ALMOST CONTINUITY OF MAPPINGS

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This note stems from the following theorem of T. Husain [1]:

THEOREM H (Theorem 4 of [1]). *Let* E *be a metric Baire space and* f *a real valued function on* E. *Then the set of points of almost continuity in* E *is dense (everywhere) in* E.

Our purpose is to set this result in its most natural context, relax some very restricted hypotheses, and to supply a direct proof. More precisely, we shall prove that the metrizability of E in Theorem H may be removed, and that the range space may be generalized from the (Euclidean) space of real numbers to *any* topological space satisfying the second axiom of countability [2].

Definition. Let X and Y be topological spaces, a mapping $f : X \to Y$ is said to be *almost continuous* at $x \in X$ if and only if for each neighbourhood V of f(x), $\underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V)$ is a neighbourhood of x; f is *almost continuous* if it is almost continuous at each of $x \in X$.

A subset A of a topological space X is *dense* (in X) if $\underline{\text{Cl}}\ A = X$; a subset of X is called a *set of the second category* if it is not the union of a countable family of sets E_n such that each $\underline{\text{Int}}\ \underline{\text{Cl}}\ E_n = \square$ (*the empty set*). A topological space is said to be a *Baire space* (or to satisfy the *condition of Baire*) provided the intersection of each countable family of open dense subsets is dense. Every nonempty Baire space is a set of the second category, but the converse is not true.

THEOREM 1. *If* $f : X \to Y$ *is a mapping from a Baire space* X *to a topological space* Y *which satisfies the second axiom of countability, then the mapping* f *is almost continuous on a dense subset of* X.

Proof. Let $\{B_n : n = 1, 2, 3, \ldots\}$ be a countable basis for the open sets in Y. For each n, denote $E_n = \underline{\text{Cl}}\ f^{-1}(B_n) \setminus \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(B_n)$, then $\underline{\text{Int}}\ \underline{\text{Cl}}\ E_n = \square$ for each n; and thus, the set $E = \bigcup_{n=1}^{\infty} E_n$

is a set of the first category. But, if f is not almost continuous at x, then there exists a B_n such that $f(x) \in B_n$ and that x is not in $\underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(B_n)$ so that x must be in E. Hence, f is almost continuous on $X \setminus E$, which, as a complement of a first category subset of a Baire space, is dense in X.

A Moore space is a topological space that has a sequence $\mathcal{G}_n$ $(n = 1, 2, 3, \ldots)$ of collections of basic open sets (called regions) satisfying 1, 2, 3, and 4 of Axiom 1 of [3]. The following lemma might have been known, but we are unable to cite a source of print.

LEMMA. Every Moore space is a Baire space.

Proof. Let D_n $(n = 1, 2, 3, \ldots)$ be a sequence of open dense sets in a Moore space X, let x be an arbitrary point in X, and let G be an open set containing x. It must be shown that $G \cap (\bigcap_{n=1}^{\infty} D_n)$ is not empty. Since D_1 is dense and G is a nonempty open set, $D_1 \cap G \neq \square$. Consequently, by (2) and (3) of Moore's Axiom 1, there exists a $G_1 \in \mathcal{G}_1$ such that

$$\square \neq G_1 \subset \underline{\text{Cl}}\ G_1 \subset D_1 \cap G.$$

Since $G_1 \cap D_2 \neq \square$, by the same argument above, there exists a $G_2 \in \mathcal{G}_2$ such that

$$\square \neq G_2 \subset \underline{\text{Cl}}\ G_2 \subset D_2 \cap G_1.$$

Continuing this process inductively, a descending sequence G_n $(n = 1, 2, 3, \ldots)$ of nonempty regions is constructed which satisfies $G_n \in \mathcal{G}_n$ and $\underline{\text{Cl}}\ G_{n+1} \subset G_n \cap D_{n+1}$ for all n. Thus, by (4) of Moore's Axiom 1, the sequence $\underline{\text{Cl}}\ G_n$ $(n = 1, 2, 3, \ldots)$ has a common point, say y. Finally, $\underline{\text{Cl}}\ G_{n+1} \subset G_n \cap D_{n+1}$ for all n imply that the point y must be in $G \cap (\bigcap_{n=1}^{\infty} D_n)$, as was to be proven.

THEOREM 2. If X is a Moore space and if Y is a topological space satisfying the second axiom of countability, then every mapping from X to Y is almost continuous on a dense subset of X.

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