infty E_s$. Then $E$ is a Lebesgue null set. To prove the theorem it suffices to construct $(\mu_n)$ with $\hat{\mu}_n(\gamma) \to 0$ a.e.

$\gamma$ and $\hat{\mu}_n(\gamma) \longrightarrow 0$ for all $\gamma \in E \setminus \{1\}$, but $\hat{\mu}_n(\gamma_0)$ does not converge for some $\gamma_0$ of infinite order. Indeed, then

$$\begin{aligned}
\|\mu_n f_s\|_2^2 &= \sum_{k,\ell \in Z} \mu_n(k)\mu_n(\ell)\langle f_s \circ \tau^{k-\ell}, f_s\rangle \\
&= \sum_{k,\ell \in Z} \mu_n(k)\mu_n(\ell)\nu_s(k-\ell) \\
&= \int_T |\hat{\mu}_n(\gamma)|^2 \, d\nu_s(\gamma) \\
&= \int_T |\hat{\mu}_n(\gamma)|^2 \, d\nu_s^1(\gamma) + \int_T |\hat{\mu}_n(\gamma)|^2 \, d\nu_s^2(\gamma) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty
\end{aligned}$$

for all $s \geq 1$. It follows easily that $\lim_{n\to\infty} \mu_n f = \int f \, dm$ in $L_2$-norm for all $f \in L_2(X)$.

To do the above, choose $\gamma_0 \in T \setminus \{1\}$ of infinite order with $\gamma_0 \notin E$ and $\gamma_0^{-1} \notin E$. Then let $K_n$ be an open arc about $\gamma_0$ and let $L_n = \gamma_0^{-1} K_n$. Assume $K_n$ is symmetric about $\gamma_0$ so that $L_n^{-1} = L_n$. Choose $L_n \cap K_n = \emptyset$ and $m_T(K_n) \longrightarrow 0$ as $n \longrightarrow \infty$ too. Let $\varphi_n = c_n(1_{K_n} + 1_{L_n}) * (1_{K_n^{-1}} + 1_{L_n})$ with $c_n > 0$. If $h_n = 1_{K_n} + 1_{L_n}$, then $\varphi_n = c_n h_n * h_n^*$. Let $\psi_n = h_n * h_n^*$. We have $\hat{\varphi}_n \geq 0$. But also, $\psi_n \in L_2(T) * L_2(T)$ so $\hat{\psi}_n \in \ell_1(Z)$ and thus $\sum_{k\in Z} \hat{\psi}_n(k) < \infty$. Choose $c_n$ so that $\sum_{k\in Z} c_n \hat{\psi}_n(k) = \sum_{h\in Z} \hat{\varphi}_n(h) = 1$. This makes $\omega_n = \hat{\varphi}_n$ a probability measure on $Z$. Actually, if $\check{\omega}_n(\gamma) = \sum_{k\in Z} \omega_n(k)\gamma^k$, then $\check{\omega}_n(1) = 1$ and $\check{\omega}_n(1) = \varphi_n(1) = c_n \psi_n(1) = c_n \int_T |h_n|^2 \, dm_T$. So $c_n = \frac{1}{m_T(K_A) + m_T(L_n)} = \frac{1}{2m_T(K_n)}$.

These calculations show that

$$\begin{aligned}
\hat{\omega}_n(\gamma_0^{-1}) = \check{\omega}_n(\gamma_0) &= c_n \int_T h_n(\lambda) h_n(\gamma_0^{-1}\lambda) \, dm_T(\lambda) \\
&\geq c_n \int_T 1_{K_n}(\lambda) 1_{\gamma_0 L_n}(\lambda) \, d\lambda.
\end{aligned}$$

But we chose $\gamma_0 L_n = K_n$ and so we have $\hat{\omega}_n(\gamma_0^{-1}) \geq \frac{m_T(K_n)}{2m_T(K_n)} = \frac{1}{2}$. However, for all values of $\gamma \in T$, $\gamma \notin \{1, \gamma_0, \gamma_0^{-1}\}$, $\hat{\omega}_n(\gamma^{-1}) = \check{\omega}_n(\gamma) = c_n h_n * h_n^*(\gamma) = 0$ for $n$ sufficiently large because supp $h_n$ and supp$({}_\gamma h_n)$ will be disjoint. Since $\gamma_0$ and $\gamma_0^{-1}$ are not in $E$, $\hat{\omega}_n(\gamma) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\gamma \in E \setminus \{1\}$. Clearly also $\hat{\omega}_n(\gamma) \longrightarrow 0$ $m_T$ a.e. $\gamma$.

Now, let $(\mu_n)$ be the sequence $(\omega_n)$ above intertwined with any other sequence of probability measure $(\omega_n')$ with $\lim_{n\to\infty} \hat{\omega}_n'(\gamma) = 0$ for all $\gamma \in T \setminus \{1\}$. Then $(\mu_n)$ is not universally $L_2$-norm good, but will be $L_2$-norm good for the $L_2$ space of the particular dynamical system above. ■

REMARK 37. a) The same argument as above can be carried out simultaneously for any given sequence of separable dynamical systems. This is by way of emphasizing again the essentially non-metric nature of $(\mu_n)$ being universally $L_2$-norm good as in Theorem 1.

b) It would be interesting to carry out the above construction with $\mu_n$ of the form $\frac{1}{n}\sum_{k=1}^n \delta_{m_k}$ for some sequence $(m_k)$. That is, given $E \subset T$, a Lebesgue null set, can we construct $(n_k)$ which is not universally $L_2$-norm good, but such that $E$ is contained in the set of $\gamma \in T$ such that $\frac{1}{n}\sum_{k=1}^n \gamma^{m_k} \longrightarrow 0$ as $n \longrightarrow \infty$

c) It is probably also possible to prove Proposition 36 with a sequence $(\mu_n)$ such that no subsequence of it is universally $L_2$-norm good.

The same technique as was used in the proof of Proposition 36 can be used to give an example relevant to Lemma 22.

EXAMPLE 38. There exists a uniformly dissipative sequence $(\mu_n)$ such that $(\hat{\mu}_n)$ does not tend to 0 a.e. To see this, we construct $\mu_n$ like in Proposition 36 but with $\gamma_0$ variable. Indeed fix $\varepsilon_m = \frac{1}{m}$ and an open symmetric arc $L_m$ about 1 with $m_T(L_m) = \varepsilon_m$. Fix $\gamma$ such that $\gamma^{-1}L_m \cap L_m = \emptyset$. Then let $\varphi_m = c_m h_m * h_m^*$ where $h_m = 1_{\gamma^{-1}L_m} + 1_{L_m}$ and $c_m$ is a normalizing constant. Let $\mu_m = \hat{\varphi}_m$. We showed $\mu_m$ is a probability measure and $\hat{\mu}_m(\gamma^{\pm 1}) \geq \frac{1}{2}$. So there is an open symmetric arc $E_m$ about 1 such that for all $\lambda \in \gamma^{\pm 1}E_m$, $\hat{\mu}_m(\lambda) \geq \frac{1}{4}$. This arc has a length which can be chosen independently of $\gamma$.

Now, with $\varepsilon_m$ fixed, choose $\gamma_1, \ldots, \gamma_{N_m}$ such that for each $i$, $\gamma_i^{-1}L_m \cap L_m = \emptyset$, but $\bigcup_{i=1}^{N_m} \gamma_i E_m \cup \bigcup_{i=1}^{N_m} \gamma_i^{-1} E_m$ covers as much of $T \setminus L_m$ as is otherwise possible. The set that can be covered this way is $T \setminus \tilde{L}_m$ where $\tilde{L}_M$ is an open symmetric arc about 1 with $m_T(\tilde{L}_m) \leq 4\varepsilon_m$. Arrange $\big(\mu(\varepsilon_m, \gamma_i) : i = 1, \ldots, N_m, m \geq 1\big)$ in a lexicographical ordering $(\mu_n)$. Then for all $\gamma \neq 1$, for $\infty$ many $n$, $|\hat{\mu}_n(\gamma)| \geq \frac{1}{4}$. So $(\hat{\mu}_n)$ does not converge to 0 for any $\gamma \in T$. The sequence $(\mu_n)$ is uniformly dissipative through. This can be seen by applying Lemma 22 directly of course. But for any probability measure $\mu$, $\mu(k) = \int_T \hat{\mu}(\gamma)\gamma^k \, dm_T(\gamma)$. So, in this case, one can calculate that if $\mu_n = \mu(\varepsilon_m, \gamma_i)$, for some $m \geq 1$ and $i = 1, \ldots, N_m$, then

$$\begin{aligned}\sup_{k \in Z} \mu_n(k) &\leq \int_T c_m h_m * h_m^* \, dm_T \\ &= \frac{1}{2\varepsilon_m} \int_T h_m * h_m^* \, dm_T \\ &= \frac{1}{2\varepsilon_m} \Big( \int_T h_m \, dm_T \Big)^2 = 2\varepsilon_m.\end{aligned}$$

As a final counterpoint to the previous results, we should discuss some examples of classes of measures $(\mu_n)$ for which the limit function is necessarily not zero at most countably often. First, let us make this definition.

DEFINITION 39. A bounded sequence $\mathbf{u} = (u_k)$ is *Hartman almost periodic* if for all $\gamma \in T$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k \gamma^k = h_{\mathbf{u}}(\gamma)$ exists.

This concept is discussed briefly in Kahane [7] where Corollary 42 is stated. It is clear that if $(m_k)$ is a sequence in $Z^+$, $m_1 < m_2 < m_3 < \cdots$, and $\lim_{n\to\infty} \frac{\#\{m_k : m_k \leq n\}}{n}$ exists and is not zero, then the indicator sequence $(u_k)$ of $(m_k)$ is Hartman almost periodic if and only if $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \gamma^{m_k}$ exists for all $\gamma \in T$. It turns out that for such sequences, or weighted sequences, generally the limit functions are not zero at most countably often. First here is an elementary fact that we will use.

LEMMA 40. *The sequence* $(n_k)$ *has lower density* $\liminf_{n\to\infty} \frac{\#\{m_k : m_k \leq n\}}{n} > 0$ *if and only if* $\sup_{n\geq 1} \frac{m_n}{n} < \infty$.

PROOF. If $\sup_{n\geq 1} \frac{m_n}{n} \leq C$, then $m_n \leq Cn$ for all $n \geq 1$. So if $m_n > M$ then $Cn > M$. Hence, the first $n$, say $n_0$, with $m_{n_0} > M$ has $n_0 > \frac{M}{C}$. That is, $\#\{m_n \leq M\} = n_0 - 1 \geq$

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$[\frac{M}{C}] \geq \frac{M}{C} - 1$. Thus, $\frac{\#\{m_n \leq M\}}{M} \geq \frac{1}{C} - \frac{1}{M}$ and $\liminf_{M\to\infty} \frac{\#\{m_n\leq M\}}{M} \geq \frac{1}{C}$. Conversely, if $\liminf_{M\to\infty} \frac{\#\{m_n \leq M\}}{M} \geq \delta > 0$, then for sufficiently large $M$, $\#\{m_n \leq m_N\} = N \geq \frac{\delta}{2} m_N$. This means, $\frac{m_N}{N} \leq \frac{2}{\delta}$ for $N$ large, and hence $\sup_{N\geq 1} \frac{m_N}{N} < \infty$. ∎

The following theorem and its proof are due to M. Boshernitzan whom the author wishes to thank for letting it be included here. The author previously only was able to give a more difficult proof that the set in question is countable.

THEOREM 41. *If $(u_k)$ is bounded, then for all $\delta > 0$, $E_\delta = \{\gamma \in T : \liminf_{n\to\infty} \frac{1}{n} |\sum_{k=1}^n u_k \gamma^k| \geq \delta\}$ is finite.*

PROOF. Suppose $\gamma_1, \ldots, \gamma_M$ are distinct elements with $\bar{\gamma}_i \in E_\delta$ for $i = 1, \ldots, M$. Then for $i \neq j$, $i, j = 1, \ldots, M$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \gamma_i^k \bar{\gamma}_j^k = 0$. Consider the finite-dimensional subspace $H$ of $\ell_\infty^{\mathbb{C}}(Z^+)$ spanned by the sequences $(u_k : k \geq 1)$ and $(\gamma_i^k : k \geq 1)$, for $i = 1, \ldots, M$. By choosing a suitable subsequence $(n_s)$, we can arrange for $\lim_{s\to\infty} \frac{1}{n_s} \sum_{k=1}^{n_s} u_k \gamma_i^k$ to exist for all $i = 1, \ldots, M$, and simultaneously for $\lim_{s\to\infty} \frac{1}{n_s} \sum_{k=1}^{n_s} |u_k|^2$ to exist. This allows us to define a complex inner-product $\langle \cdot, \cdot \rangle$ in $H$ by $\langle \mathbf{a}, \mathbf{b} \rangle = \lim_{s\to\infty} \frac{1}{n_s} \sum_{k=1}^{n_s} a_k \bar{b}_k$, for all $\mathbf{a}, \mathbf{b} \in H$, $\mathbf{a} = (a_k : k \geq 1)$, $\mathbf{b} = (b_k : k \geq 1)$. This makes $H$ a Hilbert space.

Now by Bessel's inequality, with $\gamma_i = (\gamma_i^k : k \geq 1)$, $i = 1, \ldots, M$,

$$\begin{aligned} M\delta^2 &\leq \sum_{i=1}^M \lim_{s\to\infty} \frac{1}{n_s} \Big| \sum_{k=1}^{n_s} u_k \bar{\gamma}_i^k \Big|^2 \\ &= \sum_{i=1}^M |\langle \mathbf{u}, \gamma_i \rangle|^2 \\ &\leq |\langle \mathbf{u}, \mathbf{u} \rangle|^2 \\ &\leq \lim_{s\to\infty} \frac{1}{n_s} \sum_{k=1}^{n_s} |u_j|^2 \\ &\leq \sup_{k\geq 1} |u_k|^2 < \infty. \end{aligned}$$

Hence, $M \leq (\sup_{k\geq 1} |u_k|^2)/\delta^2$. This proves that $M$ is bounded by a constant independent of the choice of $\gamma_1, \ldots, \gamma_M$ and hence that $E_\delta$ is finite. ∎

COROLLARY 42. *If $(u_k)$ is bounded Hartman almost periodic sequence, then the limit function $h(\gamma) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n u_k \gamma^k$ is not zero at most countably often.*

COROLLARY 43. *If $(m_n)$ has positive lower density, then for all $\delta > 0$, $E_\delta = \{\gamma \in T : \liminf_{n\to\infty} \frac{1}{n} |\sum_{k=1}^n \gamma^{m_k}| \geq \delta\}$ is finite.*

PROOF. Because of Lemma 40, $(m_n/n)$ is bounded. Let $(u_k)$ be the indicator function of $(m_n)$. then for $m_k \leq n < m_{k+1}$,

$$\begin{aligned} \frac{1}{n} \Big| \sum_{k=1}^n u_k \gamma^k \Big| &= \frac{1}{n} \Big| \sum_{\ell=1}^k \gamma^{m_\ell} \Big| \\ &\geq \Big( \frac{k}{m_{k+1}} \Big) \frac{1}{k} \Big| \sum_{\ell=1}^k \gamma^{m_\ell} \Big|. \end{aligned}$$

Hence, $\{\gamma \in T : \liminf_{k\to\infty} \frac{1}{k}|\sum_{\ell=1}^{k} \gamma^{m_\ell}| > \delta\} \subset \{\gamma \in T : \liminf_{n\to\infty} \frac{1}{n}|\sum_{k=1}^{n} u_k\gamma^k| \geq \frac{1}{C}\delta\}$ if $C = \sup_{k\geq 1} \frac{m_{k+1}}{k}$. Thus by Theorem 41, this result holds. ∎

COROLLARY 44. *Suppose $(m_n)$ is a universally $L_2$-norm good sequence with positive lower density. Then for all weakly mixing dynamical systems $(X,\beta,m,\tau)$, if $f \in L_2(X)$, then $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f \circ \tau^{m_k} = \int f\, dm$ in $L_2$-norm.*

PROOF. By Corollary 43, and Corollary 16, this is proved. ∎

REMARK 45. See Boshernitzan [3] for many closely related theorems about sequences $(n_k)$ for which the limit function $h(\gamma) = \lim_{n\to\infty} \frac{1}{n}|\sum_{k=1}^{n} \gamma^{n_k}|$ is not zero at most countably often. It is observed there, and is essentially the point of Corollary 43, that for such $(n_k)$, if $\nu \in M(T)$ is continuous, then $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\hat{\nu}(n_k)| = 0$. Also, an interesting characterization of when $(n_k)$ has this property is given in [3] in terms of the behavior of subsequences of the sequence of averages $\frac{1}{n}\sum_{k=1}^{n} \gamma^{n_k}$.

## REFERENCES


1. A. Bellow, R. Jones and J. Rosenblatt, *Almost everywhere convergence of weighted averages*, Math. Ann. **293**(1992), 399–426.
2. J. R. Blum and D. L. Hanson, *On the mean ergodic theorem for subsequences*, Bull. Amer. Math. Soc. **66**(1960) 308–311.
3. M. Boshernitzan, *Slow uniform distribution*, preprint.
4. I. P. Cornfeld, S. V. Fornin and Ya G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1972.
5. W. Feller, *An Introduction to Probabilty Theory and its Applications, Vol. II*, John Wiley and Sons, New York, 1971.
6. R. Jones, J. Rosenblatt and A. Tempel'man, *Ergodic theorems for group actions*, Illinois J. Math, to appear.
7. J-P. Kahane, *Some Random Series of Functions*, 2nd. Edition, Cambridge University Press, Cambridge, 1985.
8. L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.
9. L-Å. Lindahl and F. Poulsen, *Thin Sets in Harmonic Analysis*, Marcel Dekker, New York, 1971.
10. W. Parry, *Topics in Ergodic Theory*, Cambridge University Press, Cambridge, 1981.
11. K. Petersen, *Ergodic Theory*, Cambridge University Press, Cambridge, 1983.
12. J. Rosenblatt, *Universally bad sequences in ergodic theory, Almost Everywhere Convergence, II*, Proceedings of the Conference on A.E. Convergence and Probability Theory, Fall, 1989, Northwestern University, Academic Press, New York, 1991, 227–246.
13. K. Schmidt, *Asymptotic properties of unitary representations and mixing*, Proc. London Math. Soc. (3) **48**(1984) 445–460.
14. P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York-Heidelberg-Berlin, 1982.


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