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Kappa classes on KSBA spaces

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Abstract

We define kappa classes on moduli spaces of Kollár-Shepherd-Barron-Alexeev (KSBA)-stable varieties and pairs, generalizing the Miller–Morita–Mumford classes on moduli of curves, and computing them in some cases where the virtual fundamental class is known to exist, including Burniat and Campedelli surfaces. For Campedelli surfaces, an intermediate step is finding the Chow (same as cohomology) ring of the GIT quotient $(\mathbb{P}^2)^7//SL(3)$.

Contents

1. Introduction

The Miller–Morita–Mumford (MMM) classes, or kappa classes, are some very basic objects in enumerative geometry of the moduli spaces $\overline{M}_{g,n}$ of stable curves. For example, according to Mumford's conjecture, as proved by Madsen and Weiss [MW07], the stable cohomology group of M_g is $\mathbb{Q}[\kappa_1, \kappa_2, \dots]$. These classes were introduced by Mumford in [Mum83]. Morita [Mor87]

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defined equivalent classes on M_g from a topological point of view, and Miller [Mil86] showed that $\mathbb{Q}[\kappa_1,\kappa_2,\dots]$ embeds into the stable cohomology of M_g in degrees $\le g/3$.

In [Don20], Donaldson asked if it were possible to extend enumerative geometry of $\overline{M}_g$ to the Kollár-Shepherd-Barron-Alexeev (KSBA) spaces, the moduli spaces of stable varieties which are higher-dimensional analogs of stable curves. He outlined a definition of the virtual fundamental class on the moduli space of stable surfaces, which was subsequently developed by Jiang [Jia22].

In § 2, we extend the definition of the kappa classes to the KSBA spaces and ask some basic questions about them. The rest of the paper is devoted to computing them in several cases (working over $\mathbb{C}$) where the moduli spaces of stable surfaces are known explicitly, such as products of curves [vO06], Campedelli surfaces [AP23] and Burniat surfaces [AP23, AH23].

2. Definition of kappa classes

A KSBA-stable pair $(X, D=\sum_i a_iD_i)$ consists of an equidimensional variety X and integral Weil divisors D_i taken with rational coefficients $0<a_i\le 1$ such that X is deminormal (and, in particular, has only double crossings in codimension 1), D_i are Mumford divisors (so do not contain components of the double locus of X), the pair (X,D) has semi-log-canonical singularities and the divisor K_X+D is ample.

We refer to Kollár [Kol23, Definition 8.13] for the definition of the moduli functor, which is quite delicate and involves an important notion of a K-flat family of Mumford divisors. The main result of [Kol23] is that after fixing the basic invariants, dimension $d=\dim X$, the coefficient set $\mathbb{M}a=(a_i)$ and the volume $\nu=(K_X+D)^d$, this moduli functor admits a projective coarse moduli space $\mathrm{SP}(\mathbb{M}a,d,\nu)$. The moduli stack $\mathcal{SP}(\mathbb{M}a,d,\nu)$ is a proper Deligne–Mumford stack.

Lemma 2.1. *For any family $f\colon (X,D)\to S$ of KSBA-stable pairs, there exists a well-defined $\mathbb{Q}$-line bundle $K_{X/S}+D$ on X which is functorial, i.e. compatible with the base change $S'\to S$.*

Proof. It is known that there exists an open subset $j\colon U\to X$ such that:

(i) for any fiber X_s, one has $\operatorname{codim} X_s\setminus U_s\ge 2$;
(ii) $U\to S$ is Gorenstein and fibers have, at worst, simple double crossings;
(iii) the divisors $D_i|_U$ are Cartier and lie in the smooth locus of U;
(iv) for some $N\in\mathbb{N}$, $Na_i\in\mathbb{Z}$ and the sheaf $L_N:=j_*(\omega_{U/S}^{\otimes N}(ND))$ is invertible.

Define $K_{X/S}+D:=\frac{1}{N}L_N$. This definition is independent of taking further multiples of N and choosing another open subset U with the above properties.

For any base change $S'\to S$, the open set $j'\colon U'=U\times_S S'\to X'=X\times_S S'$ has the same properties and $L'_N=j'_*(\omega_{U'/S'}^{\otimes N}(ND'))$ is the pullback of L_N, because formation of $\omega_{U/S}$ and $\mathcal{O}_U(D_i)$ commutes with base changes. □

Corollary 2.2. *On the universal family $(\mathcal{X},\mathcal{D})\to\mathcal{SP}(\mathbb{M}a,d,\nu)$ over the moduli stack there is a canonical $\mathbb{Q}$-line bundle $K_{\mathcal{X}/\mathcal{SP}}+\mathcal{D}$.*

Mumford [Mum83] defined the kappa classes κ_i on $\overline{\mathcal{M}}_g$ as the pushforwards of the cycles $K^{i+1}_{\mathcal{X}/\overline{\mathcal{M}}_g}$ in the universal family $f\colon\mathcal{X}\to\overline{\mathcal{M}}_g$. Similarly, Arbarello and Cornalba [AC96, AC98] defined κ_i on $\overline{\mathcal{M}}_{g,n}$ as the pushforwards of $(K_{\mathcal{X}/\overline{\mathcal{M}}_{g,n}}+\mathcal{D})^{i+1}$ in the universal family $f\colon(\mathcal{X},\mathcal{D}=\sum_{k=1}^n\mathcal{D}_k)\to\overline{\mathcal{M}}_{g,n}$. In both cases we are greatly helped by the fact that $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ are

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smooth Deligne–Mumford stacks, so κ_i can be considered to be cocycles in $A^i(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$ and $H^{2i}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$.

We would like to define the kappa classes on KSBA spaces similarly, as the pushforwards of $(K_{\mathcal{X}/\mathcal{SP}}+\mathcal{D})^{i+d}$. The question is: in what generality does this definition make sense and which properties does it have? We propose several versions.

Definition 2.3 (κ_0 and κ_1). Obviously, for any family κ_0 can be defined simply as $\nu=(K_{X_s}+D_s)^d\in\mathbb{Q}$, the volume of a general fiber, and $\kappa_1=f_*(K_{X/S}+D)^{d+1}$ is known as the CM line bundle; see e.g. [FR06, WX14, PX17].

Definition 2.4 (Cycle version). For any family $f\colon (X,D)\to S$ with equidimensional base S, define the cycles $\kappa_i(S)\in A_{\dim S-i}(S)$ as proper pushforwards

$$\kappa_i(S)=f_*\left((K_{X/S}+D)^{i+d}\cap[X]\right)\quad\text{under}\qquad f_*\colon A_m(X)\to A_m(S),$$

where $m=\dim X-i-d=\dim S-i$. Consider the following commutative square.

$$\begin{tikzcd}
(X,D) \arrow[d, "f"'] & (X',D') \arrow[l, "g'"'] \arrow[d, "f'"] \\
S & S' \arrow[l, "g"]
\end{tikzcd}$$

Here $g\colon S'\to S$ is a proper generically finite morphism of degree e, with reduced S,S'. Then g' is also a proper generically finite morphism of degree e. We have that $g'_*[X']=e[X]$, and by the projection formula

$$\begin{aligned}
e\kappa_i(S)&=f_*\left((K_{X/S}+D)^{i+d}\cap g'_*[X']\right)\\
&=f_*g'_*\left((K_{X'/S'}+D')^{i+d}\cap[X']\right)\\
&=g_*f'_*\left((K_{X'/S'}+D')^{i+d}\cap[X']\right)=g_*\kappa_i(S').
\end{aligned}$$

The same definition, and with the same functoriality, also works for the morphisms of DM stacks using the intersection theory on stacks [Vis89]. In particular, let $\mathcal{SP}'$ be an irreducible component of $\mathcal{SP}_{\mathrm{red}}$, or its normalization. Then we get cycles $\kappa_i(\mathcal{SP}')\in A_{\dim\mathcal{SP}'-i}(\mathcal{SP}')$ and $\kappa_i(\mathrm{SP}')\in A_{\dim\mathrm{SP}'-i}(\mathrm{SP}')$ on its coarse moduli space.

Definition 2.5 (Smooth moduli stack version). If $\mathcal{SP}'$ is a smooth (necessarily proper) Deligne–Mumford stack then, just as for $\overline{\mathcal{M}}_{g,n}$, we can identify the group $A_{\dim(\mathcal{SP}')-i}(\mathcal{SP}')_{\mathbb{Q}}$ with $A^i(\mathcal{SP}')_{\mathbb{Q}}$ and define κ_i in $A^i(\mathcal{SP}')_{\mathbb{Q}}$. If $\mathcal{SP}'$ is a global quotient of a smooth projective variety $[V:\Gamma]$ by a finite group, then $A^i(\mathcal{SP}')_{\mathbb{Q}}=A^i(V)^{\Gamma}_{\mathbb{Q}}$. Using the cycle map, we also get kappa classes in $H^{2i}(\mathcal{SP}',\mathbb{Q})=H^{2i}(V,\mathbb{Q})^{\Gamma}$.

Even if $\mathcal{SP}'$ is not smooth, we can define κ_i for all resolutions of singularities $\mathcal{S}\to\mathcal{SP}'$ over $\mathcal{SP}'$, with $\mathcal{S}$ a DM stack. This definition is functorial in $\mathcal{S}$.

Definition 2.6 (Lci morphisms). Suppose that the morphism $f\colon X\to S$ is lci (e.g. families of stable curves are lci) and that S is smooth and proper. Using the identification $A_{\dim S-i}(S)=A^i(S)$ we get the kappa classes in $A^i(S)$. If $g\colon S'\to S$ is an arbitrary morphism from another smooth and proper variety S', the functoriality of the refined Gysin homomorphism in homology [Ful84, Proposition 6.6] implies that for a base change $g\colon S'\to S$ one has $\kappa_i(S')=g^*\kappa_i(S)$. In particular, we get kappa classes on a resolution of singularities of $\mathcal{SP}'$ in a functorial way.

Definition 2.7 (Cohomological version). Without assuming that f is lci, for any family of stable pairs $f\colon (X,D)\to S$ over a smooth, not necessarily proper S, we can use Gysin pushforward $H^i(X,\mathbb{Q})\to H^i(S,\mathbb{Q})$ defined as the composition

$$H^{2(i+d)}(X) \xrightarrow{\cap[X]} H^{\mathrm{BM}}_{2\dim X-2i-2d}(X) \xrightarrow{f_*} H^{\mathrm{BM}}_{2\dim S-2i} \xrightarrow{\sim} H^{2i}(S),$$

where H^{BM}_* is a Borel–Moore homology (see e.g. [Ful97, Appendix B]). I do not know if this Gysin pushforward has enough functorial properties to imply $\kappa_i(S') = g^*\kappa_i(S)$.

Definition 2.8 (Almost lci morphisms). Few KSBA-stable surfaces have lci singularities. But for most of them the index-1 Gorenstein covers are lci. An important idea from Jiang [Jia22] is to utilize the DM stack of index-1 covers to define the virtual fundamental class of SP. Using Jiang's idea, if all the singularities appearing in a certain compact moduli space admit index-1 lci covers, one may define κ_i as a Gysin pushforward of the pullback of $(K_{X/S}+D)^{i+d}$ from X to its index-1 covering stack.

At this point an educated reader will certainly think of many other ways to define kappa classes in cohomology, e.g. using operational Chow rings and bivariant theories, étale Borel–Moore homology. I think all of them deserve serious consideration.

In what follows, I assume that we have well-defined kappa classes in cohomology as in one of the above ways. In the three examples considered later in this paper the moduli stacks are smooth and we use Definition 2.5 to compute κ_i.

The semipositivity results for families of stable pairs [Fuj18, KP17] imply that $K_{X/S}+D$ is nef, by a standard argument. It follows that whenever κ_i are defined in cohomology, they are nef. In fact, κ_1 is the CM line bundle, known to be ample. This is true for $\overline{M}_g$ by [Mum77], for $\overline{M}_{g,n}$ by [Cor93] and, in general, by [PX17].

A special and very interesting case is the KSBA compactification of the moduli of log Calabi–Yau pairs $(X,\Delta+\epsilon B)$ such that, generically, $K_X+\Delta\sim_{\mathbb{Q}} 0$ and B is $\mathbb{Q}$-Cartier and ample. By [KX20, Bir23], after fixing basic numerical invariants there exists $\epsilon_0>0$ such that for any $0<\epsilon<\epsilon_0$ the compactification for the stable pairs $(X,\Delta+\epsilon B)$ does not depend on ϵ. Some concrete cases are the compactified moduli spaces of toric and abelian varieties [Ale02] and of K3 surfaces [AE23].

Definition 2.9. For a family $f\colon (X,\Delta+\epsilon B)\to S$ of KSBA-stable log Calabi–Yau pairs with $0<\epsilon\ll 1$, the generalized Hodge $\mathbb{Q}$-line bundle λ is defined by the condition $K_{X/S}+\Delta = f^*(\lambda)$. By functoriality this defines $\mathbb{Q}$-line bundles on the moduli stack and on its coarse moduli space.

Obviously, $\kappa_i(\epsilon)$ are polynomials in ϵ, λ and $f_*(B^{i+d})$. For example,

$$\kappa_1(\epsilon)/\epsilon^d = (n+1)\nu(B)\lambda + \epsilon f_* B^{d+1},$$

where $\nu(B)=B_s^d$ is the volume of a general fiber. Nefness of $\kappa_1(\epsilon)$ for $0<\epsilon\ll 1$ implies that λ is nef as well. For toric pairs $(X,\Delta+\epsilon B)$ with toric boundary Δ, one has $\lambda=0$. For abelian and K3 pairs $(X,\epsilon B)$, λ is the pullback of the ample Hodge bundle on the Satake–Baily–Borel compactification. So it is true in these cases. This was also proved for degenerations of pairs $(\mathbb{P}^2,D)$ in [ABB+23]. One may therefore ask if λ is always semiample.

3. Products of curves

Consider surfaces of the form $X=C_g\times C_h$ which are products of smooth curves of genus g and h. Obviously, the stable limits of one-parameter degenerations of such surfaces are products of

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two stable curves. By van Opstall [vO05], there is an irreducible component of the moduli of stable surfaces isomorphic to $\overline{M}_g \times \overline{M}_h$ if $g \neq h$, or a quotient of it by an involution if $g = h$. This construction is further extended by [vO06] to finite quotients of $C_g \times C_h$.

For a universal family $\mathcal{X}$ over the stack $\overline{\mathcal{M}}_g \times \overline{\mathcal{M}}_h$, the line bundle $\omega_{\mathcal{X}/\overline{\mathcal{M}}_g \times \overline{\mathcal{M}}_h}$ is simply $p_1^*(\omega_{\mathcal{C}_g/\overline{\mathcal{M}}_g}) + p_2^*(\omega_{\mathcal{C}_h/\overline{\mathcal{M}}_h})$, and the kappa classes on $\overline{M}_g \times \overline{M}_h$ are merely appropriate sums of monomials in the pullbacks of kappa classes from $\overline{M}_g$ and $\overline{M}_h$. So this case is reduced to $\overline{M}_g$. The cases of quotients of $C_g \times C_h$ can be treated similarly; at the stack level there is not much difference.

4. Kappa classes on moduli of $\mathbb{Z}_2^k$-covers

In §§ 4 and 5, we treat the cases of Burniat and Campedelli surfaces described in [AP23] and [AH23]. These are surfaces of general type that are certain branched $\mathbb{Z}_2^k$-covers ($k = 2, 3$) of pairs $(Y, \frac{1}{2}D)$, $D = \sum_i D_i$. The stable surfaces on the boundary are $\mathbb{Z}_2^k$-covers of stable pairs $(Y, \frac{1}{2}D)$. In each case, the compactified coarse moduli space of surfaces X is a finite quotient of the compactified fine moduli space for the pairs $(Y, \frac{1}{2}D)$ by a symmetry group permuting the labels of the D_i.

Any family $f\colon X \to S$ after a finite base change $S' \to S$ can be written as a $\mathbb{Z}_2^k$-cover $X' \to (Y', \frac{1}{2}D')$, where $X' = X \times_S S'$, $(Y', D') = (Y, D) \times_S S'$ and the $\mathbb{Q}$-line bundle $\omega_{X'/S'}$ is the pullback of $\omega_{Y'/S'}(\frac{1}{2}D')$. Thus, the kappa classes for the covers X are proportional to the kappa classes for the pairs $(Y, \frac{1}{2}D)$ by some multiples that are powers of 2. So it is enough to study the kappa classes for the pairs $(Y, \frac{1}{2}D)$.

5. Burniat surfaces

Burniat surfaces are certain surfaces of general type of degree $3 \leq d = K_X^2 \leq 6$ with $p_g = q = 0$ which can be obtained as $\mathbb{Z}_2^2$-covers of degree-d del Pezzo surfaces ramified in a set of 12 curves coming from a particular configuration of lines in $\mathbb{P}^2$.

Primary Burniat surfaces are those of degree 6; they are covers of Cremona surface $\Sigma = \mathrm{Bl}_3\mathbb{P}^2$. Secondary Burniat surfaces have degrees 5 and 4, they are $\mathbb{Z}_2^2$-covers of del Pezzo surfaces of degrees 5 and 4 obtained by further blowups of Σ at the points where some three of the 12 curves pass through the same point.

An explicit KSBA compactification of the moduli space of primary Burniat surfaces was described in [AP23]. Using it, explicit KSBA compactifications for the moduli of secondary Burniat surfaces were described in [AH23].

For Burniat surfaces of degrees 6 and 5 and for the non-nodal Burniat surfaces of degree 4, the above papers give compactifications of the entire irreducible components in the moduli space of surfaces of general type. In the nodal degree 4 and 3 cases they are closed subsets of irreducible components or larger dimensions. We do not discuss the nodal cases here.

As was pointed out to me by Yunfeng Jiang, for numerical applications the most interesting degrees are 4 and 5. Indeed, for degree d the dimension of the compactification $\overline{M}_d^{\mathrm{Bur}}$ is $d - 2$. On the other hand, by [Don20, Jia22] the dimension of the virtual fundamental class is $10\chi(\mathcal{O}_X) - 2K_X^2 = 10 - 2d$. Thus, for $d = 6$ the virtual fundamental class is zero, and for $d = 5$ it is a multiple of a point.

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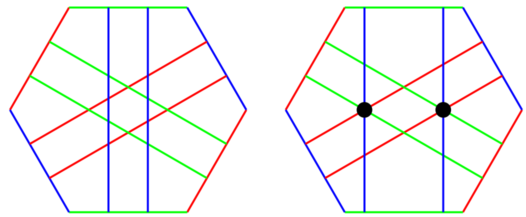

Figure 1. Burniat configurations of degree 6 and 4 non-nodal cases.

In §§ 5.1 and 5.2, we consider the degree 6 and 4 non-nodal cases, respectively. Here, the virtual fundamental case has dimension 2 and coincides with $[\overline{M}_4^{\mathrm{Bur}}]$. In § 5.3, we compute the kappa classes on $\overline{M}_4^{\mathrm{Bur}}$.

5.1 Degree 6

Burniat surfaces X_6 of degree 6 are $\mathbb{Z}_2^2$-covers of a Cremona surface $Y_6^{\mathrm{tor}} = \Sigma = \mathrm{Bl}_3\mathbb{P}^2$ ramified in a configuration of 12 curves shown in the left panel of Figure 1. A $\mathbb{Z}_2^2$-cover is determined by three divisors R, G, B satisfying certain conditions (see [AP23]). We use the primary colors red, green and blue to draw them. In this case, $R = \sum_{i=0}^3 R_i$, $G = \sum_{i=0}^3 G_i$, $B = \sum_{i=0}^3 B_i$. The curves with $i = 0, 3$ form the toric boundary $D_{6,\mathrm{bry}}$ of Σ. The curves with $i = 1, 2$ form the interior divisor $D_{6,\mathrm{int}}$. The total branch divisor of $X_6 \to Y_6$ is $D_6 = D_{6,\mathrm{bry}} + D_{6,\mathrm{int}} = R + G + B$.

In [AP23] is a construction of a compactified moduli space for the pairs $(Y_6, \frac{1}{2}D_6)$, which we will denote by $\overline{M}_6$ here. It comes with a universal family $(\mathcal{Y}_6^{\mathrm{tor}}, \frac{1}{2}\mathcal{D}_6) \to \overline{M}_6$. Then the compactified moduli space $\overline{M}_6^{\mathrm{Bur}}$ of degree 6 Burniat surfaces is the quotient of $\overline{M}_6$ by a finite group $S_3 \ltimes S_2^4$ shuffling the labels of R_i, G_i, B_i.

In more detail, $(\mathcal{Y}_6, \frac{1}{2}\mathcal{D}_6) \to \overline{M}_6$ is obtained from an explicit morphism of toric varieties $\mathcal{Y}_6^{\mathrm{tor}} \to \overline{M}_6^{\mathrm{tor}}$ by a series of smooth blowups, followed by a contraction to the relative canonical model. The morphism $\overline{M}_6 \to \overline{M}_6^{\mathrm{tor}}$ is a composition of a blowup ρ_1 at the central point $1 \in \mathbb{C}^{*4} \subset \overline{M}_6^{\mathrm{tor}}$, followed by a blowup ρ_2 along six disjoint $\mathbb{P}^1$. A family $\mathcal{Y}_6' \to \overline{M}_6$ is obtained from $\mathcal{Y}_6^{\mathrm{tor}}$ by doing the base changes under ρ_1, ρ_2 and additional smooth blowups in the fibers. On $\mathcal{Y}_6'$, the divisor $K_{\mathcal{Y}_6'/\overline{M}_6} + \frac{1}{2}D_6$ is relatively big and nef over $\overline{M}_6$. The universal family $\mathcal{Y}_6 \to \overline{M}_6$ is its relative canonical model.

The boundary divisor $\mathcal{D}_{6,\mathrm{bry}}^{\mathrm{tor}}$ is the union of the boundary curves on the fibers; it is the horizontal part of the toric boundary of $\mathcal{Y}_6^{\mathrm{tor}}$. The interior divisor $\mathcal{D}_6^{\mathrm{tor}}$ on $\mathcal{Y}_6^{\mathrm{tor}}$ is constructed in [AP23, Sec. 4] as follows. In addition to the map $p_1 \colon \mathcal{Y}_6^{\mathrm{tor}} \to \overline{M}_6^{\mathrm{tor}}$, there a second projection, a birational morphism $p_2 \colon \mathcal{Y}_6^{\mathrm{tor}} \to V_{P_6}$ to a projective toric variety V_{P_6} defined by a lattice polytope P_6 that is the convex hull of a 46-point set A_6. Under this projection, the fibers Y_6 become closed subvarieties of V_{P_6}. Then $\mathcal{D}_{\mathrm{int}}$ is a section of $p_2^*\mathcal{O}_{V_{P_6}}(2) \otimes p_1^*\mathcal{O}_{\overline{M}_6}(-F)$ for a certain effective divisor F on $\overline{M}_6$ that is defined in the proof of [AP23, Proposition 4.20].

5.2 Degree 4

Non-nodal Burniat surfaces of degree 4 are defined as follows. One considers the special configurations for which the triples of the curves (R_1, G_1, B_1) and (R_2, G_2, B_2) pass through common points, as in the right panel of Figure 1. Let $Y \to \Sigma$ be the blowup at these points. The strict preimages of R, G, B give a $\mathbb{Z}_2^2$-cover $\pi \colon X \to Y$ that is a Burniat surface of degree 4. Note that the exceptional divisors are not included in the branch divisor of π.

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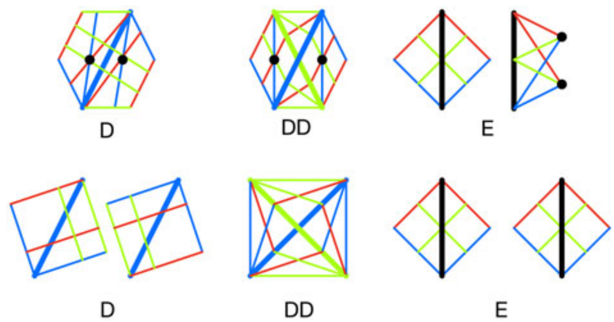

FIGURE 2. Surfaces over $Z \subset \overline{M}_6$ of degrees 6 and 4.

The compactified moduli space $\overline{M}_4^{\mathrm{Bur}}$ of degree 4 Burniat surfaces was constructed in [AH23] as an $S_3 \ltimes S_2^2$-quotient of the compactified moduli space $\overline{M}_4$ of pairs $(Y, \frac{1}{2}\mathcal{D})$, as follows.

Remark 5.1. As we only consider the non-nodal degree 4 case in this paper, in order to simplify the notation, in this section we write simply D, Z, $Y\ldots$ instead of D_{4a}, Z_{4a}, $Y_{4a}\ldots$, as in [AH23]. To distinguish the parent degree 6 case, we keep the subscripts there: D_6, Z_6, $Y_6\ldots$.

There exists a closed subvariety $Z \subset \overline{M}_6$, a complete intersection of two divisors, over which the curves (R_1, G_1, B_1) and (R_2, G_2, B_2) are incident. The degenerate pairs appearing in this family are shown in the upper row of Figure 2. The restricted family $\mathcal{Y}_6|_Z$ comes with two disjoint sections s_1, s_2. Let $\mathcal{Y}' \to Z$ be the blowup of $\mathcal{Y}_6|_Z$ along s_1, s_2.

The variety Z is the strict preimage of a toric variety $Z^{\mathrm{tor}} \subset \mathcal{Y}^{\mathrm{tor}}$ under the blowup ρ_1. It turns out that $Z^{\mathrm{tor}} \simeq \Sigma$ and $Z = \mathrm{Bl}_1\Sigma$. The divisor $K_{\mathcal{Y}'/Z} + \frac{1}{2}\mathcal{D}'$ is relatively nef over Z; let $\mathcal{Y}''$ be its relative canonical model. The degenerate fibers appearing in $\mathcal{Y}'' \to Z$ are shown in the lower row of Figure 2. They are a union of two $\mathbb{P}^1 \times \mathbb{P}^1$ glued along the diagonal, a union of four $\mathbb{P}^2$ and another union of two $\mathbb{P}^1 \times \mathbb{P}^1$ with a different configuration of branch divisors.

Over the exceptional divisor of $\mathrm{Bl}_1\Sigma \to \Sigma$ (the divisor of type E) all the fibers are isomorphic, so the family $(\mathcal{Y}'', \frac{1}{2}\mathcal{D}'') \to Z$ descends to a family $(\mathcal{Y}, \frac{1}{2}\mathcal{D}) \to \overline{M}_4 = Z^{\mathrm{tor}} = \Sigma$. Thus, the final family $\mathcal{Y} \to \Sigma$ is obtained from the toric family $\mathcal{Y}^{\mathrm{tor}} \to \Sigma$ as follows. There is a sequence of smooth blowups

$$\mathcal{Y}^{\mathrm{tor}} = \mathcal{Y}_0 \xleftarrow{\beta_1} \mathcal{Y}_1 \xleftarrow{\beta_2} \mathcal{Y}_2 \xleftarrow{\beta_3} \mathcal{Y}_3 \xleftarrow{\beta_4} \mathcal{Y}_4 = \mathcal{Y}', \tag{1}$$

in which

(i) β_1 is the blowup of the fiber $(\mathcal{Y}_6^{\mathrm{tor}})_1 \simeq \Sigma$ of $\mathcal{Y}_6^{\mathrm{tor}}$ over $1 \in \Sigma$ (this is the base change $\mathcal{Y}_1 = \mathcal{Y}_0 \times_\Sigma \mathrm{Bl}_1\Sigma \to \mathcal{Y}_0$);

(ii) β_2 is the blowup of $\mathbb{P}^1 = \beta^{-1}(1)$, preimage of the central point $1 \in (\mathcal{Y}_6^{\mathrm{tor}})_1$;

(iii) β_3 and β_4 are the blowups of the sections $s_i = R_i \cap G_i \cap B_i$, $i = 1, 2$.

This sequence is followed by a contraction $\mathcal{Y}' \to \mathcal{Y}''$ followed by a contraction $\mathcal{Y}'' \to \mathcal{Y}$ covering the contraction $\mathrm{Bl}_1\Sigma \to \Sigma$.

5.3 Kappa classes

THEOREM 5.2. *In* $A^*(\Sigma)$ *one has* $\kappa_0 = 1$, $\kappa_1 = \mathcal{O}_\Sigma(1) = \mathcal{O}(-K_\Sigma)$, $\kappa_2 = \frac{47}{4} \cdot [\mathrm{pt}]$.

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Proof. The class κ_0 is the degree of the divisor $K_Y + \frac{1}{2}D$ on a general fiber, so $\kappa_0 = (K_Y + \frac{1}{2}D)^2 = (-\frac{1}{2}K_Y)^2 = 1$.

For the rest, we begin by computing the divisor $K_{\mathcal{Y}^{\mathrm{tor}}/\Sigma} + \frac{1}{2}\mathcal{D}^{\mathrm{tor}}$. Let us denote the toric boundary of the toric variety $\mathcal{Y}^{\mathrm{tor}}$ by Δ. Denote the part of Δ that maps to the toric boundary of Σ by Δ^{ver} and the remaining part by Δ^{hor}. Obviously, one has $\Delta^{\mathrm{ver}} = p_1^*(\Delta_\Sigma) = p_1^*\mathcal{O}_\Sigma(1)$ for the projection $\mathcal{Y}^{\mathrm{tor}} \to Z^{\mathrm{tor}} = \Sigma$.

As explained above, the family $\mathcal{Y}_6^{\mathrm{tor}} \to \overline{M}_6^{\mathrm{tor}}$ comes with a second projection $p_2\colon \mathcal{Y}_6^{\mathrm{tor}} \to V_{P_6}$ and the fibers Y_6^{tor} are closed subvarieties sweeping out V_{P_6}. Restricting this family to Z gives a family that sweeps out a smaller toric variety V for the lattice polytope P obtained by an appropriate projection of P_6. By Lemma 5.4, the projection $\mathcal{Y}_6^{\mathrm{tor}} \to V$ is small, since both varieties have 18 toric boundary divisors. By Lemma 5.3, P is reflexive with a unique interior point. This implies that $\Delta = p_2^*\Delta_V$ and $-K_V = \Delta_V = \mathcal{O}_V(1)$. Restricting the divisor $\mathcal{D}_{6,\mathrm{int}}$ on $\overline{M}_6^{\mathrm{tor}}$ to the family over Z^{tor} gives $\mathcal{O}(\mathcal{D}_{\mathrm{int}}) = p_2^*\mathcal{O}_V(2) \otimes p_1^*\mathcal{O}_\Sigma(-1)$. Putting this together, writing additively, and for convenience mixing up sheaves and divisors, we get

$$\begin{aligned}
&K_{\mathcal{Y}^{\mathrm{tor}}/\Sigma} = -\Delta^{\mathrm{hor}}, \quad \Delta = p_2^*\mathcal{O}_V(1), \quad \mathcal{D}_{\mathrm{bry}} = \Delta^{\mathrm{hor}},\\
&\mathcal{D}_{\mathrm{int}} = p_2^*\mathcal{O}_V(2) - p_1^*\mathcal{O}_\Sigma(1) = 2\Delta - \Delta^{\mathrm{ver}},\\
&K_{\mathcal{Y}^{\mathrm{tor}}/\Sigma} + \tfrac{1}{2}\mathcal{D}^{\mathrm{tor}} = -\Delta^{\mathrm{hor}} + \tfrac{1}{2}(\Delta^{\mathrm{hor}} + 2\Delta - \Delta^{\mathrm{ver}}) = \tfrac{1}{2}\Delta = p_2^*\mathcal{O}_V(\tfrac{1}{2}).
\end{aligned}$$

By symmetry, κ_1 is a multiple of $\mathcal{O}_\Sigma(1)$. To find this multiple it is enough to find its intersection with a boundary (-1)-curve C on Σ, which equals $(K_{\mathcal{Y}} + \frac{1}{2}\mathcal{D})^3$ on the divisor $F = f^{-1}(C) \subset \mathcal{Y}$. To compute it, we can ignore the blowup ρ_1 since it does not touch F. We can also compute on the family $\mathcal{Y}'$ since the contraction $\mathcal{Y}' \to \mathcal{Y}''$ from a relative minimal model to a relative canonical model is crepant.

The restriction of $\mathcal{Y}^{\mathrm{tor}}$ to F has two irreducible components corresponding to the two surfaces $\mathrm{Bl}_1\mathbb{F}_1$ in case D of Figure 2. Each component maps birationally to a boundary divisor of V. Thus, the degree of $p_2^*\mathcal{O}_V(1)$ on F is twice the degree of $\mathcal{O}_V(1)$ on a boundary divisor of V. The latter degree is the lattice volume of the corresponding facet of P, which by Lemma 5.3 is 7. So the degree of $K_{\mathcal{Y}^{\mathrm{tor}}/\Sigma} + \frac{1}{2}\mathcal{D}$ on F is $2 \cdot \frac{7}{8}$.

Restricting $\mathcal{Y}^{\mathrm{tor}}$ to C gives a family $\mathcal{Y}_C^{\mathrm{tor}}$ with two disjoint sections corresponding to the two special points. The family $\mathcal{Y}'_C$ is obtained from it by blowups at the two special sections, one in each irreducible component of $\mathcal{Y}_C^{\mathrm{tor}}$. One has

$$K_{\mathcal{Y}'_C/C} + \tfrac{1}{2}\mathcal{D}|_F = \beta^*(K_{\mathcal{Y}_C^{\mathrm{tor}}/C} + \tfrac{1}{2}\mathcal{D}_C^{\mathrm{tor}}) - \tfrac{1}{2}E_1 - \tfrac{1}{2}E_2,$$

where E_1, E_2 are the exceptional divisors. Using the blowup formula [Ful84, 3.3.4] and Lemma 5.5, we get that the degree of $K_{\mathcal{Y}/\Sigma} + \frac{1}{2}\mathcal{D}$ on F is $2 \cdot \frac{7-3}{8} = 1$. So, $\kappa_1 = \mathcal{O}_\Sigma(1)$.

To compute $\kappa_2 = (K_{\mathcal{Y}'/\mathrm{Bl}_1\Sigma} + \frac{1}{2}\mathcal{D}')^4$, we can compute on $\mathcal{Y}' \to \mathrm{Bl}_1\Sigma$ using the functorial property of the diagram (1). On $\mathcal{Y}^{\mathrm{tor}}$ one has $\left(K_{\mathcal{Y}^{\mathrm{tor}}/\Sigma} + \frac{1}{2}\mathcal{D}^{\mathrm{tor}}\right)^4 = \mathcal{O}_V\left(\frac{1}{2}\right)^4 = \frac{18\cdot 7}{2^4} = \frac{63}{8}$ by Lemma 5.3. Then we trace how this number changes under the four blowups in (1) using [Ful84, 3.3.4]. $\square$

The next two lemmas are proved by direct computations with polytopes.

LEMMA 5.3. *The polytope P is a reflexive four-dimensional polytope with the f-vector* $(1,30,84,72,18,1)$ *and a unique interior point. Its* 18 *facets are isomorphic three3-dimensional polytopes with* 8 *vertices and lattice volume* 7. *One has* $\mathrm{vol}(P) = 18\cdot 7$.

LEMMA 5.4. *The toric family $\mathcal{Y}^{\mathrm{tor}}$ is a projective toric variety for a four4-dimensional lattice polytope $P + 14P_\Sigma$ with the f-vector* $(1,42,96,72,18,1)$.

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Here, P_Σ is the hexagon corresponding to the toric variety $(\Sigma, \mathcal{O}_\Sigma(1))$, and $14P_\Sigma$ is the fiber polytope coming from the construction of the toric family in [AP23].

LEMMA 5.5. *For each of the special sections $s_1 = R_1^{\mathrm{tor}} \cap G_1^{\mathrm{tor}} \cap B_1^{\mathrm{tor}}$ and $s_2 = R_2^{\mathrm{tor}} \cap G_2^{\mathrm{tor}} \cap B_2^{\mathrm{tor}}$ of $\mathcal{Y}^{\mathrm{tor}} \to \Sigma$, the normal bundle is trivial, i.e. equal to $\mathcal{O}_\Sigma^{\oplus 2}$.*

Proof. Consider the union U of torus orbits in $\mathcal{Y}^{\mathrm{tor}}$ containing s_1. Since U comes with a section and a free action of the vertical torus $\mathbb{C}^{*2} = \ker(\mathbb{C}^{*4} \to \mathbb{C}^{*2})$, one has $U = \Sigma \times \mathbb{C}^{*2}$. So $N_{s_1/U} = \mathcal{O}_{s_1}^{\oplus 2}$, and the same works for s_2. $\square$

6. Campedelli surfaces

The Campedelli surfaces considered in [AP23] are surfaces of general type with $K^2 = 2$ and $p_g = 0$ which are $\mathbb{Z}_2^3$-covers of $\mathbb{P}^2$ ramified in seven lines in general position. The branch data for this cover consists of these seven lines D_g, $g \in \mathbb{Z}_2^3 \setminus 0$. The moduli space has dimension 6, which coincides with the dimension of the virtual fundamental class, equal to $10\chi - 2K_X^2$. So the virtual fundamental class in this case is $[\overline{\mathcal{M}}_{\mathrm{Cam}}]$.

By [AP23], the moduli stack $\overline{\mathcal{M}}_{\mathrm{Cam}}$ is a global quotient $[\overline{M} : \Gamma]$, where $\overline{M}$ is the compactified moduli space of labeled log canonical pairs $(\mathbb{P}^2, \sum_{g \in \mathbb{F}_2^3 \setminus 0} \frac{1}{2} D_g)$ and $\Gamma = \mathrm{GL}(3, \mathbb{F}_2^3)$. Further, $\overline{M}$ in this case is the GIT quotient $(\mathbb{P}^2)^7//\mathrm{SL}(3)$ for the symmetric polarization $(1, \dots, 1)$. The sets of stable and semistable points in this case coincide and the SL(3)-action on it is free. Therefore, $(\mathbb{P}^2)^7//\mathrm{SL}(3)$ is smooth.

The universal family $(\mathcal{Y}, \sum \frac{1}{2} D_g) \to \overline{M}$ is itself a GIT quotient of a family of hyperplane arrangements in $(\mathbb{P}^2)^7 \times (\mathbb{P}^2)^\vee$ by the action of the group SL(3) for the polarization $(1, \dots, 1, \epsilon)$, $0 < \epsilon \ll 1$.

As a first step, I compute the Chow ring of $\overline{M} = (\mathbb{P}^2)^7//\mathrm{SL}(3)$. Then the rational Chow ring of the stack $\overline{\mathcal{M}}_{\mathrm{Cam}}$ is identified with its Γ-invariant subring.

6.1 Chow ring

Let $X = (\mathbb{P}^2)^7$ with the diagonal action of $G = \mathrm{SL}(3)$. Consider the GIT quotient $X//G$ for the symmetric polarization $(1, \dots, 1)$. In this case, the stable and semistable loci coincide and the G-action on X^{s} is free, so the cohomology ring of $X//G$ can be identified with the equivariant cohomology ring $H_G(X^{\mathrm{ss}}, \mathbb{Q})$ and the Chow ring of $X//G$ with the equivariant Chow ring $A_G(X^{\mathrm{ss}})_\mathbb{Q}$.

Remark 6.1. As Michel Brion explained to me, for any semisimple group G with a maximal torus T and a G-variety V that admits a T-invariant cell decomposition, the G-equivariant Chow ring $A_G^*(V)_\mathbb{Q}$ and the G-equivariant cohomology ring $H_G^*(V, \mathbb{Q})$ coincide. Indeed, for the T-equivariant versions the cycle map $A_T^*(V)_\mathbb{Q} \to H_T^*(V, \mathbb{Q})$ is an isomorphism by [Bri97], and the G-equivariant versions A_G^* and H_G^* are the Weyl group invariant subrings of these.

THEOREM 6.2. *One has*

$$A^*(X//G)_\mathbb{Q} = H^*(X//G, \mathbb{Q}) = \mathbb{Q}[z_1, \dots, z_7, c_2, c_3]/J,$$

with the generators of degree $(1, \dots, 1, 2, 3)$ and the ideal J generated by the relations:

(i) $z_1^3 + c_2 z_1 + c_3$;

(ii) $\sigma_3(z_1, z_2, z_3, z_4, z_5) - \sigma_1(z_1, z_2, z_3, z_4, z_5) c_2 + c_3$;

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(iii) $\sigma_4(z_1,z_2,z_3,z_4,z_5) - \sigma_2(z_1,z_2,z_3,z_4,z_5)c_2 + c_2^2 + \sigma_1(z_1,z_2,z_3,z_4,z_5)c_3$;
(iv) $(z_1^2z_2^2 + z_2^2z_3^2 + z_3^2z_1^2) + (z_1+z_2+z_3)(z_1z_2z_3 - c_3) + (z_1^2+z_2^2+z_3^3)c_2 + c_2^2$;

and the ones obtained from them by permuting the variables $z_1,\dots,z_7$. Here, σ_k are the elementary symmetric polynomials.

Moreover, the relations of types (i),(ii),(iv) and a single relation of type (iii), or the sum of all relations of type (iii), suffice. The relations of types (i),(ii) are independent.

The dimensions of A^i for $i=0,\dots,6$ are $(1,7,29,64,29,7,1)$.

Proof. This is a direct application of Brion's paper [Bri91]. Let $T\subset G$ be the maximal torus. One has

$$A_T(\cdot) = S = \mathbb{Q}[\epsilon_0,\epsilon_1,\epsilon_2]/(\epsilon_0+\epsilon_1+\epsilon_2), \quad A_G(\cdot) = S^W = \mathbb{Q}[c_2,c_3], \quad W = S_3,$$

where c_2 and c_3 are the Chern characters of the representation V with $\mathbb{P}^2=\mathbb{P}(V)$, the elementary symmetric polynomials in ϵ_i. Then the equivariant cohomology ring $A_T^*(X)$ equals $S[z_1,\dots,z_7]$ modulo the basic relations $(z_i+\epsilon_0)(z_i+\epsilon_1)(z_i+\epsilon_2)$. Similarly, $A_G^*(X)$ equals $S^W[z_1,\dots,z_7]$ modulo the basic relations $z_i^3+c_2z_i+c_3$. Here, $z_i = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$ for the ith $\mathbb{P}^2$, cf. Remark 6.3.

Let $X_T^{\rm ss}$ denote the set of semistable points for the action of T. Then by [Bri91, Theorem 2.1], $A_T^*(X_T^{\rm ss}) = A_T^*(X)/I$, and the ideal I is described with the help of the maximal unstable sets. Up to the permutation by $S_3\times S_7$ of indices, they are

$$\{0\}^3,\ \{0,1,2\}^4 \quad \text{and} \quad \{1,2\}^5,\ \{0,1,2\}^2,$$

which in a transparent way correspond to the instability conditions of seven lines on $\mathbb{P}^2$ (see [AP23]): when three lines coincide or five lines pass through a common point. Then I up to permutation by $S_3\times S_7$ is generated by the expressions

$$\begin{aligned}&(z_1+\epsilon_1)(z_1+\epsilon_2)(z_2+\epsilon_1)(z_2+\epsilon_2)(z_3+\epsilon_1)(z_3+\epsilon_2)\\ \text{and}\quad &(z_1+\epsilon_0)(z_2+\epsilon_0)(z_3+\epsilon_0)(z_4+\epsilon_0)(z_5+\epsilon_0).\end{aligned}$$

Then, by [Bri91, Section 1.2], one has $A_G^*(X^{\rm ss}) = A^*G(X)/p(I)$, where p is the antisymmetrization operator

$$p = D^{-1}\sum_{w\in W}(-1)^{\operatorname{sign}(w)}w, \quad D = (\epsilon_0-\epsilon_1)(\epsilon_1-\epsilon_2)(\epsilon_2-\epsilon_0).$$

Moreover, given generators f_i of I and an additive basis $\langle g_j\rangle$ for the harmonic module $\mathcal{H}$, the ideal $p(I)$ is generated by $p(f_ig_j)$. For $G=\mathrm{SL}(3)$ the harmonic module is $\mathcal{H} = \langle 1,\epsilon_i,\epsilon_i^2-\epsilon_j^2, D\rangle$. The rest is a computation in sagemath [Sag22]. □

Remark 6.3. In [Bri91] Grothendieck's convention for a projective space $\mathbb{P}V$ as the space of one1-dimensional quotient of V is followed. We follow the convention that $\mathbb{P}V$ is the space of lines in V, more common in the literature on equivariant cohomology. Then for us $z_i = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$, and in [Bri91], $z_i = c_1(\mathcal{O}_{\mathbb{P}^2}(-1))$.

6.2 GL(3, 2)-invariants

Denote by $s_k = \sigma_k(z_1,\dots,z_7)$ the elementary symmetric polynomials in $z_1,\dots,z_7$. Also, denote by s_3', respectively s_3'', the sums of $z_iz_jz_k$ with distinct i,j,k such that the indices i,j,k considered as points of the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$ are incident, respectively are not incident. Obviously, s_3' and s_3'' are $\mathrm{GL}(3,\mathbb{F}_2)$-invariant but not S_7-invariant. One has $s_3 = s_3' + s_3''$ and denotes $t = s_3'' - 4s_3'$.

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Theorem 6.4. *For the ring of invariants $A^*(X//G)^{\mathrm{GL}(3,2)}$, the dimensions of the invariants of degree $0,\dots,6$ are $(1,1,3,4,3,1,1)$ with an additive basis*

$$1 \quad s_1 \quad s_1^2, c_2, s_2 \quad s_1^3, c_2s_1, s_2s_1, t \quad c_2^2, c_2s_2, s_2^2 \quad c_2^2s_1 \quad c_2^3.$$

The algebra $A^*(X//G)^{\mathrm{GL}(3,2)}$ is generated by s_1, c_2, s_2, t with relations

$$\begin{aligned}&ts_1, \quad tc_2, \quad ts_2, \quad t^2+126c_2^3,\\ &45s_1^4-1246c_2^2+1090c_2s_2-240s_2^2, \quad 15c_2s_1^2-28c_2^2+35c_2s_2-10s_2^2,\\ &3s_2s_1^2-49c_2^2+46c_2s_2-11s_2^2, \quad 5c_2s_2s_1-16c_2^2s_1, \quad 5s_2^2s_1-59c_2^2s_1.\end{aligned}$$

Proof. The irreducible characters of $\mathrm{GL}(3,2)$ are χ_1, χ_3, $\chi_{\bar{3}}$, χ_6, χ_7, χ_8, where the subscript denotes the dimension and χ_1 is the trivial representation. There are two basic permutation representations of S_7 on the 7 variables $z_1,\dots,z_7$ and on the 21 monomials z_iz_j with $i\neq j$. The induced $\mathrm{GL}(3,2)$-representations are

$$\chi_7^{\mathrm{p}}=\chi_1+\chi_6, \quad \chi_{21}^{\mathrm{p}}=\chi_1+2\chi_6+\chi_8.$$

From the relations of Theorem 6.2, the representations on A^i for $i=1,2,3$ are:

(i) $A^1=\chi_7^{\mathrm{p}}=\chi_1+\chi_6$, invariant: s_1;

(ii) $A^2=\mathrm{Sym}^2(\chi_7^{\mathrm{p}})+\chi_1\cdot c_2=3\chi_1+3\chi_6+\chi_8$, invariants s_1^2, s_2, c_2;

(iii) (from the generators and relations of type (i) and (ii))

$$\begin{aligned}A^3&=\mathrm{Sym}^3(\chi)+\chi_7^{\mathrm{p}}\cdot c_2+\chi_1\cdot c_3-\chi_7^{\mathrm{p}}-\chi_{14}^{\mathrm{p}}\\ &=(4\chi_1+7\chi_6+2\chi_7+3\chi_8)+\chi_1-(\chi_1+2\chi_6+\chi_8)\\ &=4\chi_1+5\chi_6+2\chi_7+2\chi_8.\end{aligned}$$

From this and Poincare duality, for $\dim(H^{2i})^{\mathrm{GL}(3,2)}$ we get $1,1,3,4,3,1,1$. By hard Lefschetz, $s_1^{2i}H^{6-2i}\simeq H^{6+2i}$. This gives additive bases in the invariant subspaces of $H^8=A^4$, $H^{10}=A^5$, $H^{12}=A^6$, and we check that our choices (leading to smaller formulas) also give bases. For H^6, we get a subspace $s_1\cdot(H^4)^{\mathrm{GL}(3,2)}=\langle s_1^3, c_2s_1, s_2s_1\rangle$. Since these vectors are S_7-invariant and t is not, adding t completes them to a basis of $(H^6)^{\mathrm{GL}(3,2)}$.

I checked the algebra relations and the fact that they suffice in sagemath. $\square$

Remark 6.5. The smaller subring of invariants $A^*(X//G)^{S_7}$ has $(1,1,3,3,3,1,1)$ for the dimensions of the graded pieces. It is generated by s_1, c_2, s_2 with the same relations, dropping those involving t.

For § 6.3, I note the relation $c_3=\frac{1}{7}(2s_1^3-6s_2s_1+17c_2s_1)$.

6.3 Kappa classes

The moduli stack $\overline{\mathcal{M}}_{\mathrm{Cam}}$ is a global quotient $[\overline{M}:\mathrm{GL}(3,2)]$, where $\overline{M}=(\mathbb{P}^2)^7//\mathrm{SL}(3,\mathbb{C})$. Let $f\colon\mathcal{Y}\to\overline{M}$ be the universal family of stable pairs $(Y,\sum_{i=1}^7\frac{1}{2}B_i)$ (see [AP23]). The kappa classes on $\overline{M}$ are

$$\kappa_l=f_*L^{l+2} \quad \text{where} \quad L=\omega_{\mathcal{Y}/\overline{M}}\left(\sum_{i=1}^{7}\frac{1}{2}B_i\right).$$

I am grateful to William Graham for explaining to me how to do pushforward in equivariant cohomology, which is used in the proof of the next theorem.

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Theorem 6.6. *In the Chow ring $A^*(\overline{M}, \mathbb{Q})$ described above, one has*

$$\begin{aligned}
2^2\kappa_0 &= 1,\\
2^3\kappa_1 &= 3s_1,\\
2^4\kappa_2 &= 6s_1^2 - c_2,\\
2^5\kappa_3 &= 10s_1^3 - 5s_1c_2 + c_3,\\
2^6\kappa_4 &= 15s_1^4 - 15s_1^2c_2 + c_2^2 + 6s_1c_3,\\
2^7\kappa_5 &= 21s_1^5 - 35s_1^3c_2 + 7s_1c_2^2 + 21s_1^2c_3 - 2c_2c_3,\\
2^8\kappa_6 &= 28s_1^6 - 70s_1^4c_2 + 28s_1^2c_2^2 - c_2^3 + 56s_1^3c_3 - 16s_1c_2c_3 + c_3^2.
\end{aligned}$$

Proof. Over $X = \mathbb{P}(V)^7$ we have the universal family $X \times \mathbb{P}(V^*)$ with seven divisors $\mathcal{B}_i$, and the family over $\overline{M}$ is a quotient by a free action of $G = \mathrm{SL}(V)$. Therefore, it suffices to compute in $A_G^*(X)$. Denote $h = \mathcal{O}_{\mathbb{P}(V^*)}(1)$. Each $\mathcal{B}_i$ is the incidence divisor in $\mathbb{P}(V) \times \mathbb{P}(V^*)$ and is linearly equivalent to $z_i + h$. Therefore,

$$L = \omega_{X \times \mathbb{P}(V^*)/X}\left(\sum_{i=1}^{7} \frac{1}{2} B_i\right) = -3h + \sum_{i=1}^{7} \frac{1}{2}(h + z_i) = \frac{1}{2}(h + s_1).$$

By projection formula, to compute f_*L^{i+2} it suffices to know f_*h^k under the homomorphism $A_G^*(\mathbb{P}(V^*)) \to A_G^*(\cdot) = S$ induced by the morphism $\mathbb{P}(V^*) \to \mathrm{pt}$. For $s \in S^W$ one has $f_*(s) = f_*(hs) = 0$, $f_*(h^2 s) = s$, and the pushforwards of higher powers of h follow by recursively using the basic relation $h^3 + c_2h - c_3 = 0$. The rest is an easy computation. ☐

Note that s_1 is the ample line bundle that comes with the GIT quotient construction, the $\mathcal{O}(1)$ on the Proj of the graded algebra of invariants.

Data Availability Statement

No supplemental data is provided with this article.

Author Contributions

The author confirms the sole responsibility for the conception of the study, presented results, and manuscript preparation.

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Conflicts of Interest

None

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