Canad. Math. Bull. Vol. 17 (2), 1974.

NUMERICAL RANGE AND CONVEX SETS*

BY
FREDRIC M. POLLACK

The numerical range $W(T)$ of a bounded linear operator T on a Hilbert space H is defined by

$$W(T) = \{(Tx, x) \mid \|x\| = 1, x \in H\}.$$

$W(T)$ is always a convex subset of the plane [1] and clearly $W(T)$ is bounded since it is contained in the ball of radius $\|T\|$ about the origin. Which non-empty convex bounded subsets of the plane are the numerical range of an operator? The theorem we prove below shows that every non-empty convex bounded subset of the plane is $W(T)$ for some T. To prove this theorem we need the following lemma:

LEMMA *Let D be a convex set in the plane and let $r_0 \in \bar{D} - D$. It is then impossible to find sequences of complex numbers $\{\alpha_n\}$ and $\{z_n\}$ which have the following properties:*

$$\alpha_n > 0, \qquad n = 1, 2, 3, \ldots \tag{1}$$

$$\sum_{n=1}^{\infty} \alpha_n = 1, \tag{2}$$

$$z_n \in D, \qquad \sum_{n=1}^{\infty} \alpha_n z_n = r_0. \tag{3}$$

Proof. We may assume without loss of generality that $r_0=0$, the origin, and that $\text{Re}(z)\geq 0$ for all $z \in D$. Thus assume that sequences $\{\alpha_n\}$ and $\{z_n\}$ exist and satisfy properties (1), (2), (3). Since $\alpha_n>0$ and $\text{Re}(D)\geq 0$, each z_n must be pure imaginary for if some z_n has a non-zero real part then $\sum_{n=1}^{\infty} \alpha_n z_n=0$ must have a nonzero real part and this is absurd. Now $0 \notin D$ and therefore no $z_n=0$. Since D is convex, we must have either $\text{Im}(z_n)>0$ for all n or $\text{Im}(z_n)<0$ for all n (otherwise $0 \in D$). But if for example $\text{Im}(z_n)>0$ for all n then $\text{Im}(\sum_{n=1}^{\infty} \alpha_n z_n)=\text{Im}(0)>0$ which is impossible, and thus the lemma is proved.

THEOREM. *Any bounded convex nonempty subset of the plane is the numerical range of an operator.*

Proof. Let D be a bounded convex non-empty subset of the plane. If D consists of exactly one point, say $D=\{\lambda_0\}$, then $W(\lambda_0 I)=D$. If D has more than one point then since it is convex it has precisely $2^{\aleph_0}$ points. Let H be a Hilbert space with an

Received by the editors.
AMS subject classifications: Primary 4710, Secondary 5230.

* Partially supported by NSF Grant #GP-6727.

https://doi.org/10.4153/CMB-1974-058-3 Published online by Cambridge University Press

orthonormal basis of cardinality $2^{\aleph_0}$ and let this basis be indexed by the points of D. Define an operator T on H by $Te_\lambda = \lambda e_\lambda$ for all λ in D. T is a bounded operator since the set D is a bounded set. We will show that $W(T)=D$.

If $\lambda \in D$ then $\lambda = (Te_\lambda, e_\lambda) \in W(T)$ and thus $D \subset W(T)$. Now if $x_0 \in H$, $\|x_0\|=1$, then there exists an at most countable sequence of non-zero complex numbers, $\{\beta_n\}$, such that $x_0 = \sum_{n=1}^\infty \beta_n e_{\lambda_n}$. Since $\|x_0\|^2=1$ we must have $\sum_{n=1}^\infty |\beta_n|^2=1$. Let

$$r_0 = (Tx_0, x_0) = \left(T\left(\sum_{n=1}^\infty \beta_n e_{\lambda_n}\right), \sum_{n=1}^\infty \beta_n e_{\lambda_n}\right) = \left(\sum_{n=1}^\infty \beta_n \lambda_n e_{\lambda_n}, \sum_{n=1}^\infty \beta_n e_{\lambda_n}\right) = \sum_{n=1}^\infty |\beta_n|^2 \lambda_n.$$

We have to show $r_0 \in D$, and as a first step we will show that $r_0 \in \bar{D}$. Let $\alpha_n = |\beta_n|^2$ and let $\varepsilon_n = 1 - \sum_{k=1}^{n-1} \alpha_k$. We have $r_0 = \sum_{n=1}^\infty \alpha_n \lambda_n$, $\varepsilon_n > 0$, $\varepsilon_n + (\sum_{k=1}^{n-1} \alpha_k) = 1$. Finally let $\gamma_n = (\sum_{k=1}^{n-1} \alpha_k \lambda_k) + \varepsilon_n \lambda_0$, where λ_0 is any element of D. Clearly $\gamma_n \in D$ since D is convex. Now $|\gamma_n - r_0| = |-(\sum_{k \geq n} \alpha_k \lambda_k) + \varepsilon_n \lambda_0| \leq |\sum_{k \geq n} \alpha_k \lambda_k| + \varepsilon_n |\lambda_0|$. But $\lim_{n\to\infty} |\sum_{k\geq n} \alpha_k \lambda_k| = 0$, and $\lim_{n\to\infty} \varepsilon_n |\lambda_0| = 0$. Thus $r_0 = \lim_{n\to\infty} \gamma_n$ and since $\gamma_n \in D$ we have $r_0 \in \bar{D}$. If we assume to the contrary that r_0 is not in D then we have $r_0 \in \bar{D} - D$. By the previous lemma, however, since $r_0 = \sum_{n=1}^\infty \alpha_n \lambda_n$, $\alpha_n > 0$, $\sum_{n=1}^\infty \alpha_n = 1$, $\lambda_n \in D$ this is impossible. Thus $r_0 \in D$. This completes the proof.

Reference

1. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, 1967.

https://doi.org/10.4153/CMB-1974-058-3 Published online by Cambridge University Press