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The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of $K3$ surfaces

François Charles and Eyal Markman

Abstract

We prove the standard conjectures for complex projective varieties that are deformations of the Hilbert scheme of points on a $K3$ surface. The proof involves Verbitsky's theory of hyperholomorphic sheaves and a study of the cohomology algebra of Hilbert schemes of $K3$ surfaces.

1. Introduction

An *irreducible holomorphic symplectic manifold* is a simply connected compact Kähler manifold X, such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic two-form (see [Bea83, Huy99]).

Let S be a $K3$ surface and $S^{[n]}$ the Hilbert scheme (or Douady space) of length n zero-dimensional subschemes of S. Beauville proved in [Bea83] that $S^{[n]}$ is an irreducible holomorphic symplectic manifold of dimension $2n$. If X is a smooth compact Kähler manifold deformation-equivalent to $S^{[n]}$, for some $K3$ surface S, then we say that X is of $K3^{[n]}$*-type*. The variety X is then an irreducible holomorphic symplectic manifold. The odd Betti numbers of X vanish [Got94].

The moduli space of manifolds of $K3^{[n]}$-type is smooth and 21-dimensional, if $n \geqslant 2$, while that of $K3$ surfaces is 20-dimensional [Bea83]. It follows that if S is a $K3$ surface, a general Kähler deformation of $S^{[n]}$ is not of the form $S'^{[n]}$ for a $K3$ surface S'. The same goes for projective deformations. Indeed, a general projective deformation of $S^{[n]}$ has Picard number 1, whereas for a projective S, the Picard number of $S^{[n]}$ is at least 2.

In this note, we prove the standard conjectures for projective varieties of $K3^{[n]}$-type. Let us recall general facts about the standard conjectures.

In the paper [Gro69] of 1968, Grothendieck states those conjectures concerning the existence of some algebraic cycles on smooth projective algebraic varieties over an algebraically closed ground field. Here we work over $\mathbb{C}$. The Lefschetz standard conjecture predicts the existence of algebraic self-correspondences on a given smooth projective variety X of dimension d that give an inverse to the operations

$$H^i(X) \to H^{2d-i}(X)$$

given by the cup-product $d-i$ times with a hyperplane section, for all $i \leqslant d$. Above and throughout the rest of the paper, the notation $H^i(X)$ stands for singular cohomology with rational coefficients.

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Over the complex numbers, the Lefschetz standard conjecture implies all the standard conjectures. If it holds for a variety X, it implies that numerical and homological equivalence coincide for algebraic cycles on X, and that the Künneth components of the diagonal of $X \times X$ are algebraic. We refer to [Kle68] for a detailed discussion.

Though the motivic picture has tremendously developed since Grothendieck's statement of the standard conjectures, very little progress has been made in their direction. The Lefschetz standard conjecture is known for abelian varieties [Lie68], and in degree 1, where it reduces to the Hodge conjecture for divisors. The Lefschetz standard conjecture is also known for varieties X, for which $H^*(X)$ is isomorphic to the Chow ring $A^*(X)$, see [Kle94]. Varieties with the latter property include flag varieties, and smooth projective moduli spaces of sheaves on rational Poisson surfaces [ES93, Mar07].

In the paper [Ara06], Arapura proves that the Lefschetz standard conjecture holds for uniruled threefolds, unirational fourfolds, the moduli space of stable vector bundles over a smooth projective curve, and for the Hilbert scheme $S^{[n]}$ of every smooth projective surface, see [Ara06, Corollaries 4.3, 7.2 and 7.5]. He also proves that if S is a $K3$ or abelian surface, H an ample line-bundle on S, and $\mathcal{M}$ a smooth and compact moduli space of Gieseker–Maruyama–Simpson H-stable sheaves on S, then the Lefschetz standard conjecture holds for $\mathcal{M}$ (see [Ara06, Corollary 7.9]). Those results are obtained by showing that the motive of those varieties is very close, in a certain sense, to that of a curve or a surface. Aside from those examples and ones obtained by specific constructions from them (e.g. hyperplane sections, products, projective bundles, etc.), very few cases of the Lefschetz standard conjecture seem to be known.

The main result of this note is the following statement.

Theorem 1.1. *The Lefschetz standard conjecture holds for every smooth projective variety of $K3^{[n]}$-type.*

Since the Lefschetz standard conjecture is the strongest standard conjecture in characteristic zero, we get the following corollary.

Corollary 1.2. *The standard conjectures hold for any smooth projective variety of $K3^{[n]}$-type.*

Note that by the remarks above, Theorem 1.1 does not seem to follow from Arapura's results, as a general variety of $K3^{[n]}$-type is not a moduli space of sheaves on any $K3$ surface.

Theorem 1.1 is proven in § 8. The degree 2 case of the Lefschetz standard conjecture, for projective varieties of $K3^{[n]}$-type, is already known, by results of [Cha10], combined with [Mar, Theorem 1.6]. Section 2 gives general results on the Lefschetz standard conjecture. Sections 3–5 introduce the algebraic cycles we need for the proof, while §§ 6 and 7 contain results on the cohomology algebra of the Hilbert scheme of $K3$ surfaces.

Unless otherwise specified, we will consider K-groups, cohomology groups, etc. with rational coefficients. For instance, $K(X, \mathbb{Z})$ will denote the K-group of X with integer coefficients, and $K(X)$ will by definition denote $K(X, \mathbb{Z}) \otimes \mathbb{Q}$.

2. Preliminary results on the Lefschetz standard conjecture

Let X be a smooth projective variety of dimension d. Let $\xi \in H^2(X)$ be the cohomology class of a hyperplane section of X. According to the hard Lefschetz theorem, for all $i \in \{0, \ldots, d\}$, cup-product with ξ^{d-i} induces an isomorphism

$$L^{d-i} := \cup \xi^{d-i} : H^i(X) \to H^{2d-i}(X).$$

The Lefschetz standard conjecture was first stated in [Gro69, Conjecture $B(X)$]. It is the following.

Conjecture 2.1. Let X and ξ be as above. Then for all $i \in \{0, \dots, d\}$, there exists an algebraic cycle Z of codimension i in the product $X \times X$ such that the correspondence

$$[Z]_* : H^{2d-i}(X) \to H^i(X)$$

is the inverse of $\cup \xi^{d-i}$.

If this conjecture holds for some specific i on X, we will say that *the Lefschetz conjecture holds in degree i for the variety X*. It is known that for the conjecture to hold, it is enough to require for the morphism $[Z]_*$ above to be any isomorphism, see [Kle68, Theorem 4.1].

We will derive Theorem 1.1 as a consequence of Theorem 4.1 and Corollary 6.2 below. In this section, we prove some general results we will need. The reader can consult [Ara06, §§ 1 and 4] for related arguments, and [And96] for a more general use of polarizations and semi-simplicity. Let us first state an easy lemma.

Lemma 2.2. *Let X be a smooth projective variety of dimension d. Let $i \leqslant d$ be an integer.*

(1) *Assume $i = 2j$ is even, and let $\alpha \in H^{2j}(X)$ be the cohomology class of a codimension j algebraic cycle in X. Then there exists a cycle Z of codimension $i = 2j$ in $X \times X$ such that the image of the correspondence*

$$[Z]_* : H^{2d-2j}(X) \to H^{2j}(X)$$

contains α.

(2) *Assume that X satisfies the Lefschetz standard conjecture in degree up to $i - 1$. Then $X \times X$ satisfies the Lefschetz standard conjecture in degree up to $i - 1$.*
Let j and k be two positive integers with $i = j + k$. Then there exists a cycle Z of codimension i in $(X \times X) \times X$ such that the image of the correspondence

$$[Z]_* : H^{4d-i}(X \times X) \to H^i(X)$$

contains the image of $H^j(X) \otimes H^k(X)$ in $H^{j+k}(X) = H^i(X)$ by cup-product.

Proof. Let $\alpha \in H^{2j}(X)$ be the cohomology class of a codimension j algebraic cycle T in X. Let Z be the codimension $i = 2j$ algebraic cycle $T \times T$ in $X \times X$. Since the image in $H^i(X) \otimes H^i(X)$ of the cohomology class of Z in $H^{2i}(X \times X)$ is $\alpha \otimes \alpha$, the image of the correspondence

$$[Z]_* : H^{2d-i}(X) \to H^i(X)$$

is the line generated by α. This proves (1).

Let us prove the first part of (2). We repeat some of Kleiman's arguments in [Kle68]. Assume that X satisfies the Lefschetz standard conjecture in degree up to $i - 1$. We want to prove that $X \times X$ satisfies the Lefschetz standard conjecture in degree up to $i - 1$. By induction, we only have to prove that $X \times X$ satisfies the Lefschetz standard conjecture in degree $i - 1$.

For any j between 0 and $i - 1$, there exists a codimension j algebraic cycle Z_j in $X \times X$ such that the correspondence

$$[Z_j]_* : H^{2d-j}(X) \to H^j(X)$$

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is an isomorphism. For k between 0 and $2d$, let $\pi^k \in H^{2d-k}(X) \otimes H^k(X) \subset H^{2d}(X \times X)$ be the kth Künneth component of the diagonal. By [Kle68, Lemma 2.4], the assumption on X implies that the elements $\pi^0, \ldots, \pi^{i-1}, \pi^{2d-i+1}, \ldots, \pi^{2d}$ are algebraic. Identifying the π^j with the correspondence they induce, this implies that for all j between 0 and $i-1$, the projections

$$\pi^j : H^*(X) \to H^j(X) \hookrightarrow H^*(X)$$

and

$$\pi^{2d-j} : H^*(X) \to H^{2d-j}(X) \hookrightarrow H^*(X)$$

are given by algebraic correspondences. Replacing the correspondence $[Z_j]_*$ by $[Z_j]_* \circ \pi_{2d-j}$, which is still algebraic, we can thus assume that the morphism

$$[Z_j]_* : H^{2d-k}(X) \to H^{2j-k}(X)$$

induced by $[Z_j]$ is zero unless $k = j$.

Now consider the codimension $i-1$ cycle Z in $(X \times X) \times (X \times X)$ defined by

$$Z = \sum_{j=0}^{i-1} Z_j \times Z_{i-1-j}.$$

We claim that the correspondence

$$[Z]_* : H^{4d-i+1}(X \times X) \to H^{i-1}(X \times X)$$

is an isomorphism.

Fix j between 0 and $i-1$. The hypothesis on the cycles Z_j implies that the correspondence

$$[Z_j \times Z_{i-1-j}]_* : H^{4d-i+1}(X \times X) \to H^{i-1}(X \times X)$$

maps the subspace $H^{2d-k}(X) \otimes H^{2d-i+1+k}(X)$ of $H^{4d-i+1}(X \times X)$ to zero unless $k = j$, and it maps $H^{2d-j}(X) \otimes H^{2d-i+1+j}(X)$ isomorphically onto $H^j(X) \otimes H^{i-1-j}(X)$. The claim follows, as does the first part of (2).

For the second statement, let j and k be as in the hypothesis. Since j (respectively k) is smaller than or equal to $i-1$, X satisfies the Lefschetz standard conjecture in degree j (respectively k). As a consequence, there exists a cycle T (respectively T') in $X \times X$ such that the morphism

$$[T]_* : H^{2d-j}(X) \to H^j(X)$$

(respectively $[T']_* : H^{2d-k}(X) \to H^k(X)$) is an isomorphism. Consider now the projections p_{13} and p_{23} from $X \times X \times X$ to $X \times X$ forgetting the second and first factor respectively, and let Z in $CH^i(X \times X \times X)$ be the intersection of p_{13}^*T and p_{23}^*T'. We view Z as a correspondence between the first two factors and the third. Since the cohomology class of Z is just the cup-product of that of p_{13}^*T and p_{23}^*T', it follows that the image of the correspondence

$$[Z]_* : H^{4d-i}(X \times X) \to H^i(X)$$

contains the image of $H^j(X) \otimes H^k(X)$ in $H^{j+k}(X) = H^i(X)$ by cup-product. □

The following result appears in [Cha10, Proposition 8].

Theorem 2.3. *Let X be a smooth projective variety of dimension d, and let $i \leqslant d$ be an integer. Then the Lefschetz conjecture is true in degree i for X if and only if there exists a disjoint union S of smooth projective varieties of dimension $l \geqslant i$ satisfying the Lefschetz conjecture in degree*

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up to $i-2$ and a codimension i cycle Z in $X \times S$ such that the morphism

$$[Z]_* : H^{2l-i}(S) \to H^i(X)$$

induced by the correspondence Z is surjective.

The following statement is an immediate corollary of Lemma 2.2 and Theorem 2.3.

COROLLARY 2.4. *Let X be a smooth projective variety of dimension d, and let $i \leqslant d$ be an integer. Suppose that X satisfies the Lefschetz standard conjecture in degree up to $i-1$.*

Let $A^i(X) \subset H^i(X)$ be the subspace of classes, which belong to the subring generated by classes of degree $<i$, and let $\mathrm{Alg}^i(X) \subset H^i(X)$ be the subspace of $H^i(X)$ generated by the cohomology classes of algebraic cycles.[1]

Assume that there is a cycle Z of codimension i in $X \times X$ such that the image of the morphism

$$[Z]_* : H^{2d-i}(X) \to H^i(X)$$

maps surjectively onto the quotient space $H^i(X)/[\mathrm{Alg}^i(X) + A^i(X)]$. Then X satisfies the Lefschetz standard conjecture in degree i.

Proof. We use Lemma 2.2. Let $\alpha_1, \ldots, \alpha_r$ be a basis for $\mathrm{Alg}^i(X)$. We can find codimension i cycles $Z_1, \ldots, Z_r$ in $X \times X$ and $(Z_{j,k})_{j,k>0,j+k=i}$ in $(X \times X) \times X$, such that the image of the correspondence

$$[Z_l]_* : H^{2d-i}(X) \to H^i(X)$$

contains α_l for $1 \leqslant l \leqslant r$, and such that the image of the correspondence

$$[Z_{j,k}]_* : H^{4d-i}(X \times X) \to H^i(X)$$

contains the image of $H^j(X) \otimes H^k(X)$ in $H^{j+k}(X) = H^i(X)$, for $j+k=i$.

We proved that $X \times X$ satisfies the Lefschetz standard conjecture in degree up to $i-1$. The disjoint union of the cycles $Z \times X$, $(Z_l \times X)_{1 \leqslant l \leqslant r}$ and $(Z_{j,k})_{j,k>0,j+k=i}$ in a disjoint union of copies of $(X \times X) \times X$ thus satisfies the hypothesis of Theorem 2.3 (we took products with X in order to work with equidimensional varieties). Indeed, the space generated by the images in $H^i(X)$ of the correspondences $[Z_{j,k}]_*$ contains $A^i(X)$ by definition. Adding the images in $H^i(X)$ of the $[Z_l \times X]_*$, which generate a space containing $\mathrm{Alg}^i(X)$, and the image in $H^i(X)$ of $[Z \times X]_*$, which maps surjectively onto $H^i(X)/[A^i(X) + \mathrm{Alg}^i(X)]$, we get the whole space $H^i(X)$.

This ends the proof, and shows that X satisfies the Lefschetz standard conjecture in degree i. □

The strategy formulated in Corollary 2.4 will be used in the rest of this paper to prove Theorem 1.1.

COROLLARY 2.5. *Let X be a smooth projective variety with cohomology algebra generated by classes in degree less than i, and assume that X satisfies the Lefschetz standard conjecture in degree up to i. Then X satisfies the standard conjectures.*

Proof. Using induction and taking $Z=0$, the previous corollary shows that X satisfies the Lefschetz standard conjecture, hence all the standard conjectures, since we work in characteristic zero. □

[1] Note that this subspace is zero unless i is even.

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Note that the Lefschetz conjecture is true in degree 1, because it is a consequence of the Lefschetz theorem on $(1,1)$-classes. The preceding corollary hence allows us to recover the Lefschetz conjecture for abelian varieties which was proved in [Lie68].

3. Moduli spaces of sheaves on a $K3$ surface

Let S be a projective $K3$ surface. Denote by $K(S,\mathbb{Z})$ the topological K-group of S, generated by topological complex vector bundles. The K-group of a point is $\mathbb{Z}$ and we let $\chi : K(S,\mathbb{Z}) \to \mathbb{Z}$ be the Gysin homomorphism associated to the morphism from S to a point. The group $K(S,\mathbb{Z})$, endowed with the *Mukai pairing*

$$(v,w) := -\chi(v^\vee \otimes w),$$

is called the *Mukai lattice* and denoted by $\Lambda(S)$. Mukai identifies the group $K(S,\mathbb{Z})$ with $H^*(S,\mathbb{Z})$, via the isomorphism sending a class F to its *Mukai vector* $\mathrm{ch}(F)\sqrt{td_S}$. Using the grading of $H^*(S,\mathbb{Z})$, the Mukai vector of F is

$$(\mathrm{rank}(F), c_1(F), \chi(F) - \mathrm{rank}(F)), \tag{3.1}$$

where the rank is considered in H^0 and $\chi(F) - \mathrm{rank}(F)$ in H^4 via multiplication by the orientation class of S. The homomorphism $\mathrm{ch}(\bullet)\sqrt{td_S} : \Lambda(S) \to H^*(S,\mathbb{Z})$ is an isometry with respect to the Mukai pairing on $\Lambda(S)$ and the pairing

$$((r',c',s'),(r'',c'',s'')) = \int_S c' \cup c'' - r' \cup s'' - s' \cup r''$$

on $H^*(S,\mathbb{Z})$ (by the Hirzebruch–Riemann–Roch theorem). Mukai defines a weight 2 Hodge structure on the Mukai lattice $H^*(S,\mathbb{Z})$, and hence on $\Lambda(S)$, by extending that of $H^2(S,\mathbb{Z})$, so that the direct summands $H^0(S,\mathbb{Z})$ and $H^4(S,\mathbb{Z})$ are of type $(1,1)$ (see [Muk87]).

Let $v \in \Lambda(S)$ be a primitive class with $c_1(v)$ of Hodge-type $(1,1)$. There is a system of hyperplanes in the ample cone of S, called v-walls, that is countable but locally finite [HL97, ch. 4C]. An ample class is called *v-generic*, if it does not belong to any v-wall. Choose a v-generic ample class H. Let $\mathcal{M}_H(v)$ be the moduli space of H-stable sheaves on the $K3$ surface S with class v. When non-empty, the moduli space $\mathcal{M}_H(v)$ is a smooth projective irreducible holomorphic symplectic variety of $K3^{[n]}$ type, with $n = ((v,v)+2)/2$. This result is due to several people, including Huybrechts, Mukai, O'Grady, and Yoshioka. It can be found in its final form in [Yos01].

Over $S \times \mathcal{M}_H(v)$ there exists a universal sheaf $\mathcal{F}$, possibly twisted with respect to a non-trivial Brauer class pulled-back from $\mathcal{M}_H(v)$. Associated to $\mathcal{F}$ is a class $[\mathcal{F}]$ in $K(S \times \mathcal{M}_H(v),\mathbb{Z})$ (see [Mar07, Definition 26]). Let π_i be the projection from $S \times \mathcal{M}_H(v)$ onto the ith factor. Assume that $(v,v) > 0$. The second integral cohomology $H^2(\mathcal{M}_H(v),\mathbb{Z})$, its Hodge structure, and its Beauville–Bogomolov pairing [Bea83], are all described by Mukai's Hodge-isometry

$$\theta : v^\perp \longrightarrow H^2(\mathcal{M}_H(v),\mathbb{Z}), \tag{3.2}$$

given by $\theta(x) := c_1(\pi_{2_!}\{\pi_1^!(x^\vee) \otimes [\mathcal{F}]\})$ (see [Yos01]). Above, $\pi_{2_!}$ and $\pi_1^!$ are the Gysin and pull-back homomorphisms in K-theory.

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4. An algebraic cycle

Let $\mathcal{M} := \mathcal{M}_H(v)$ be a moduli space of stable sheaves on the $K3$ surface S as in § 3, so that $\mathcal{M}$ is of $K3^{[n]}$-type, $n \geqslant 2$. Assume that there exists an untwisted universal sheaf $\mathcal{F}$ over $S \times \mathcal{M}$. Denote by π_{ij} the projection from $\mathcal{M} \times S \times \mathcal{M}$ onto the product of the ith and jth factors. Denote by E^i the relative extension sheaf

$$\mathcal{E}xt^i_{\pi_{13}}(\pi_{12}^*\mathcal{F}, \pi_{23}^*\mathcal{F}). \tag{4.1}$$

Let $\Delta \subset \mathcal{M} \times \mathcal{M}$ be the diagonal. Then E^1 is a reflexive coherent $\mathcal{O}_{\mathcal{M}\times\mathcal{M}}$-module of rank $2n-2$, which is locally free away from Δ, by [Mar, Proposition 4.5]. The sheaf E^0 vanishes, while E^2 is isomorphic to $\mathcal{O}_\Delta$. Set $\kappa(E^1) := \mathrm{ch}(E^1)\exp[-c_1(E^1)/(2n-2)]$. Then $\kappa(E^1)$ is independent of the choice of a universal sheaf $\mathcal{F}$. Let $\kappa_i(E^1)$ be the summand in $H^{2i}(\mathcal{M} \times \mathcal{M})$. Then $\kappa_1(E^1) = 0$.

There exists a suitable choice of $\mathcal{M}_H(v)$, one for each n, so that the sheaf E^1 over $\mathcal{M}_H(v) \times \mathcal{M}_H(v)$ can be deformed, as a *twisted* coherent sheaf, to a sheaf $\widetilde{E}^1$ over $X \times X$, for every X of $K3^{[n]}$-type [Mar]. See [Cal00] for the definition of a family of twisted sheaves. We note here only that such a deformation is equivalent to a flat deformation of $\mathcal{E}nd(E^1)$, as a reflexive coherent sheaf, together with a deformation of its associative algebra structure. Verbitsky's theory of hyperholomorphic reflexive sheaves plays a central role in the construction of the deformation, see [Ver99]. The characteristic class $\kappa_i(\widetilde{E}^1)$ is well defined for twisted sheaves [Mar]. Furthermore, $\kappa_i(\widetilde{E}^1)$ is a rational class of weight (i, i), which is algebraic, whenever X is projective. The construction is summarized in the following statement.

Theorem 4.1 [Mar, Theorem 1.6]. *Let X be a smooth projective variety of $K3^{[n]}$-type. Then there exists a smooth and proper family $p : \mathcal{X} \to C$ of irreducible holomorphic symplectic varieties, over a simply connected reduced (possibly reducible) projective curve C, points $t_1, t_2 \in C$, isomorphisms $\mathcal{M} \cong p^{-1}(t_1)$ and $X \cong p^{-1}(t_2)$, with the following property. Let $q : \mathcal{X} \times_C \mathcal{X} \to C$ be the natural morphism. The flat section of the local system $R^*q_*\mathbb{Q}$ through the class $\kappa(E^1)$ in $H^*(\mathcal{M} \times \mathcal{M})$ is algebraic in $H^*(\mathcal{X}_t \times \mathcal{X}_t)$, for every projective fiber $\mathcal{X}_t$, $t \in C$, of p.*

5. A self adjoint algebraic correspondence

In this section, we introduce the algebraic correspondences on moduli spaces of sheaves on $K3$ surfaces that will be used in the proof of the main result. Starting with the sheaf E^i as above, we construct correspondences with suitable properties with respect to Mukai pairings and monodromy operators.

Given a topological space X denote by $K(X)$ its topological K-group with rational coefficients. We keep the notation $\mathcal{M} := \mathcal{M}_H(v)$ of § 4. As earlier, let $(x, y) := -\chi(x^\vee \otimes y)$ be the Mukai pairing on $K(M)$. Given a class α in $H^*(\mathcal{M})$ denote by $\alpha^\vee$ the image of α via the automorphism of $H^*(\mathcal{M})$ acting on $H^{2i}(\mathcal{M})$ via multiplication by $(-1)^i$. The Mukai pairing on $H^*(\mathcal{M})$ is by definition

$$(\alpha, \beta) := -\int_{\mathcal{M}} \alpha^\vee \beta.$$

Define

$$\mu : K(\mathcal{M}) \longrightarrow H^*(\mathcal{M}) \tag{5.1}$$

by $\mu(x) := \mathrm{ch}(x)\sqrt{td_{\mathcal{M}}}$. Then μ is an isometry, by the Hirzebruch–Riemann–Roch theorem.

Remark 5.1. Note that the graded direct summands $H^i(\mathcal{M})$ of the cohomology ring $H^*(\mathcal{M})$ satisfy the usual orthogonality relation with respect to the Mukai pairing: $H^i(\mathcal{M})$ is orthogonal to $H^j(\mathcal{M})$, if $i+j \neq 4n$.

Given two $\mathbb{Q}$-vector spaces V_1 and V_2, each endowed with a non-degenerate symmetric bilinear pairing $(\bullet, \bullet)_{V_i}$, and a homomorphism $h : V_1 \to V_2$, we denote by $h^\dagger$ the adjoint operator, defined by the equation

$$(x, h^\dagger(y))_{V_1} := (h(x), y)_{V_2},$$

for all $x \in V_1$ and $y \in V_2$. We consider $H^*(\mathcal{M})$, $K(\mathcal{M})$, and $K(S)$, all as vector spaces over $\mathbb{Q}$ endowed with the Mukai pairing.

Let $[E^i] \in K(\mathcal{M} \times \mathcal{M})$ be the class of the sheaf E^i, $i = 1, 2$, given in (4.1). Set

$$[E] := [E^2] - [E^1].$$

Let π_i be the projection from $\mathcal{M} \times \mathcal{M}$ onto the ith factor, $i = 1, 2$. The algebraic correspondence $[E]$ induces a morphism

$$\tilde{f}' : K(\mathcal{M}) \to K(\mathcal{M}).$$

It satisfies $\tilde{f}'(x) := \pi_{2_!}(\pi_1^!(x) \otimes [E])$, where $\pi_{2_!}$ and $\pi_1^!$ are the Gysin and pull-back homomorphisms in K-theory. Let p_i be the projection from $S \times \mathcal{M}$ onto the ith factor. Similarly, the algebraic correspondences $[\mathcal{F}]$ and $[\mathcal{F}^\vee]$ induce morphisms

$$\phi' : K(S) \to K(\mathcal{M})$$

with $\phi'(\lambda) := p_{2_!}(p_1^!(\lambda) \otimes [\mathcal{F}])$ and

$$\psi' : K(\mathcal{M}) \to K(S)$$

with $\psi'(x) := p_{1_!}(p_2^!(x) \otimes [\mathcal{F}^\vee])$, where $\mathcal{F}^\vee$ is the dual class.

We have the following identities

$$\psi' = (\phi')^\dagger, \tag{5.2}$$

$$\tilde{f}' = \phi' \circ \psi'. \tag{5.3}$$

Equality (5.2) is a K-theoretic analogue of the following well known fact in algebraic geometry. Let $\Phi : D^b(S) \to D^b(\mathcal{M})$ be the Fourier–Mukai functor with kernel $\mathcal{F}$. Set $\mathcal{F}_R := \mathcal{F}^\vee \otimes p_1^*\omega_S[2]$ and let $\Psi : D^b(\mathcal{M}) \to D^b(S)$ be the Fourier–Mukai functor with kernel $\mathcal{F}_R$. Then Ψ is the right adjoint functor of Φ (see [Muk81] or [Huy06, Proposition 5.9]). The classes of $\mathcal{F}^\vee$ and $\mathcal{F}_R$ in $K(S \times \mathcal{M})$ are equal, since ω_S is trivial. The equality (5.2) is proven using the same argument as its derived-category analogue. Equality (5.3) expresses the fact that the class $[E]$ is the convolution of the classes of $\mathcal{F}^\vee$ and $\mathcal{F}$.

The identities (5.2) and (5.3) imply that $\tilde{f}'$ is *self adjoint*. Set

$$f' := \mu \circ \tilde{f}' \circ \mu^\dagger.$$

Then f' is the self adjoint endomorphism given by the algebraic class

$$(\pi_1^*\sqrt{td_{\mathcal{M}}})\,\mathrm{ch}([E])(\pi_2^*\sqrt{td_{\mathcal{M}}})$$

in $H^*(\mathcal{M} \times \mathcal{M})$.

In order to use monodromy arguments in § 7, we will not work with f' directly, but rather with a normalization of f' which satisfies a monodromy-invariance property. Set $\alpha := \exp((-c_1(\phi'(v^\vee)))/(2n-2))$. Note that α is the Chern character of a $\mathbb{Q}$-line-bundle.

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Let $\tau_\alpha : H^*(\mathcal{M}) \to H^*(\mathcal{M})$ be cup-product with α, i.e., $\tau_\alpha(x) := x \cup \alpha$. Then τ_α is an isometry. Hence, $\tau_\alpha^\dagger = \tau_\alpha^{-1}$. Define

$$\begin{aligned} \phi &:= \tau_\alpha \circ \mu \circ \phi', \\ \psi &:= \psi' \circ \mu^\dagger \circ \tau_\alpha^{-1}, \\ f &:= \phi \circ \psi. \end{aligned}$$

Then f is the self adjoint endomorphism given by the algebraic class

$$\pi_1^* \exp\left(\frac{c_1(\phi'(v^\vee))}{2n-2}\right)(\pi_1^*\sqrt{td_{\mathcal{M}}})\,\mathrm{ch}([E])(\pi_2^*\sqrt{td_{\mathcal{M}}})\pi_2^* \exp\left(\frac{-c_1(\phi'(v^\vee))}{2n-2}\right). \tag{5.4}$$

Finally, let $h_i : H^*(\mathcal{M}) \to H^{2i}(\mathcal{M})$ be the projection, and $e_i : H^{2i}(\mathcal{M}) \to H^*(\mathcal{M})$ the inclusion. Set

$$f_i := h_i \circ f \circ e_{2n-i}.$$

Note that f_i is induced by the Künneth component in $H^{2i}(\mathcal{M}) \otimes H^{2i}(\mathcal{M})$ of the class given in (5.4).

6. Generators for the cohomology ring and the image of f_i

Let $A^{2i} \subset H^{2i}(\mathcal{M})$ be the subspace of classes, which belong to the subring generated by classes of degree $<2i$. Set

$$\overline{H}^{2i}(\mathcal{M}) := H^{2i}(\mathcal{M})/[\mathbb{Q} \cdot c_i(T\mathcal{M}) + A^{2i}].$$

Proposition 6.1. *The composition*

$$H^{4n-2i}(\mathcal{M}) \xrightarrow{f_i} H^{2i}(\mathcal{M}) \to \overline{H}^{2i}(\mathcal{M}) \tag{6.1}$$

is surjective, for $i \geqslant 2$.

The proposition is proven below after Claim 7.2. Let $g_i : H^{4n-2i}(\mathcal{M}) \to H^{2i}(\mathcal{M})$ be the homomorphism induced by the graded summand of degree $4i$ of the cycle

$$-(\pi_1^*\sqrt{td_{\mathcal{M}}})\kappa(E^1)(\pi_2^*\sqrt{td_{\mathcal{M}}}). \tag{6.2}$$

Denote by $\bar{f}_i : H^{4n-2i}(\mathcal{M}) \to \overline{H}^{2i}(\mathcal{M})$ the homomorphism given in (6.1) and define $\bar{g}_i : H^{4n-2i}(\mathcal{M}) \to \overline{H}^{2i}(\mathcal{M})$ similarly in terms of g_i.

Corollary 6.2. *The equality $\bar{g}_i = \bar{f}_i$ holds for $i \geqslant 2$. In particular, $\bar{g}_i$ is surjective, for $i \geqslant 2$.*

Proof. The equality $c_1([E]) = \pi_1^* c_1(\phi'(v^\vee)) - \pi_2^* c_1(\phi'(v^\vee))$ is proven in [Mar, Lemma 4.3]. Hence, the difference between the two classes (5.4) and (6.2) is $\mathrm{ch}(\mathcal{O}_\Delta)\pi_1^* td_{\mathcal{M}}$. Now $\mathrm{ch}_j(\mathcal{O}_\Delta) = 0$, for $0 \leqslant j < 2n$. Hence, $f_i = g_i$, for $0 \leqslant i \leqslant n-1$. The quotient group $\overline{H}^{2i}(\mathcal{M})$ vanishes, for $i > n-1$, by [Mar02, Lemma 10, part 4]. Consequently, $\bar{f}_i = 0 = \bar{g}_i$, for $i \geqslant n$. □

Set $\phi_i := h_i \circ \phi$. We abuse notation and identify h_i with the endomorphism $e_i \circ h_i$ of $H^*(\mathcal{M})$. Similarly, we identify e_{2n-i} with the endomorphism $e_{2n-i} \circ h_{2n-i}$ of $H^*(\mathcal{M})$. With this notation we have

$$h_i^\dagger = e_{2n-i}.$$

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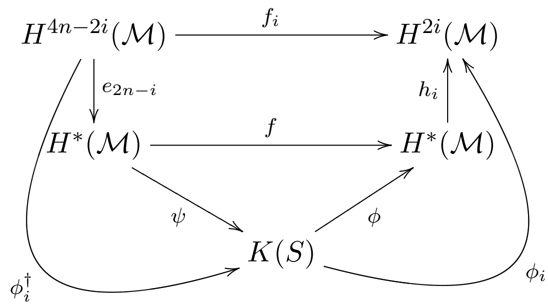
We get the following commutative diagram.



(6.3)

The two main ingredients in the proof of Proposition 6.1 are the following theorem and the monodromy equivariance of diagram (6.3) reviewed in § 7.

Theorem 6.3. *The composite homomorphism*

$$K(S) \xrightarrow{\phi_i} H^{2i}(\mathcal{M}) \to \overline{H}^{2i}(\mathcal{M})$$

is surjective, for all $i \geqslant 1$.

Proof. The subspaces $\mathrm{ch}_i(\phi'(K(S)))$, $i \geqslant 1$, generate the cohomology ring $H^*(\mathcal{M})$, by [Mar02, Corollary 2]. When $\mathcal{M} = S^{[n]}$, this was proven independently in [LQW02]. The same statement holds for the subspaces $\phi_i(K(S)) + \mathrm{span}\{c_i(T\mathcal{M})\}$. Indeed, $\phi_1(K(S)) = \mathrm{ch}_1(\phi'(K(S))) = H^2(\mathcal{M})$, since $\phi'(\lambda^\vee)$ is a class of rank 0, for $\lambda \in v^\perp$, and so $c_1(\phi'(\lambda^\vee)) = \phi_1(\lambda^\vee)$, for $\lambda \in v^\perp$. Now $\mathrm{ch}_1(\phi'([v^\perp]^\vee)) = H^2(\mathcal{M})$, since Mukai's isometry given in (3.2) is surjective. For $i > 1$, the subspaces $\mathrm{ch}_i(\phi'(K(S)))$ and $\phi_i(K(S))$ are equal modulo the subring generated by $H^2(\mathcal{M})$ and the Chern classes of $T\mathcal{M}$. The surjectivity of the composite homomorphism follows. □

Claim 6.4. *If* ϕ_i *is injective, then* $\mathrm{Im}(f_i) = \mathrm{Im}(\phi_i)$.

Proof. The assumption implies that $\phi_i^\dagger$ is surjective. Furthermore, we have $f_i = \phi_i \circ \phi_i^\dagger$. The equality $\mathrm{Im}(f_i) = \mathrm{Im}(\phi_i)$ follows. □

In the next section we will prove an analogue of the above claim, without the assumption that ϕ_i is injective (see Claim 7.2).

7. Monodromy

Recall that the Mukai lattice $\Lambda(S)$ is a rank 24 integral lattice isometric to the orthogonal direct sum $E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}$, where $E_8(-1)$ is the negative definite E_8 lattice and U is the unimodular rank 2 hyperbolic lattice [Muk87]. Recall that $\mathcal{M}$ is the moduli space $\mathcal{M}_H(v)$. Denote by $O^+\Lambda(S)_v$ the subgroup of isometries of the Mukai lattice, which send v to itself and preserve the spinor norm. The *spinor norm* is the character $O\Lambda(S) \to \{\pm 1\}$, which sends reflections by -2 vectors to 1 and reflections by $+2$ vectors to -1. The group $O^+\Lambda(S)_v$ acts on $\Lambda(S)$ and on $K(S) \cong \Lambda(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ via the natural action.

Let $D_S : K(S) \to K(S)$ be given by $D_S(\lambda) = \lambda^\vee$.

Theorem 7.1. (i) *[Mar08, Theorem 1.6] There exists a natural homomorphism*

$$\mathrm{mon} : O^+\Lambda(S)_v \longrightarrow \mathrm{GL}[H^*(\mathcal{M})],$$

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introducing an action of $O^+\Lambda(S)_v$ on $H^*(\mathcal{M})$ via monodromy operators. Denote the image of $g \in O^+\Lambda(S)_v$ by mon_g.

(ii) *[Mar08, Theorem 3.10] The equation*

$$\mathrm{mon}_g(\phi(\lambda^\vee)) = \phi([g(\lambda)]^\vee) \tag{7.1}$$

holds for every $g \in O^+\Lambda(S)_v$, for all $\lambda \in \Lambda(S)$. Consequently, the composite homomorphism

$$K(S) \xrightarrow{D_S} K(S) \xrightarrow{\phi} H^*(M)$$

is $O^+\Lambda(S)_v$ equivariant.[2]

Set $w := D_S(v)$. We have the orthogonal direct sum decomposition

$$K(S) = \mathbb{Q}w \oplus w^\perp_{\mathbb{Q}}$$

into two distinct irreducible representations of $O^+\Lambda(S)_v$, where we consider a new action of $O^+\Lambda(S)_v$ on $K(S)$, i.e., the conjugate by D_S of the old one. So $g \in O^+\Lambda(S)_v$ acts on $K(S)$ via $D_S \circ g \circ D_S$. Let $\pi_w : K(S) \to \mathbb{Q}w$ and $\pi_{w^\perp} : K(S) \to w^\perp_{\mathbb{Q}}$ be the orthogonal projections. Let $\phi_w : \mathbb{Q}w \to H^*(\mathcal{M})$ be the restriction of ϕ to $\mathbb{Q}w$ and $\phi_{w^\perp} : w^\perp_{\mathbb{Q}} \to H^*(\mathcal{M})$ the restriction of ϕ to $w^\perp_{\mathbb{Q}}$. We have

$$\pi_w \circ \psi = (\phi_w)^\dagger \quad \text{and} \quad \pi_{w^\perp} \circ \psi = (\phi_{w^\perp})^\dagger.$$

Set $\phi_{i,w} := h_i \circ \phi_w$ and $\phi_{i,w^\perp} := h_i \circ \phi_{w^\perp}$. Then

$$\begin{aligned} (\phi_{i,w})^\dagger &= \pi_w \circ \psi \circ e_{2n-i}, \\ (\phi_{i,w^\perp})^\dagger &= \pi_{w^\perp} \circ \psi \circ e_{2n-i}. \end{aligned}$$

Claim 7.2. *The homomorphisms f_i and ϕ_i in diagram (6.3) have the same image in $H^{2i}(\mathcal{M})$.*

Proof. Clearly, $\phi_{i,w}$ is injective, if it does not vanish. We observe next that the same is true for $\phi_{i,w^\perp}$. This follows from the fact that $\phi_{i,w^\perp}$ is equivariant with respect to the action of the group $O^+\Lambda(S)_v$ (Theorem 7.1, part (ii)). Now $w^\perp_{\mathbb{Q}}$ is an irreducible representation of $O^+\Lambda(S)_v$. Hence, $\phi_{i,w^\perp}$ is injective, if and only if it does not vanish. We have

$$\begin{aligned} \phi_i &= \phi_{i,w} \circ \pi_w + \phi_{i,w^\perp} \circ \pi_{w^\perp}, \\ (\phi_i)^\dagger &= (\phi_{i,w})^\dagger + (\phi_{i,w^\perp})^\dagger, \\ f_i &= \phi_{i,w} \circ (\phi_{i,w})^\dagger + \phi_{i,w^\perp} \circ (\phi_{i,w^\perp})^\dagger. \end{aligned}$$

Furthermore, the image of $(\phi_{i,w})^\dagger$ is equal to $\mathbb{Q}w$, if $\phi_{w,i}$ does not vanish, and the image of $(\phi_{i,w^\perp})^\dagger$ is equal to $w^\perp_{\mathbb{Q}}$, if $\phi_{i,w^\perp}$ does not vanish. Hence, the image of $\phi_{i,w} \circ (\phi_{i,w})^\dagger$ is equal to the image of $\phi_{i,w}$ and the image of $\phi_{i,w^\perp} \circ (\phi_{i,w^\perp})^\dagger$ is equal to the image of $\phi_{i,w^\perp}$. Thus, the image of f_i is equal to the sum of the images of $\phi_{i,w}$ and $\phi_{i,w^\perp}$. The latter is precisely the image of ϕ_i. □

Proof of Proposition 6.1. Follows immediately from Theorem 6.3 and Claim 7.2. □

[2] The appearance of $\lambda^\vee$ instead of λ as an argument of ϕ in (7.1), as well as in equation (3.2) for Mukai's isometry, is due to the fact that we use the Mukai pairing to identify $\Lambda(S)$ with its dual. So the class of $[\mathcal{F}] \otimes p_2^* \exp((-c_1(\phi'(v^\vee)))/(2n-2))$ in $K(S \times \mathcal{M}) \cong K(S) \otimes K(\mathcal{M})$ is $O^+\Lambda(S)_v$ invariant with respect to the usual action of $O^+\Lambda(S)_v$ on the first factor, and the monodromy action on the second.

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8. Proof of the main theorem

We can now prove the main result of this note. We use the notation of § 2.

Proof of Theorem 1.1. Let X be a smooth projective variety of $K3^{[n]}$-type. According to Theorem 4.1, there exists a smooth and proper family $p : \mathcal{X} \to C$ of irreducible holomorphic symplectic varieties, over a connected reduced projective curve C, points $t_1, t_2 \in C$, isomorphisms $\mathcal{M} \cong \mathcal{X}_{t_1}$ and $X \cong \mathcal{X}_{t_2}$, where $\mathcal{X}_t$ denotes the fiber of p at t.

Additionally, if $q : \mathcal{X} \times_C \mathcal{X} \to C$ is the natural morphism, we can choose the family so that there exists a flat section s of the local system $R^* q_* \mathbb{Q}$ through the class $\kappa([E^1])$ in $H^*(\mathcal{M} \times \mathcal{M})$ which is algebraic in $H^*(\mathcal{X}_{t_2} \times \mathcal{X}_{t_2})$.

The map $t \mapsto td_{\mathcal{X}_t} \in H^*(\mathcal{X}_t)$ induces a global section of the local system $R^* p_* \mathbb{Q}$. It follows that there exists a flat section r of the local system $R^* q_* \mathbb{Q}$ through the class

$$(\pi_1^* \sqrt{td_{\mathcal{M}}})\kappa([E^1])(\pi_2^* \sqrt{td_{\mathcal{M}}}) \in H^*(\mathcal{M} \times \mathcal{M}),$$

where π_1 and π_2 are the two projections $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, such that $r(t_2)$ is algebraic in $H^*(\mathcal{X}_{t_2} \times \mathcal{X}_{t_2})$.

Let us denote by Z_i an algebraic cycle in $H^{2i}(X \times X)$ with cohomology class the degree $2i$ component of $r(t_2)$.

Using the cycles Z_i, we prove by induction on $i \leqslant n$ that X satisfies the Lefschetz standard conjecture in degree $2i$ for every integer i, recall that the cohomology groups of X vanish in odd degrees. This is obvious for $i = 0$.

Let $i \leqslant n$ be a positive integer. Assume that the Lefschetz conjecture holds for X in degree up to $2i - 1$, and let $A^{2i}(X) \subset H^{2i}(X)$ be the subspace of classes, which belong to the subring generated by classes of degree strictly less than $2i$. By Corollary 2.4, we only need to show that the composition

$$H^{4n-2i}(X) \xrightarrow{[Z_i]_*} H^{2i}(X) \to H^{2i}(X)/[A^{2i}(X) + \mathbb{Q} \cdot c_i(TX)] \tag{8.1}$$

is surjective.

The degree $4i$ component of r induces a morphism of local systems

$$R^{2n-4i} p_* \mathbb{Q} \to R^{2i} p_* \mathbb{Q},$$

the fiber of which at t_2 is induced by Z_i. Furthermore, the quotient map

$$H^{2i}(X) \to H^{2i}(X)/[A^{2i}(X) + \mathbb{Q} \cdot c_i(TX)]$$

extends to a surjective map of local systems

$$R^{2i} p_* \mathbb{Q} \to L,$$

where L is the local system such that $L_t = H^{2i}(\mathcal{X}_t)/[A^{2i}(\mathcal{X}_t) + \mathbb{Q} \cdot c_i(T\mathcal{X}_t)]$ for every point t in C.

With this notation, the map (8.1) is the fiber at t_2 of the composition

$$R^{2n-4i} p_* \mathbb{Q} \to R^{2i} p_* \mathbb{Q} \to L. \tag{8.2}$$

Since the curve C is connected, it follows that the map (8.1) is surjective if and only if the fiber at t_1 of (8.2) is surjective. By definition, the fiber at t_1 of the map (8.2) is the map $\bar{g}_i$ of Corollary 6.2, where it is proven that it is surjective. This concludes the proof. □

Remark 8.1. Note that the proof of the main result of this note makes essential use of deformations of hyperkähler varieties along twistor lines, and that a general deformation of a

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hyperkähler variety along a twistor line is never algebraic, see [Huy99, 1.17]. Though the standard conjectures deal with projective varieties, we do not know a purely algebraic proof of the result of this note.

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