COMPACT DIFFERENCES OF COMPOSITION OPERATORS

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Abstract

Let ϕ and ψ be analytic self-maps of the open unit disk. Each of them induces a composition operator, C_ϕ and C_ψ respectively, acting between weighted Bergman spaces of infinite order. We show that the difference $C_\phi - C_\psi$ is compact if and only if both operators are compact or both operators are not compact and the pseudohyperbolic distance of the functions ϕ and ψ tends to zero if $|\phi(z)| \to 1$ or $|\psi(z)| \to 1$.

1. Introduction

Let $\phi, \psi : D \to D$ analytic maps. Each such map induces through composition a linear composition operator $C_\phi(f) = f \circ \phi$ and $C_\psi(f) = f \circ \psi$. Let v and w be strictly positive continuous and bounded functions (*weights*) on the unit disk D of the complex plane. We are interested in differences of composition operators defined on weighted Bergman spaces of infinite order,

$$H_v^\infty := \left\{ f \in H(D); \, \|f\|_v = \sup_{z \in D} v(z)|f(z)| < \infty \right\},$$

where $H(D)$ denotes the space of all analytic functions on D.

In [7] MacCluer *et al.* showed that the difference of two composition operators acting on the space $H^\infty(D)$ of bounded analytic functions on D is compact if and only if both operators are compact or if both operators are not compact and the pseudohyperbolic distance of the analytic self-maps ϕ and ψ of D tends to zero if $|\phi(z)| \to 1$ or $|\psi(z)| \to 1$. The aim of this article is to show that under some assumptions on the weights this remains true in the setting of weighted Bergman spaces of infinite order. In [3] we already characterized the essential norm and

therefore also the compactness of the difference of composition operators in terms of the weights. Here we want to show how compactness of the difference is related to the compactness of the single composition operators.

2. Notation and auxiliary results

For notation on composition operators we refer the reader to the monographs [4] and [8]. For differences of composition operators we need the pseudohyperbolic metric $\rho(z, w) := |\varphi_z(w)|$ for $z, w \in D$, where $\varphi_z(w) = (z - w)/(1 - \bar{z}w)$ is the Möbius transformation of D.

Let B_v^∞ be the closed unit ball of the space H_v^∞. Very often the so-called *associated weights* are needed to formulate results on weighted Bergman spaces of infinite order. For a weight v the associated weight $\tilde{v}$ is given by

$$\tilde{v}(z) = 1/(\sup\{|f(z)|, \ f \in B_v^\infty\}) = 1/\|\delta_z\|_{H_v^{\infty'}},$$

where δ_z denotes the point evaluation of z. By [1] we know that the associated weight $\tilde{v}$ has the following properties:

(i) $\tilde{v}$ is continuous;
(ii) $\tilde{v} > v > 0$;
(iii)   for every $z \in D$ we can find $f_z \in B_v^\infty$ such that $|f_z(z)| = 1/(\tilde{v}(z))$.

We are especially interested in the radial weights which satisfy the following condition (which is due to Lusky, see [6]):

$$\inf_k \frac{v(1 - 2^{-k-1})}{v(1 - 2^{-k})} > 0. \tag{L1}$$

Lusky showed that each of the following weights satisfies the condition (L1):

$$v_p(z) = (1 - |z|)^p, \quad 0 < p < \infty,$$
$$w_1(z) = (1 - \log(1 - |z|))^{-\beta}, \quad \beta > 0,$$
$$w_2(z) = (1 - |z|)^p (1 - \log(1 - |z|))^{-\beta}, \quad 0 < p < \infty, \beta > 0.$$

Radial weights with $\lim_{|z| \to 1} v(z) = 0$ are called *typical weights*.

For a better understanding we want to collect some auxiliary results that we need to prove our main result.

THEOREM 1 (Bonet *et al.* [2]). *Let ϕ be an analytic self-map of D.*

(a)   *Let v and w be radial weights. Then the following assertions are equivalent.*

   (i)   *The operator $C_\phi : H_v^\infty \to H_w^\infty$ is bounded.*
   (ii) $\sup_{z \in D} (w(z))/(\tilde{v}(\phi(z))) = M < \infty.$

(b)   *Let v and w be typical weights. Then the following statements are equivalent.*

   (i)   *The operator $C_\phi : H_v^\infty \to H_w^\infty$ is compact.*
   (ii) $\lim_{|z| \to 1} (w(z))/(\tilde{v}(\phi(z))) = 0.$

LEMMA 2 (Bonet *et al.* [3]). *Let* v *be a radial weight satisfying the Lusky condition* (*L1*) *and let* $f \in H_v^\infty$. *Then there exists a constant* C_v (*depending on the weight* v) *such that*

$$|f(z) - f(p)| \leq C_v \|f\|_v \max\left\{\frac{1}{v(z)}, \frac{1}{v(p)}\right\} \rho(z, p),$$

for all $z, p \in D$.

LEMMA 3 (Lindström and Wolf [5]). *Let* v *be a radial weight on* D *satisfying the Lusky condition* (*L1*) *such that* v *is continuously differentiable with respect to* $|z|$. *Then* $v(p)/v(z) \to 1$ *when* $\rho(p, z) \to 1$.

THEOREM 4 (Cowen and MacCluer [4, Proposition 3.11]). *Let* ϕ *and* ψ *be analytic self-maps of the disk. If* $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ *is bounded, then* $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ *is compact if and only if whenever* $(h_n)_{n \in \mathbb{N}}$ *is a bounded sequence in* H_v^∞ *with* $h_n \to 0$ *uniformly on compact subsets of* D, *then* $\|(C_\phi - C_\psi)h_n\|_w \to 0$.

3. Main result

THEOREM 5. *Let* v *and* w *be typical weights such that* v *is continuously differentiable with respect to* $|z|$ *and satisfies* (*L1*). *We assume that* $C_\phi, C_\psi : H_v^\infty \to H_w^\infty$ *both are bounded. Then* $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ *is compact if and only if one of* (a) *or* (b) *holds:*

(a) C_ϕ *and* C_ψ *are compact;*
(b) C_ϕ *and* C_ψ *both are not compact and*

$$\lim_{|\phi(z)| \to 1} \rho(\phi(z), \psi(z)) = \lim_{|\psi(z)| \to 1} \rho(\phi(z), \psi(z)) = 0.$$

PROOF. Suppose (a) is not true. Without loss of generality, we may assume that C_ϕ is not compact.

Moreover, first we assume that $\lim_{|\phi(z)| \to 1} \rho(\phi(z), \psi(z)) \neq 0$. Then there is a sequence $(z_n)_{n \in \mathbb{N}} \subset D$ such that $|\phi(z_n)| \to 1$ (and hence $|z_n| \to 1$) and

$$\rho(\phi(z_n), \psi(z_n)) \geq \alpha > 0 \quad \text{for every } n \in \mathbb{N}.$$

Furthermore, since C_ϕ is not compact and v and w both are radial, by Theorem 1 it follows that $w(z_n)/\tilde{v}(\phi(z_n)) \geq \beta > 0$ for every $n \in \mathbb{N}$. Next we choose an increasing sequence $(\alpha_n)_{n \in \mathbb{N}}$ of natural numbers going to infinity such that $|\phi(z_n)|^{\alpha_n} \geq \gamma > 0$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we can find $f_n \in B_v^\infty$ such that $|f_n(\phi(z_n))| = 1/\tilde{v}(\phi(z_n))$. Now set

$$h_n(z) := z^{\alpha_n} \varphi_{\psi(z_n)}(z) f_n(z).$$

Hence $(h_n)_n \subset B_v^\infty$ and $(h_n)_n$ tends to zero on the compact sets. Since $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is compact, by Theorem 4 $(\|(C_\phi - C_\psi)h_n\|_w)_{n \in \mathbb{N}}$ must tend to zero. But

$$\|(C_\phi - C_\psi)h_n\|_w \geq |\phi(z_n)|^{\alpha_n} \frac{w(z_n)}{\tilde{v}(\phi(z_n))} \rho(\phi(z_n), \psi(z_n)) \geq \alpha\beta\gamma > 0.$$

This is a contradiction.

Next, we assume that $\lim_{|\psi(z)| \to 1} \rho(\phi(z), \psi(z)) \neq 0$. Then there is a sequence $(z_n)_{n \in \mathbb{N}} \subset D$ such that $|\psi(z_n)| \to 1$ (and hence $|z_n| \to 1$ and thus $|\phi(z_n)| \to 1$) and

$$\rho(\phi(z_n), \psi(z_n)) \geq \alpha > 0 \quad \text{for every } n \in \mathbb{N}.$$

Since w and v are radial and C_ϕ is not compact, Theorem 1 yields

$$w(z_n)/v(\phi(z_n)) \geq \beta > 0$$

for every $n \in \mathbb{N}$. From now on the proof is analogous to the proof of the previous case.

It remains to show that C_ψ is not compact. By Lemma 3 we know that

$$\left| \frac{\tilde{v}(\phi(z))}{\tilde{v}(\psi(z))} - 1 \right| \to 0 \quad \text{if } \rho(\phi(z), \psi(z)) \to 0.$$

Hence C_ψ is not compact, since C_ϕ is not compact.

For the converse, by Theorem 4 we have to show $\lim_{n \to \infty} \|(C_\phi - C_\psi)h_n\|_w = 0$ for $(h_n)_{n \in \mathbb{N}}$ bounded and $h_n \to 0$ uniformly on compact subsets of D. It is enough to consider $(h_n)_{n \in \mathbb{N}} \subset B_v^\infty$. By Lemma 2 we can find $C > 0$ such that

$$|h_n(\phi(z)) - h_n(\psi(z))| \leq C \max\left\{ \frac{1}{\tilde{v}(\phi(z))}, \frac{1}{\tilde{v}(\psi(z))} \right\} \rho(\phi(z), \psi(z)).$$

By Theorem 1 and the assumption that C_ϕ and C_ψ are bounded, we can choose $M > 0$ such that $\sup_{z \in D} w(z) \max\{1/\tilde{v}(\phi(z)), 1/\tilde{v}(\psi(z))\} \leq M$. Given $\varepsilon > 0$ we may find $r < 1$ such that

$$|\phi(z)| \geq r \Longrightarrow \rho(\phi(z), \psi(z)) < \frac{\varepsilon}{CM} \quad \text{and}$$
$$|\psi(z)| \geq r \Longrightarrow \rho(\phi(z), \psi(z)) < \frac{\varepsilon}{CM}.$$

Thus, we obtain

$$w(z)|h_n(\phi(z)) - h_n(\psi(z))| < \varepsilon \quad \text{if } |\phi(z)| \geq r \text{ or } |\psi(z)| \geq r,$$

and, by almost uniform convergence of $(h_n)_{n \in \mathbb{N}}$ to zero and boundedness of w, we can find $n_0 \in \mathbb{N}$ such that

$$w(z)|h_n(\phi(z)) - h_n(\psi(z))| < \varepsilon \quad \text{for } n \geq n_0, |\psi(z)| \leq r, |\phi(z)| \leq r.$$

Hence the claim follows. $\square$

For examples on compact and non-compact differences of composition operators acting between weighted Bergman spaces of infinite order we refer the reader to [3].

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