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Composition operators

R.K. Singh and B.S. Komal

A study of centered composition operators on l^2 is made in this paper. Also the spectrum of surjective composition operators is computed. A necessary and sufficient condition is obtained for the closed unit disc to be the spectrum of a surjective composition operator.

1. Preliminaries

Let $L^2(\lambda)$ be the Hilbert space of all square integrable functions on a σ-finite measure space (X, S, λ) and let ϕ be a non-singular measurable transformation from X into itself. Then the equation $C_\phi f = f \circ \phi$ for every $f \in L^2(\lambda)$ defines a linear transformation. If C_ϕ happens to be a bounded operator on $L^2(\lambda)$, then we call it a composition operator. If $X = N$, the set of all non-zero positive integers and λ is the counting measure on the family of all subsets of N, then $L^2(\lambda) = l^2$, the Hilbert space of all square summable sequences.

In this note we have studied composition operators on l^2. The second section characterises centered composition operators while the third section is devoted to the study of the spectrum of a surjective composition operator. If H is a Hilbert space, then $B(H)$ denotes the Banach algebra of all bounded linear operators on H.

2. Centered composition operators on l^2

Let H be a complex Hilbert space, $T \in B(H)$, and let

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$s_T = \{(T^*)^k T^k : k \in N\} \cup \{T^k(T^*)^k : k \in N\}$. Then T is defined to be a centered operator if any two elements of s_T commute. These operators have been studied by Morrel and Muhly [4] in detail. We give a necessary and sufficient condition for a composition operator to be centered.

THEOREM 2.1. *Let ϕ be a mapping from N into itself such that $C_\phi \in B(l^2)$. Then C_ϕ is centered if and only if f_0^k is constant on $(\phi^p)^{-1}(\{n\})$ for every $n \in N$ and $p \in N$, where f_0^k is the Radon-Nikodym derivative of the measure $\lambda(\phi^k)^{-1}$ with respect to the measure λ .*

For the proof of the theorem we need the following lemma.

LEMMA 2.2. *If ϕ is a measurable transformation from a measure space (X, S, λ) into itself such that $C_\phi \in B(L^2(\lambda))$, then*

$$(C_\phi^*)^k C_\phi^k = M_{f_0^k} \quad \text{for every } k \in N ,$$

where $M_{f_0^k}$ is the multiplication operator induced by f_0^k .

Proof. Since $C_\phi \in B(L^2(\lambda))$, it is easy to show that $C_{\phi^k} = C_\phi^k \in B(L^2(\lambda))$, where ϕ^k is obtained by composing ϕ k-times. If f and g are any two elements in $L^2(\lambda)$ and $k \in N$, then

$$\begin{aligned}\left\langle (C_\phi^*)^k C_\phi^k f, g\right\rangle &= \left\langle C_\phi^k f, C_\phi^k g\right\rangle \\ &= \langle C_{\phi^k} f, C_{\phi^k} g\rangle \\ &= \int_X f \circ \phi^k \cdot \overline{g \circ \phi^k} d\lambda \\ &= \int_X (f \cdot \bar{g}) \circ \phi^k d\lambda \\ &= \int_X f \cdot \bar{g} d\lambda(\phi^k)^{-1}\end{aligned}$$

$$= \int_X f \cdot \bar{g} \cdot f_0^k d\lambda$$

$$= \langle M_{f_0^k} f, g \rangle .$$

This shows that $(C_\phi^*)^k C_\phi^k = M_{f_0^k}$. Hence the proof of the lemma is complete.

Proof of theorem. Suppose that the condition of the theorem holds. Let $A, B \in s_{C_\phi}$. Then $A = C_\phi^{*k} C_\phi^k$ or $A = C_\phi^l C_\phi^{*l}$ and $B = C_\phi^{*p} C_\phi^p$ or $B = C_\phi^m C_\phi^{*m}$ for some k, l, p, and m in N . If $A = C_\phi^{*k} C_\phi^k$ and $B = C_\phi^{*p} C_\phi^p$, then from the above lemma it follows that $AB = M_{f_0^k} M_{f_0^p} = BA$.

If $A = C_\phi^{*k} C_\phi^k$ and $B = C_\phi^m C_\phi^{*m}$ and if $e^{(n)}$ is the nth basis vector defined by $e^{(n)}(q) = \delta_{nq}$ (the Kronecker delta), then

$$(AB)e^{(n)} = C_\phi^{*k} C_\phi^k C_\phi^m C_\phi^{*m} e^{(n)}$$

$$= M_{f_0^k} C_\phi^m e^{(\phi^m(n))} \quad \left(\text{by definition of } C_\phi^* \ [8]\right)$$

$$= f_0^k(n) \chi_{\{(\phi^m)^{-1}(\{\phi^m(n)\})\}} ,$$

where χ_E denotes the characteristic function of the set E . A similar computation shows that $BAe^{(n)} = f_0^k(n)\chi_{\{(\phi^m)^{-1}(\{\phi^m(n)\})\}}$. Thus $AB = BA$. Suppose now that $A = C_\phi^l C_\phi^{*l}$ and $B = C_\phi^m C_\phi^{*m}$ and without loss of generality assume $m \le l$. Then

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$$
\begin{aligned}
ABe^{(n)} &= C_\phi^l C_\phi^{*l} C_\phi^m C_\phi^{*m} e^{(n)} \\
&= C_\phi^l C_\phi^{*l-m} M_{f_0^m} X_{\{\phi^m(n)\}} \\
&= f_0^m(\phi^m(n)) C_\phi^l C_\phi^{*l-m} X_{\{\phi^m(n)\}} \\
&= f_0^m(\phi^m(n)) X_{\{(\phi^l)^{-1}(\{\phi^l(n)\})\}} .
\end{aligned}
$$

Also

$$
\begin{aligned}
BAe^{(n)} &= C_\phi^m C_\phi^{*m} C_\phi^l C_\phi^{*l} e^{(n)} \\
&= C_\phi^m M_{f_0^m} C_\phi^{l-m} C_\phi^{*l} e^{(n)} \\
&= C_\phi^m M_{f_0^m} X_{\{(\phi^{l-m})^{-1}(\{\phi^l(n)\})\}} \\
&= f_0^m(\phi^m(n)) X_{\{(\phi^l)^{-1}(\{\phi^l(n)\})\}} .
\end{aligned}
$$

This shows that $AB = BA$.

On the other hand, suppose the condition of the theorem is not true. Then there exist $n_1, n_2 \in (\phi^p)^{-1}(\{n\})$ such that $f_0^k(n_1) \neq f_0^k(n_2)$ for some $p, k, n \in N$. If $A = C_\phi^{*k} C_\phi^k$ and $B = C_\phi^p C_\phi^{*p}$, then $ABe^{(n_1)} = f_0^k \cdot X_{\{(\phi^p)^{-1}(\{\phi^p(n_1)\})\}}$ and $BAe^{(n_1)} = f_0^k(n_1) X_{\{(\phi^p)^{-1}(\{n\})\}}$. Since $f_0^k(n_1) \neq f_0^k(n_2)$, we can conclude that $AB \neq BA$. Hence C_ϕ is not centered. This completes the proof of the theorem.

3. Spectrum of a composition operator on l^2

This section is devoted to the study of the spectrum of a composition operator on l^2 . The set of all complex numbers will be denoted by C and the set D defined by $D = \{\lambda : \lambda \in C \text{ and } |\lambda| \leq 1\}$ is called the closed unit disc. The symbol $\sigma(T)$ stands for the spectrum of T . The

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unit circle will be denoted by c .

THEOREM 3.1. *If $\phi : N \to N$ is an injection which is not a surjection, then $\sigma(C_\phi) = D$.*

Proof. Since ϕ is not a surjection, there is an $n_1 \in N$ such that $\lambda\left\{\phi^{-1}(\{n_1\})\right\} = 0$. Let $\phi^m(n_1) = n_{m+1}$ for $m \in N$ and let $M = \operatorname{span}\{e^{(n_m)} : m \in N\}$. Then M is a closed subspace of l^2 . By the projection theorem $l^2 = M \oplus M^\perp$. Since M is a reducing subspace of C_ϕ , $C_\phi = C_\phi|_M \oplus C_\phi|_{M^\perp}$, where $C_\phi|_E$ denotes the restriction of C_ϕ to the subspace E . Define the transformation S from l^2 into M by

$$Se^{(m)} = e^{(n_m)} \quad \text{for every } m \in N .$$

Then S is a bounded linear transformation. Also S is invertible and $C_\phi|_M = SU^*S^{-1}$, where U^* is the adjoint of the unilateral shift U . Since similar operators have the same spectrum [3, Problem 60], we have $\sigma(C_\phi|_M) = \sigma(U^*)$. Thus $\sigma(C_\phi|_M) = D$ by the solution to Problem 67 of [3]. Since $\sigma(C_\phi) = \sigma(C_\phi|_M) \cup (C_\phi|_{M^\perp})$, $D \subset \sigma(C_\phi)$. From a corollary to Theorem 2.1 of [8], $\|C_\phi\| = 1$ and hence $\sigma(C_\phi) \subset D$. Thus $\sigma(C_\phi) = D$.

The following two theorems compute the spectrum of invertible composition operators. We know that C_ϕ is invertible if and only if ϕ is invertible [8, Theorem 2.2].

THEOREM 3.2. *Let $C_\phi \in B(l^2)$ be invertible, and assume for every $n \in N$ there exists an $m \in N$ such that $\phi^m(n) = n$. Let $m_n = \inf\{m : \phi^m(n) = n\}$ and $Q = \{m_n : n \in N\}$. Then*

$$\phi(C_\phi) = \bigcup_{q\in Q} \{\lambda : \lambda^q = 1\} .$$

Proof. For $q \in Q$, let $E_q = \{n : n \in N \text{ and } \phi^q(n) = n\}$. Then $M_q = \operatorname{span}\left\{e^{(p)} : p \in E_q\right\}$ is a reducing subspace of C_ϕ . Since ϕ is

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invertible, the family $\{M_q \mid q \in Q\}$ is a disjoint orthogonal family of subspaces which spans l^2. Thus l^2 can be written as $l^2 = \sum_{q \in Q} \oplus M_q$. Since $\phi^q = I$ on E_q, it follows that $I = \sum_{q \in Q} \oplus I|_{M_q} = \sum_{q \in Q} \oplus C_\phi^q|_{M_q}$. From this we can conclude that

$$\sigma(I) = \overline{\bigcup_{q \in Q} \sigma\left(C_\phi^q|_{M_q}\right)}, \text{ [3, p. 80]}$$
$$= \overline{\bigcup_{q \in Q} \left(C_\phi|_{M_q}\right)^q} \text{ (by the spectral mapping theorem).}$$

This shows that $\sigma\left(C_\phi|_{M_q}\right) = 1$ for every $q \in Q$. Hence $\sigma\left(C_\phi|_{M_q}\right) = \{\lambda : \lambda^q = 1\}$. Since $C_\phi = \sum_{q \in Q} \oplus C_\phi|_{M_q}$, $\sigma\left(C_\phi\right) = \bigcup_{q \in Q} \sigma\left(C_\phi|_{M_q}\right)$. Hence $\sigma\left(C_\phi\right) = \bigcup_{q \in Q} \{\lambda : \lambda^q = 1\}$.

COROLLARY. *If ϕ is periodic with period m, then $\sigma\left(C_\phi\right) = \{\lambda : \lambda^m = 1\}$.*

THEOREM 3.3. *If $C_\phi \in B\left(l^2\right)$ is invertible and if for some $n \in N$, there does not exist any $m \in N$ such that $\phi^m(n) = n$, then $\sigma\left(C_\phi\right) = c$.*

Proof. Let $n_0 \in N$ be such that $\phi^m\left(n_0\right) \neq n_0$ for all $m \in N$. For $m \in N$, let $n_m = \phi^m\left(n_0\right)$ and $n_{-m} = \left(\phi^m\right)^{-1}\left(\{n_0\}\right)$. If $E = \{n_m : m \in Z\}$, where Z is the set of all integers, and $l_E^2 = \operatorname{span}\{e^{(p)} : p \in E\}$, then

$$(1) \qquad l^2 = l_E^2 \oplus l_{N\setminus E}^2 .$$

If the transformation S from $l^2(Z)$ into l_E^2 is defined as $Se^{(m)} = e^{\left(n_m\right)}$ for every $m \in Z$, then S is a bounded linear invertible transformation and $C_\phi|_{l_E^2} = SW^*S^{-1}$, where W is the bilateral shift.

Thus by [3, Problem 60], $\sigma\left(C_\phi|_{l_E^2}\right) = \sigma(W^*)$. From Problem 68 of [3], $\sigma(W^*) = c$ and from the relation (1), we have $\sigma\left(C_\phi|_{l_E^2}\right) \subset \sigma\left(C_\phi\right)$. Since C_ϕ is invertible implies that C_ϕ is unitary [8, Theorem 2.3], it follows that $\sigma\left(C_\phi\right) = c$.

COROLLARY. *C_ϕ is hermitian if and only if $\sigma\left(C_\phi\right) \subset \{-1, 1\}$.*

Proof. Suppose C_ϕ is hermitian. Then by Theorem 3 of [6], $\phi \circ \phi = I$, and hence $\sigma\left(C_\phi\right) \subset \{-1, 1\}$ by the corollary to Theorem 3.2.

Conversely, if $\sigma\left(C_\phi\right) \subset \{-1, 1\}$, then C_ϕ is invertible and so it is normal. Hence C_ϕ is hermitian in view of Corollary 1.7 of [5].

COROLLARY. *Let $\phi : N \to N$ be an injection. Then $\sigma\left(C_\phi\right) = D$ if and only if C_ϕ is not an injection.*

Proof. Suppose C_ϕ is not an injection. Then ϕ is not onto. Thus by Theorem 3.1, $\sigma\left(C_\phi\right) = D$.

On the other hand, if C_ϕ is an injection, then ϕ is onto. This shows that ϕ is invertible which further shows that C_ϕ is unitary. Hence $\sigma\left(C_\phi\right) \subset c$, which is a contradiction.

Bibliography

[1] Ronald G. Douglas, *Banach algebra techniques in operator theory* (Pure and Applied Mathematics, 49. Academic Press, New York and London, 1972).

[2] Paul R. Halmos, *Measure theory* (Van Nostrand, New York, Toronto, London, 1950. Reprinted: Graduate Texts in Mathematics, 18. Springer-Verlag, New York, Heidelberg, Berlin, 1974).

[3] Paul R. Halmos, *A Hilbert space problem book* (Van Nostrand, Princeton, New Jersey; Toronto; London; 1967. Reprinted: Graduate Texts in Mathematics, **19**. Springer-Verlag, New York, Heidelberg, Berlin, 1974).

[4] Bernard B. Morrel and Paul S. Muhly, "Centered operators", *Studia Math.* (to appear).

[5] Heydar Radjavi, Peter Rosenthal, *Invariant subspaces* (Ergebnisse der Mathematik und ihrer Grenzgebiete, **77**. Springer-Verlag, Berlin, Heidelberg, New York, 1973).

[6] Raj Kishor Singh, "Normal and hermitian composition operators", *Proc. Amer. Math. Soc.* **47** (1975), 348-350.

[7] R.K. Singh, "Composition operators induced by rational functions", *Proc. Amer. Math. Soc.* **59** (1976), 329-333.

[8] R.K. Singh and B.S. Komal, "Composition operator on l^p and its adjoint", *Proc. Amer. Math. Soc.* (to appear).

Department of Mathematics,
University of Jammu,
Jammu,
India.