

ELEMENTARY OPERATORS ON PRIME C*-ALGEBRAS II†

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1. Introduction. Compact elementary operators acting on the algebra $\mathcal{L}(H)$ of all bounded operators on some Hilbert space H were characterised by Fong and Sourour in [9]. Akemann and Wright investigated compact and weakly compact derivations on C*-algebras [1], and also studied compactness properties of the sum and the product of the right and the left regular representation of an element in a C*-algebra [2]. They used the concept of a *compact* Banach algebra element due to Vala [17]: an element a in a Banach algebra A is called *compact* if the mapping $x \mapsto axa$ is compact on A . This notion has been further investigated by Ylinen [18, 19, 20], who showed in particular that a is a compact element of the C*-algebra A if $x \mapsto axa$ is weakly compact on A [19].

In Section 2 we will discuss some of the properties of the set $K(A)$ of compact elements in a C*-algebra A with emphasis on the case of a *prime* C*-algebra. This information will then be used in Section 3 to characterise compact and weakly compact elementary operators on prime C*-algebras which extends the results in [2] and [9]. As a consequence we will confirm a conjecture by Fong and Sourour that on the Calkin algebra there do not exist non-zero compact elementary operators in a much stronger form.

An *elementary operator* on a C*-algebra A is a mapping of the form $S: x \mapsto \sum_{j=1}^n a_j x b_j$, where $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in M(A)^n$ and $M(A)$ denotes the multiplier algebra of A . It has emerged that the properties of S can be described very neatly if the algebra A is *prime*; i.e., if the product IJ of any two non-zero ideals I and J of A is again non-zero. More generally, an ideal I of A is a *prime ideal* if the quotient A/I is a prime algebra. For example the ideal $\mathcal{K}(H)$ of all compact operators on a Hilbert space H is a prime ideal in $\mathcal{L}(H)$.

The main tool to obtain the above mentioned characterisations, Theorems 3.7 and 3.8, will be the following result which was proved in the first part of the paper ([13, Theorem 4.1]).

THEOREM 1.1. *Let A be a prime C*-algebra and for $a, b \in M(A)^n$ let $S = \sum_{j=1}^n M_{a_j, b_j}$ be the corresponding elementary operator. If the set $\{b_1, \dots, b_n\}$ is linearly independent, then the following conditions are equivalent:*

- (a) $S = 0$,
- (b) $a_j = 0$ for $1 \leq j \leq n$.

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This result gives full information in which way the elements $a_j, b_j \in M(A)$ can be changed without changing the generated elementary operator. In order to make this more explicit, we call the elementary operator $x \mapsto axb$, ($a, b \in M(A)$), a *(two-sided) multiplication on A* and denote it by $M_{a,b}$. The *minimal length* $l(S)$ of a non-zero elementary operator S now is the minimum of the number of multiplications whose sum represents S . In the case $S = 0$ we define $l(S) = 0$. Each representation of S by a sum of $l(S)$ multiplications is called a *minimal representation* of S . As an immediate consequence of Theorem 1.1, a representation $S = \sum_{j=1}^n M_{a_j, b_j}$ is minimal if and only if both $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are linearly independent subsets of $M(A)$, and in any other minimal representation $S = \sum_{k=1}^n M_{c_k, d_k}$ the elements c_k, d_k are linear combinations of a_j, b_j respectively whenever A is a prime C^* -algebra ([13, Proposition 4.6 and Corollary 4.7]).

2. The ideal of compact elements in a prime C^* -algebra. Throughout this paper A will denote a C^* -algebra with spectrum \hat{A} . The *reduced atomic representation* π_a of A is the direct sum $\pi_a = \bigoplus_{t \in \hat{A}} \pi_t$, where (π_t, H_t) is some irreducible representation in $t \in \hat{A}$ (cf. [15, 4.3]). If $\{A_i \mid i \in I\}$ is a family of C^* -algebras, by $\sum_i^\oplus A_i$ we will denote their *restricted direct sum*, i.e.

$$\sum_i^\oplus A_i = \left\{ x = (x_i) \in \sum_i^\oplus A_i \mid \forall \varepsilon > 0: \{i \in I \mid \|x_i\| \geq \varepsilon\} \text{ is finite} \right\}.$$

A C^* -algebra is said to be of *elementary type* if it is isomorphic to $\mathcal{K}(H)$ for some Hilbert space H . The *socle* $\text{soc}(A)$ of a C^* -algebra A is the sum of all minimal left ideals of A , unless A does not contain any minimal left ideal in which case $\text{soc}(A) = 0$. The elements in $\text{soc}(A)$ are called *finite rank elements*; it is easily seen that $a \in \text{soc}(A)$ if and only if $M_{a,a}$ is a finite rank operator on A ([5, $C^*.1.2$]).

The following proposition, entailing in particular that $K(A)$ is a closed ideal, lists several known characterisations of the compact elements of a C^* -algebra A . The equivalences (b) \Leftrightarrow (c) \Leftrightarrow (e) are implicit in [2], and (b) \Leftrightarrow (c) is [19, Theorem 3.1]. Here, they will follow from our results on elementary operators; thus the proof is postponed until Section 3.

PROPOSITION 2.1. *The following conditions on an element a of a C^* -algebra A are equivalent:*

- (a) $a \in K(A)$;
- (b) $M_{a,a}$ is compact;
- (c) $M_{a,a}$ is weakly compact;
- (d) $a \in \text{soc}(A)$;
- (e) $a \in \sum_n^\oplus I_n$, where $(I_n)_{n \in \mathbb{N}}$ is a sequence of orthogonal closed ideals of A of elementary type;
- (f) $\pi_a(a) \in \sum_t^\oplus \mathcal{K}(H_t)$.

In the case where A is equal to $\mathcal{L}(H)$, the ideal $K(A)$ coincides with $\mathcal{K}(H)$ ([16, Theorem 3]), and thus $K(A)$ is the unique minimal closed ideal of A . This remains true in prime C*-algebras.

PROPOSITION 2.2. *If A is a prime C*-algebra then $\text{soc}(A)$ is contained in every non-zero closed ideal of A . Thus, if $K(A) \neq 0$, it is the only minimal closed ideal of A .*

Proof. Let e be a minimal projection in A . Then $P_e = \{x \in A \mid xAe = 0\}$ is the unique primitive ideal in A which does not contain e ([5, BA.3.5]). Since A is prime, $P_e = 0$. Indeed, $AxAeA = 0$ implies that $AxA = 0$. Thus every non-zero primitive ideal P of A contains e , and since every minimal left ideal L is of the form $L = Ae$ for some minimal projection e , it follows that $\text{soc}(A) \subseteq P$. Since each closed ideal of A is the intersection of those primitive ideals which contain it ([15, 3.13.8]), $\text{soc}(A)$ is contained in every non-zero closed ideal. Since $K(A) = \text{soc}(A)$ by 2.1, the second assertion follows. \square

We now investigate the structure of $K(A)$ for a prime C*-algebra A . For the notions of *liminal*, *antiliminal*, and *type I* C*-algebras see [7, Section 4] and [15, Chapter 6]. We recall here that a positive element a in a C*-algebra A is called *abelian* if the range of the multiplication $M_{a,a}$ is a commutative algebra. A C*-algebra which is generated by its abelian elements is said to be *of type I_0* . Every C*-algebra contains a largest ideal of type I_0 , namely the C*-subalgebra I_{abel} generated by all abelian elements ([15, 6.1.7]).

PROPOSITION 2.3. *If A is a prime C*-algebra, then $K(A) = I_{\text{abel}}$. In particular, a prime C*-algebra A is antiliminal if and only if $K(A) = 0$.*

Proof. If A does not contain any non-zero abelian element, then there is no minimal projection in A whence $K(A) = I_{\text{abel}} = 0$. Thus we may assume that A is not antiliminal. Then, by Proposition 2.2, $K(A) \subseteq I_{\text{abel}}$. To prove the reverse inclusion take a non-zero abelian element a in A . Then

$$axaaya = ayaaxa, \quad (\text{for all } x, y \in A).$$

Thus $(a^2xa^2)ya^2 - a^2y(a^2xa^2) = 0$, for all $x, y \in A$. Since A is prime, by Theorem 1.1, for each $x \in A$ there is $\lambda(x) \in \mathbb{C}$ such that $a^2xa^2 = \lambda(x)a^2$. In particular, $M_{a^2,a^2}A \subseteq \mathbb{C}a^2$. Since $a^6 = \lambda(a^2)a^2$ with $\lambda(a^2) > 0$, the element $e := \lambda(a^2)^{-1}a^4$ is a projection in A . Thus, $a = \lambda(a^2)^{1/4}e$ is a scalar multiple of a minimal projection for

$$eAe = a^2Aa^2 \subseteq \mathbb{C}a^2 = \mathbb{C}e.$$

This proves $I_{\text{abel}} \subseteq K(A)$. \square

REMARKS. 1. Since a prime C*-algebra which contains minimal projections is primitive (see below), it follows from Proposition 2.3 that a prime C*-algebra which is not antiliminal is primitive. This, of course, is well known (cf. e.g. [4, Lemma 3.1]); however, our approach via elementary operators reveals that this is only a special case of Martindale’s characterisation of prime rings satisfying a generalised polynomial identity ([12, Theorem 3]).

2. The inclusion $K(A) \subseteq I_{\text{abel}}$ remains true without primeness. In fact, for every C^* -algebra A the ideal $K(A)$ is of type I_0 ([14, 6.14.3]). Therefore, $K(A)$ is liminal and thus contained in the largest liminal ideal I_{lim} of A (cf. [18, p. 32]). A commutative C^* -algebra without minimal projections provides an example of a liminal C^* -algebra with $K(A) = 0$. It is easy to see that, for a prime C^* -algebra A , again $K(A)$ coincides with I_{lim} .

A W^* -algebra is prime if and only if it is a factor. In addition, each closed ideal in a factor is a prime ideal. The following well-known result shows in particular that, in every factor A , the ideal $K(A)$ is also the largest ideal of type I.

PROPOSITION 2.4. *For every Hilbert space H the Calkin algebra $\mathcal{C}(H) = \mathcal{L}(H)/\mathcal{K}(H)$ is an antiliminal prime C^* -algebra.*

For an arbitrary prime C^* -algebra A the situation may be different. Apart from trivial cases, e.g. if A is liminal or antiliminal, the *generalised Calkin algebra* $A/K(A)$ need neither be antiliminal nor be prime. For an example let A be the C^* -algebra generated by $\mathcal{K}(H)$ and two orthogonal infinite dimensional projections on H . Then $A/K(A) \cong \mathbb{C}^2$.

Nevertheless, primeness will play an important role in the next section. Before we turn our attention back to elementary operators, we recall that a prime C^* -algebra A with non-zero socle is already primitive. If p is a minimal projection in A , we endow the minimal left ideal Ap with an inner product $(x | y) = y^*x$ and put $H_p = (Ap, (\cdot | \cdot))$. Then, H_p is a Hilbert space whose inner product norm coincides with the original norm on Ap ([5, p. 79]). By $\pi_p(a)x = ax$ for $a \in A$, $x \in H_p$ we can define an irreducible representation of A on H_p , which is faithful since A is prime. Therefore, A is primitive. In addition, $\pi_p(\text{soc}(A)) = \mathcal{F}(H_p)$, the finite rank operators on H_p , and $\pi(K(A)) = \mathcal{K}(H_p)$ ([5, F.4.3]).

LEMMA 2.5. *For every C^* -algebra A we have $\text{soc}(A) = \text{soc}(M(A))$ and $K(A) = K(M(A))$.*

Proof. If $pAp = \mathbb{C}p$, then $pA^{**}p = \mathbb{C}p$, and thus $pM(A)p = \mathbb{C}p$. Therefore each minimal projection in A is a minimal projection in $M(A)$, whence $\text{soc}(A) \subseteq \text{soc}(M(A))$. If $p \in M(A)$ is a minimal projection, then $\mathbb{C}p = pAp \subseteq A$, which implies that $p \in A$. Since, moreover, $M(A)p = Ap$, we conclude that $\text{soc}(M(A)) \subseteq \text{soc}(A)$. The second assertion now follows by Proposition 2.1 (d). \square

REMARK. For Banach algebras there are also different notions of compact elements, most of which coincide in the C^* -case (cf. [5, R.5]). For W^* -algebras there is another concept of compact elements due to Breuer [6]. An element a of a W^* -algebra A is called *compact relative to A* if it belongs to the closed ideal $m(A)$ generated by all finite projections in A . In general, $m(A)$ is strictly larger than $K(A)$, and therefore less suited for our purposes.

3. Compact and weakly compact elementary operators. We begin our investigations by characterising the weakly compact elementary operators on the algebra $\mathcal{L}(H)$ as

those which vanish on the Calkin algebra. Recall that a bounded linear operator T on a Banach space E is weakly compact if and only if the second adjoint T^{**} maps the second dual E^{**} of E into the natural embedding iE of E into E^{**} ([8, VI.4.2]). Recall also that the second dual $\mathcal{K}(H)^{**}$ of the C*-algebra $\mathcal{K}(H)$ can be identified with $\mathcal{L}(H)$, and that an operator $T \in \mathcal{L}(\mathcal{L}(H))$ which leaves $\mathcal{K}(H)$ invariant is ultraweakly continuous if and only if $(T_{|\mathcal{K}(H)})^{**} = T$. For such T denote by $\tilde{T} \in \mathcal{L}(\mathcal{C}(H))$ the induced operator on the Calkin algebra. The next lemma is a reformulation of [2, Proposition 2.1].

LEMMA 3.1. *Let $T: \mathcal{L}(H) \rightarrow \mathcal{L}(H)$ be an ultraweakly continuous bounded linear operator which leaves $\mathcal{K}(H)$ invariant. Then T is weakly compact if and only if $\tilde{T} = 0$.*

Proof. If $T_1 = T_{|\mathcal{K}(H)}$ denotes the restriction of T to $\mathcal{K}(H)$ then, by Gantmacher's theorem ([8, VI.4.8]), T_1 is weakly compact if and only if $T_1^{**} = T$ is weakly compact. By the remarks above, this is equivalent to $T_1^{**}\mathcal{L}(H) \subseteq \mathcal{K}(H)$; i.e. $\tilde{T} = 0$. \square

Now, we are in a position to apply Theorem 1.1 in order to unify Propositions 2.2, 2.3 and 2.7 of [2] (see also [19, Theorem 3.1] and [20, Theorem 3.1]) and to generalise them to arbitrary elementary operators.

PROPOSITION 3.2. *Let $A = \mathcal{L}(H)$, and for $a, b \in A^n$ let $S = \sum_{j=1}^n M_{a_j, b_j}$ be the corresponding elementary operator. If the set $\{b_1, \dots, b_n\}$ is linearly independent mod $\mathcal{K}(H)$ then the following conditions are equivalent:*

- (a) S is weakly compact;
- (b) $a_j \in \mathcal{K}(H)$ for all $1 \leq j \leq n$.

Proof. By Lemma 3.1, S is weakly compact if and only if the induced operator \tilde{S} vanishes on the Calkin algebra. Denoting by π the canonical homomorphism from $\mathcal{L}(H)$ onto $\mathcal{C}(H)$ we have $\tilde{S} = \sum_{j=1}^n M_{\pi(a_j), \pi(b_j)}$. Since $\mathcal{C}(H)$ is prime, by Proposition 2.4, it follows from Theorem 1.1 that $\tilde{S} = 0$ if and only if $\pi(a_j) = 0$ for all j . \square

In particular, 3.1 and 3.2 show that the primeness of $\mathcal{C}(H)$ is nothing but the assertion that $M_{a,b}$ is weakly compact if and only if a or b is compact. We recall at this place that $M_{a,b} \neq 0$ is compact on $\mathcal{L}(H)$ if and only if both a and b are compact ([16, Theorem 3]).

The following characterisation of weakly compact derivations on $\mathcal{L}(H)$ was first established by Akemann and Wright in [1, Theorem 3.1]. We obtain it here as an immediate consequence.

COROLLARY 3.3. *The following conditions on a derivation δ on $\mathcal{L}(H)$ are equivalent:*

- (a) δ is weakly compact;
- (b) $\delta\mathcal{L}(H) \subseteq \mathcal{K}(H)$;
- (c) $\delta = \delta_a$ for some $a \in \mathcal{K}(H)$.

Proof. (a) \Leftrightarrow (b) by 3.1 and (a) \Leftrightarrow (c) by 3.2. \square

Our next aim is to extend Proposition 3.2 to arbitrary prime C^* -algebras. If π is a representation of a C^* -algebra A , we denote by $\tilde{\pi}$ the extended ultraweakly continuous representation of its enveloping W^* -algebra A^{**} . Then $\tilde{\pi}$ maps the closed unit ball A_1^{**} of A^{**} onto the closed unit ball $\pi(A)_1''$ of the ultraweak closure $\pi(A)''$. If we consider A and $M(A)$ canonically embedded in A^{**} and $S = \sum_{j=1}^n M_{a_j, b_j}$ with $a_j, b_j \in M(A)$, then $S_{\tilde{\pi}} = \sum_{j=1}^n M_{\tilde{\pi}(a_j), \tilde{\pi}(b_j)}$ is the induced elementary operator on $\pi(A)''$.

LEMMA 3.4. *Let A be a C^* -algebra and let π be a representation of A . If S is a (weakly) compact elementary operator, then so is $S_{\tilde{\pi}}$ and $S_{\tilde{\pi}}\pi(A)'' \subseteq \pi(A)$.*

Proof. If S is (weakly) compact, then so is its second adjoint S^{**} ([8], VI.4.8 and VI.5.2, respectively). From

$$S_{\tilde{\pi}}\pi(A)_1'' = S_{\tilde{\pi}}\tilde{\pi}(A_1^{**}) = \tilde{\pi}S^{**}A_1^{**} \subseteq \tilde{\pi}(A) = \pi(A)$$

we obtain $S_{\tilde{\pi}}\pi(A)'' \subseteq \pi(A)$ as well as the (weak) compactness of $S_{\tilde{\pi}}$. \square

LEMMA 3.5. *If S is a weakly compact elementary operator on a C^* -algebra A , then $SA \subseteq K(A)$.*

Proof. Let π_a denote the reduced atomic representation of A . Then $\pi_a(A)'' = \sum_t^\oplus \mathcal{L}(H_t)$ ([15, 4.3.8]). If S is weakly compact on A , by Lemma 3.4 its extension $S_{\tilde{\pi}}$ is weakly compact on $\pi_a(A)''$. Denote by S_t the induced mapping on $\pi_t(A)'' = \mathcal{L}(H_t)$ for each $t \in \hat{A}$. Since S_t is weakly compact it follows by Lemma 3.1 that $S_t\mathcal{L}(H_t) \subseteq \mathcal{K}(H_t)$. By an argument similar to [1, Lemma 3.2] (see also [2, Lemma 2.4]) there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in \hat{A} such that $S_t = 0$ if $t \notin \{t_n \mid n \in \mathbb{N}\}$ and $\lim_{n \rightarrow \infty} \|S_{t_n}\| = 0$. Therefore, if $y = S_{\tilde{\pi}_a}x$ for some $x = (x_t) \in \pi_a(A)''$, then $y_t = 0$ for all $t \notin \{t_n \mid n \in \mathbb{N}\}$ and $\lim_{n \rightarrow \infty} \|y_{t_n}\| = 0$. From this and Lemma 3.4 we obtain

$$S_{\tilde{\pi}_a} \sum_t^\oplus \mathcal{L}(H_t) \subseteq \sum_t^\oplus \mathcal{K}(H_t) \cap \pi_a(A). \tag{1}$$

Our last claim is that

$$\sum_t^\oplus \mathcal{K}(H_t) \cap \pi_a(A) = \pi_a(K(A)). \tag{2}$$

This together with (1) will yield $S_{\tilde{\pi}_a}\pi_a(A)'' \subseteq \pi_a(K(A))$ from which $SA \subseteq K(A)$ follows by the faithfulness of π_a ([15, 4.3.11]).

Since $M_{a,a}$ is compact for every $a \in K(A)$, by (1),

$$M_{\pi_a(a), \pi_a(a)}\pi_a(A)'' \subseteq \sum_t^\oplus \mathcal{K}(H_t).$$

In particular, $\pi_a(a)\pi_a(a)^*\pi_a(a)$ belongs to $\sum_t^\oplus \mathcal{K}(H_t)$; i.e. $\lim_{n \rightarrow \infty} \|\pi_{t_n}(a)\|^3 = 0$ for some

sequence $(t(n))_{n \in \mathbb{N}}$ in \hat{A} and $\|\pi_t(a)\|^3 = 0$ for all $t \notin \{t(n) \mid n \in \mathbb{N}\}$. Thus $\pi_a(a) \in \sum_0^\oplus \mathcal{K}(H_t)$. Conversely, take $a \in A$ with $\pi_a(a) \in \sum_0^\oplus \mathcal{K}(H_t)$, i.e. $\pi_t(a) = 0$ for all but a countable number of indices $t(n) \in \hat{A}$, $\lim_{n \rightarrow \infty} \|\pi_{t(n)}(a)\| = 0$ and $\pi_{t(n)}(a) \in \mathcal{K}(H_{t(n)})$ for all $n \in \mathbb{N}$. Since the multiplications $M_{\pi_{t(n)}(a), \pi_{t(n)}(a)}$ are compact on $\mathcal{L}(H_{t(n)})$, the formula

$$S_m = \bigoplus_{n=1}^m M_{\pi_{t(n)}(a), \pi_{t(n)}(a)}$$

defines a sequence of compact mappings which converges uniformly to $M_{\pi_a(a), \pi_a(a)}$, as we have $\lim_{n \rightarrow \infty} \|M_{\pi_{t(n)}(a), \pi_{t(n)}(a)}\| = 0$. Thus $M_{a,a}$ is compact on A . The proof is complete. \square

REMARK. The conclusion in Lemma 3.5 cannot be reversed as the example $A = \mathcal{K}(H)$ and $S = M_{p,p}$, with p an infinite dimensional projection on H , shows.

In [9], Fong and Sourour conjectured that on the Calkin algebra $\mathcal{C}(H)$ on a separable Hilbert space H there cannot exist any non-zero compact elementary operator. This conjecture was confirmed by Apostol and Fialkow [3], and independently by Magajna [11]. Propositions 2.3 and 2.4 show that this is only a special case of the following consequence of Lemma 3.5.

COROLLARY 3.6. *There are no non-zero weakly compact elementary operators on a C*-algebra without non-zero compact elements.*

Before we go on to establish our main results, we provide the proof of Proposition 2.1 that was promised in Section 2.

Proof of 2.1. (a) \Leftrightarrow (b) by definition, and (b) \Rightarrow (c) is obvious.

(c) \Rightarrow (a) Since $M_{a,a}$ weakly compact implies that $aAa \subseteq K(A)$, by Lemma 3.5, it follows that $a \in K(A)$.

(b) \Leftrightarrow (d) by [5, C*.1.3].

(a) \Leftrightarrow (f) by identity (2) in the proof of Lemma 3.5.

(f) \Rightarrow (e) If $a \in A$ is such that $\pi_a(a) \in \sum_0^\oplus \mathcal{K}(H_t)$, then there is a sequence $(t_n)_{n \in \mathbb{N}}$ in \hat{A} with $\lim_{n \rightarrow \infty} \|\pi_{t_n}(a)\| = 0$ and $\pi_{t_n}(a) = 0$ if $t \notin \{t_n \mid n \in \mathbb{N}\}$. If $a \neq 0$ we may assume that $\pi_{t_n}(a) \neq 0$, and thus $\mathcal{K}(H_{t_n}) \cap \pi_{t_n}(A) \neq 0$. By Kadison's transitivity theorem, it follows that $\mathcal{K}(H_{t_n}) \subseteq \pi_{t_n}(A)$ (cf. [15, 6.1.4]). Hence $a \in \sum_0^\oplus I_n$, where $I_n \cong \mathcal{K}(H_{t_n})$ via π_a for each $n \in \mathbb{N}$. An argument analogous to the last part in the proof of Lemma 3.5 yields (e) \Rightarrow (b). \square

Using the results obtained so far, it is now easy to characterise the weakly compact elementary operators on a prime C*-algebra.

THEOREM 3.7. *Let A be a prime C^* -algebra and for $a, b \in M(A)^n$ let $S = \sum_{j=1}^n M_{a_j, b_j}$ be the corresponding elementary operator. If the set $\{b_1, \dots, b_n\}$ is linearly independent mod $K(A)$ then the following conditions are equivalent:*

- (a) S is weakly compact;
- (b) $a_j \in K(A)$ for all $1 \leq j \leq n$.

Proof. (b) \Rightarrow (a) This is true in every C^* -algebra and without the additional assumption on $\{b_1, \dots, b_n\}$. If $a_j \in K(A)$ for all j , then the left multiplication L_{a_j} is weakly compact for all j ([20, Theorem 3.1]). Since the weakly compact operators form an

ideal ([8, VI.4.6]), the operator $S = \sum_{j=1}^n L_{a_j} R_{b_j}$ is weakly compact.

(a) \Rightarrow (b) First assume that $K(A) = 0$. By Lemma 3.5, S is weakly compact if and only if $S = 0$. Thus the assertion follows by Theorem 1.1. Now assume $K(A) \neq 0$. If p is a minimal projection in A , then the corresponding irreducible representation (π_p, H_p) of A (see Section 2) extends to an irreducible representation of $M(A)$, denoted by the same symbol. Since $M(A)$ is prime ([13, Lemma 2.2]), π_p is faithful on $M(A)$. By hypothesis, the set $\{\pi_p(b_1), \dots, \pi_p(b_n)\}$ is linearly independent mod $\mathcal{K}(H_p)$ (which equals $\pi_p(K(A))$ by Lemma 2.5). Since $S_{\pi_p} = \sum_{j=1}^n M_{\pi_p(a_j), \pi_p(b_j)}$ is weakly compact on $\mathcal{L}(H_p)$ by Lemma 3.4, it follows that $\pi_p(a_j) \in \mathcal{K}(H_p)$ by Proposition 3.2. Thus $a_j \in K(A)$ for all $1 \leq j \leq n$. \square

The following description of compact elementary operators was proved in the case $A = \mathcal{L}(H)$ by Fong and Sourour in [9, Theorem 2].

THEOREM 3.8. *Let A be a prime C^* -algebra and for $a, b \in M(A)^n$ let $S = \sum_{j=1}^n M_{a_j, b_j}$ be the corresponding elementary operator. If S is compact and $\{b_1, \dots, b_n\}$ is linearly independent, then $a_j \in K(A)$ for all $1 \leq j \leq n$.*

Proof. If $K(A) = 0$, then S is compact if and only if $S = 0$ by Lemma 3.5. Hence, the assertion is a consequence of Theorem 1.1.

If $K(A) \neq 0$, then $K(A) = \pi^{-1}(\mathcal{K}(H))$ for some faithful irreducible representation (π, H) of A . Since S is compact if and only if the induced elementary operator S_{π} on $\mathcal{L}(H)$ is compact (Lemma 3.4), the assertion follows from the corresponding result on $\mathcal{L}(H)$ ([9, Theorem 2]). \square

COROLLARY 3.9. *A non-zero elementary operator S on a prime C^* -algebra A is compact if and only if there are linearly independent subsets $\{a_1, \dots, a_n\}$, $\{b_1, \dots, b_n\}$ in $K(A)$ such that $S = \sum_{j=1}^n M_{a_j, b_j}$.*

Proof. Let $n = l(S) \geq 1$ be the minimal length of S . If $S = \sum_{j=1}^n M_{a_j, b_j}$ is a minimal representation, both subsets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ of $M(A)$ are linearly

independent (see Section 1). If S is compact, so is $\bar{S} = \sum_{j=1}^n M_{b_j^*, a_j^*}$ since $\bar{S}x = (Sx^*)^*$. Thus, by Theorem 3.8, $\{a_1, \dots, a_n\} \subseteq K(A)$ and $\{b_1, \dots, b_n\} \subseteq K(A)$.

The converse conclusion follows from Vala's theorem ([16, Theorem 3]) stating that $M_{a,b}$ is compact whenever a and b are compact operators on H applied in an irreducible representation (π, H) of A . \square

As another consequence of Theorem 3.8, there are no non-zero compact derivations on an infinite dimensional prime C*-algebra. This result appeared (with an incorrect proof) for the case $A = \mathcal{L}(H)$ in [10]. Similarly, Corollary 3.3 can be extended to prime C*-algebras.

COROLLARY 3.10. *Every compact elementary operator on a prime C*-algebra is the norm-limit of elementary operators of finite rank.*

Proof. Let S be a compact elementary operator on the prime C*-algebra A . Without loss of generality we may assume that $l(S) = 1$. By Corollary 3.9, we thus have $S = M_{a,b}$ with a, b in $K(A) \setminus \{0\}$. Since

$$M_{a,b} = \frac{1}{4} \sum_{k=0}^3 i^k M_{(b+i^k a)^*, (b+i^k a)^*},$$

we may further assume that $a = b^*$. Since $K(A) = \overline{\text{soc}(A)}$, there is a sequence $(b_n)_{n \in \mathbb{N}}$ in $\text{soc}(A)$ with $b = \lim_{n \rightarrow \infty} b_n$. Similarly to [5, C*.1.2] one can show that each multiplication $M_{b_n^*, b_n}$ is of finite rank. Therefore, $M_{b^*, b} = \lim_{n \rightarrow \infty} M_{b_n^*, b_n}$ is approximated by finite rank operators. \square

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