RADIAL AND NON-RADIAL OSCILLATIONS OF SPHERICALLY SYMMETRIC STELLAR SYSTEMS

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ABSTRACT. In an expansion scheme in velocity space, the first order perturbations of a stellar system bear close resemblance to those of a fluid. This feature is exploited to study the structure of the Hilbert space of the linear perturbations of a stellar system, to provide a classification for the modes, and to construct ansatz for variational calculations. The first order non-radial modes appear to be trispectral in that they are predominantly derived from a scalar potential, a toroidal vector potential, or a poloidal vector potential. The eigenfrequencies and the eigenfunctions of radial ($\ell = 0$) and non-radial ($\ell = 1$) modes of polytropes and of truncated isothermal distributions are calculated. The density waves associated with these modes are also reported.

1. ANTONOV'S PROBLEM AND A TECHNIQUE OF ANALYSIS

Let $F(E)$ be a phase space equilibrium distribution for a spherically symmetric stellar system, where $E = 1/2\, v^2 + U(\underline{\underline{x}})$ is the energy integral and $U(\underline{\underline{x}})$ is the mean self-gravitational field of the system. Let $\varphi(\underline{\underline{x}},\underline{\underline{v}},t)$ be a perturbation on F and $\delta U(\underline{\underline{x}},t)$ be the gravitational field associated with it. The perturbation satisfies the linearized Liouville equation,

$$\frac{\partial\varphi}{\partial t} + D\varphi + \frac{dF}{dE} D\delta U = 0; \quad D = v_i \frac{\partial}{\partial x_i} - \frac{\partial U}{\partial x_i}\frac{\partial}{\partial v_i} \tag{1}$$

Antonov (1962) decomposed φ into symmetric and antisymmetric parts in the velocity space, eliminated the symmetric component from Eq.(1) and arrived at his celebrated eigenvalue equation to which one often resorts to study the stability of star clusters. Let $\varphi_{\pm}(\underline{\underline{x}},\underline{\underline{v}},t) = \pm\varphi_{\pm}(\underline{\underline{x}},\pm\underline{\underline{v}},t)$ be these components. They satisfy the following equations:

$$\frac{\partial\varphi_+}{\partial t} + D\varphi_- = 0 \quad , \tag{2a}$$

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$$\frac{\partial^2\varphi_-}{\partial t^2} - D^2\varphi_- - G\,\frac{dF}{dE}\,D\!\int D'\varphi'_-|\underline{x}-\underline{x}'|^{-1}\,d\tau' = 0\ , \tag{2b}$$

where a prime on a function or an operator indicates that the object in question is to be evaluated at $\underline{x}',\underline{v}'$ and $d\tau' = d\underline{x}'d\underline{v}'$. We shall consider only those cases for which dF/dE is either positive or negative for all E. Let $\varphi_- = |dF/dE|^{1/2}\,f(\underline{x},\underline{v})\exp(i\omega t)$. Introduce this expression in Eq.(2b), divide the result by $|dF/dE|^{1/2}$, multiply by $f^*\exp(-i\omega t)$ from left, and integrate over the permissible volume of the phase space. After some integrations by parts one arrives at the following variational expressions for ω^2:

$$\omega^2 = [W_1 + \mathrm{sign}(dF/dE)W_2]/S\ , \tag{3}$$

where

$$S = \int f^*f\,d\tau > 0\ , \tag{3a}$$

$$W1 = \int Df^*Df\,d\tau \geqslant 0\ , \tag{3b}$$

$$W2 = G\iint D\varphi^*_-D'\varphi'_-|\underline{x}-\underline{x}'|^{-1}\,d\tau d\tau' \gtrless\ . \tag{3c}$$

Because of the symmetry of S, $W1$ and $W2$ under the interchange of f and f^* or φ and φ^*, ω^2 is real. The W2-integral arises from perturbations in the gravitational potential and can be simplified considerably. The density perturbation is given by

$$\delta\rho(\underline{x}) = \int\varphi_+\,d\underline{v} = -\frac{1}{i\omega}\int D\varphi_-\,d\underline{v}\ . \tag{4a}$$

The gravitational potential associated with it is

$$\delta U(\underline{x}) = -G\int\delta\rho(\underline{x}')|\underline{x}-\underline{x}'|^{-1}\,d\underline{x}'\ . \tag{4b}$$

The two expressions are of course related by Poisson's equation, $\nabla^2\delta U = 4\pi G\delta\rho$. Substituting Eqs.(4) in Eq.(3c) and using Poisson's equation to carry out an integration by part yields

$$W2 = \frac{|\omega^2|}{4\pi G}\int\underline{\nabla}(\delta U^*)\cdot\underline{\nabla}(\delta U)\,d\underline{x} \geqslant 0\ . \tag{5}$$

From Eqs.(3) and (5) it is evident that S is positive definite, and $W1$ and $W2$ are positive. They can be zero iff $Df = 0$. It, immediately, follows that the system is stable, that is, $\omega^2 > 0$ if $dF/dE \geqslant 0$. This, for example is the case for the polytropic distributions, $F(E) = (-E)^{n-3/2}$, $E \leqslant 0$, for which $1/2 < n \leqslant 3/2$. Distributions with positive energy gradient are bound to vanish discontinuously at some energy; $E = 0$, say. The subtlety associated with this discontinuity is analyzed in Sobouti (1984). Polytropes with $n > 3/2$, isothermal and truncated isothermal distributions have negative energy gradients. Their stability or instability cannot be inferred in any simple manner from the above equations. It has, however, been shown that such systems are stable against perturbations leading to radial displacements in the configuration space (Dorémus et al., 1970, 1971). To the best of the author's knowledge, however, their stability against general phase-space perturbations is as yet unknown.

As to the solution of Eqs.(3), let us observe that $\underline{f}(\underline{\underline{x}},\underline{\underline{v}})$ is antisymmetric in $\underline{\underline{v}}$. A Fourier sine-transformation of $\underline{f}$ in $\underline{\underline{v}}$-space, followed by a series expansion of $\underline{\sin}(\underline{\underline{k}}.\underline{\underline{v}})$ appearing in the transformation gives a series expansion of $\underline{f}$ in $\underline{\underline{v}}$:

$$f(\underline{\underline{x}},\underline{\underline{v}}) = \xi_i(\underline{\underline{x}})v_i - \frac{1}{6}\,\xi_{ijk}(\underline{\underline{x}})v_iv_jv_k + \ldots \,, \tag{6}$$

where $\xi_i(\underline{\underline{x}})$, $\xi_{ijk}(\underline{\underline{x}})$, etc. are a vector field, a third rank tensor field, etc. The limit on $\underline{\underline{v}}$ at any space point $\underline{\underline{x}}$ is the escape velocity from that point, namely

$$v \leqslant v_e = [-2U(\underline{\underline{x}})]^{1/2} \,. \tag{7}$$

Substituting Eq.(6) in Eqs.(3) and integrating over the velocities gives a variational expression for ω^2 involving integrals in $\underline{x}$-space only. This reduction from the six-dimensional phase space to the three-dimensional configuration space clarifies conceptual ambiguities and makes the computations tractable. In a first order approximation let us keep only the first term in Eq.(6). One obtains (Sobouti, 1984):

$$S = \int\phi^2\ \underline{\xi}^*\!\cdot\underline{\xi}\ d\underline{x} \quad , \tag{8a}$$

$$W1 = \int\phi 4\left[\partial_j\xi_i^*\partial_j\xi_i + 2\partial_j\xi_i^*\partial_i\xi_j\right]d\underline{\underline{x}} + \int\phi 2\partial_{ij}^2U\xi_i^*\xi_j d\underline{\underline{x}} \tag{8b}$$

$$W2 = G\int\partial_i[\Psi\xi_i^*]\partial_j'[\psi\xi_j]'\,|\underline{\underline{x}}-\underline{\underline{x}}'|\,d\underline{\underline{x}}d\underline{\underline{x}}' \,, \tag{8c}$$

where

$$\phi 2 = \frac{4\pi}{15}\ (-2U)^{5/2} \quad , \tag{9a}$$

$$\phi(4) = \frac{4\pi}{107}\,(-2U)^{7/2} \quad , \tag{9b}$$

$$\psi = \frac{4\pi}{3}\int_0^{\sqrt{-2U}} |dF/dE|^{1/2}\ v^4 dv. \tag{9c}$$

Eqs.(8) are similar to those governing the linear oscillations of a star, in that in both cases one deals with the standing modes of a vector field $\underline{\xi}(\underline{\underline{x}})$. In a fluid $\underline{\xi}(\underline{\underline{x}})$ represents the Lagrangian displacement of a fluid element. In a stellar system it is proportional to the Lagrangian displacement of a volume element (Sobouti, 1986). By a modified version of Helmholtz's theorem (Sobouti, 1981), the vector field $\phi 2\ \underline{\xi}$ can be expressed in terms of one scalar potential and two vector potentials:

$$\phi 2\underline{\underline{\xi}} = -\phi 2\underline{\underline{\nabla}}\chi_s + \underline{\underline{\nabla}}\times\underline{\underline{A}}_1 + \underline{\underline{\nabla}}\times\underline{\underline{A}}_2,\quad \underline{\underline{\nabla}}\cdot\underline{\underline{A}}_{1,2} = 0 \quad . \tag{10}$$

Because of the solenoidal character of the vector potentials there are only two independent ones. We shall choose $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ in such a way that $\underline{\underline{\nabla}} \times \underline{\underline{A}}_1$ and $\underline{\underline{\nabla}} \times \underline{\underline{A}}_2$ are poloidal and toroidal fields, respectively. Thus we may write

$$\underline{\underline{\xi}} = \underline{\underline{\xi}}_s + \underline{\underline{\xi}}_p + \underline{\underline{\xi}}_t \tag{11}$$

where

$$\underline{\underline{\xi}}_s = -\underline{\underline{\nabla}}\chi_s \ , \tag{11a}$$

$$\underline{\underline{\xi}}_p = \underline{\underline{\nabla}} \times \underline{\underline{\nabla}} \times (\hat{r}\ \chi_p)/\phi 2 \tag{11b}$$

$$\underline{\underline{\xi}}_t = \underline{\underline{\nabla}} \times (\hat{r}\chi_t)/\phi 2 \quad , \tag{11c}$$

and χ_s, χ_p and χ_t are scalar fields. Any two components of Eq.(11) are mutually orthogonal in the sense that $\int \underline{\phi 2}\ \underline{\underline{\xi}}^*_\alpha \cdot \underline{\underline{\xi}}_\beta\ \underline{d}\ \underline{x}$, $\alpha \neq \beta = \underline{s,p,t}$. Following Elsasser (1946), the three components will be termed as scaloidal, poloidal and toroidal fields. For a spherically symmetric systems the scalars χ_s, χ_p and χ_t can be expanded in spherical harmonics and variational ansatz be chosen as linear combinations of various terms. Numerical calculations show that the standing modes fall into three categories: i) modes in which the scaloidal component is predominant. These are analogous to the p-modes of a fluid system; ii) Modes in which the poloidal term is dominant. These are the counterparts of the g-modes of a fluid; (iii) Modes which are essentially made up of toroidal fields and have very small but non-zero eigenfrequencies. These are similar to the neutral toroidal displacements of a fluid. The radial modes corresponding to spherically symmetric perturbations are of course of scaloidal type and share properties with the radial modes of a fluid star. Numerical calculations for radial modes of polytropes and truncated isothermal distributions can be found in Sobouti (1984, 1985). The non-radial modes are analyzed and computed in Sobouti (1986).

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