A NEW APPROACH TO JACOBI'S THEOREMS
VIA RAMANUJAN'S CONTINUED FRACTIONS

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In this paper, we show that Jacobi's two-square and two-triangular number theorems are immediate consequence of Ramanujan's continued fraction identities.

1. INTRODUCTION

Let $r_k(n)$ denote the number of ways the positive integer n can be represented as a sum of k squares, with representations arising from different signs and from different orders being regarded as distinct. Geometrically, $r_k(n)$ counts the number of lattice points on the k-dimensional sphere $x_1^2 + \cdots + x_k^2 = n$. Let

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, |q| < 1.$$

Then the generating function for $r_k(n)$ is

$$\phi^k(q) = \sum_{n=0}^{\infty} r_k(n) q^n.$$

A triangular number is a number of the form $k(k+1)/2$ for some non-negative integer k. Let $t_k(n)$ denote the number of representations of n as a sum of k triangular numbers. Geometrically, $2^k t_k(n)$ counts the number of lattice points on the k-dimensional sphere centred at $(-1/2, \ldots, -1/2)$ with radius $\sqrt{2n + k/4}$. Let

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, |q| < 1.$$

Then the generating function for $t_k(n)$ is

$$\psi^k(q) = \sum_{n=0}^{\infty} t_k(n) q^n.$$

One of the main problems is to find formulae for determining $r_k(n)$, in terms of simple arithmetical functions such as divisor functions. Jacobi's two-square and two-triangular-number theorems state that

(1.1)
$$r_2(n) = 4\big[d_1(n) - d_3(n)\big]$$

and

(1.2)
$$t_2(n) = d_1(4n+1) - d_3(4n+1).$$

where $d_i(m)$ is the number of divisors of m congruent to i modulo 4. Many authors gave proofs of (1.1) and (1.2) using q-series identities. Hirschhorn [8] and Ewell [6] have used Jacobi's triple product identity to obtain (1.1). Askey [3], Adiga [1], Bhargava and Adiga [4], Fine [7] and, Cooper and Lam [5] have employed Ramanujan's $_1\psi_1$ summation formula to derive (1.1) and (1.2). We show that (1.1) and (1.2) are immediate consequences of two continued fraction identities of Ramanujan.

Ramanujan's general theta function and its special cases are defined as follows:

(1.3)
$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, |ab| < 1,$$

(1.4)
$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q : -q)_\infty}{(q : -q)_\infty},$$

(1.5)
$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2 : q^2)_\infty}{(q : q^2)_\infty},$$

where $(a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n)$.

$\phi(q)$ and $\psi(q)$ have been used for a long time in the theory of elliptic functions and modular forms. The main purpose of this note is to establish continued fractions for $\phi^2(q)$ and $\psi^2(q^2)$, and to employ them to derive Jacobi's two-square and two-triangular number theorems.

2. MAIN RESULTS

THEOREM 1. *If $|q| < 1$, then*

(2.1)
$$1 + \phi^2(q) = \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3} + \cdots,$$

where

$$a_1 = 2, a_2 = -2q, a_3 = (q - q^2)^2, \ldots,$$
$$a_n = (-1)^{n-3} q^{n-3}(q + (-1)^{n-2} q^{n-1})^2, n \geqslant 3, \quad \text{and}$$
$$b_1 = 1, b_2 = 1 + q, b_3 = 1 + q^3, \ldots, b_n = 1 + q^{2n-3}, n \geqslant 3.$$

PROOF: From ([2, p. 14, Entry 11]), we have

$$\frac{(-a)_\infty(b)_\infty - (a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty + (a)_\infty(-b)_\infty} = \frac{a-b}{1-q} + \frac{(a-bq)(aq-b)}{1-q^3} + \frac{q(a-bq^2)(aq^2-b)}{1-q^5} + \cdots .$$

This continued fraction identity may be rewritten as

$$1 + \frac{(-a)_\infty(b)_\infty}{(a)_\infty(-b)_\infty} = \frac{2}{1-}\frac{a-b}{1-q} + \frac{(a-bq)(aq-b)}{1-q^3} + \frac{q(a-bq^2)(aq^2-b)}{1-q^5} + \cdots .$$

Changing q to $-q$, a to q and b to $-q$ in the above continued fraction identity and using (1.4), we complete the proof.   ▯

THEOREM 2.   *If $|q| < 1$, then*

$$(2.2) \qquad 2 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}q^{n(n+1)/2}}{1+(-q)^n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots ,$$

where a_n and b_n are as defined in Theorem 1.

PROOF: Let P_n/Q_n denote the n-th convergent of the continued fraction $a_1/b_1 + a_2/b_2 + a_3/b_3 + \cdots$. Then

$$Q_1 = 1, Q_2 = (1-q), Q_3 = (1+q^3)(1-q) + (q-q^2)^2 = (1-q)(1+q^2) \ \ \text{and}$$
$$Q_n = b_n Q_{n-1} + a_n Q_{n-2}.$$

By induction on n, we can show that

$$Q_n = \prod_{k=1}^{n-1}(1+(-1)^k q^k), n \geqslant 2.$$

We have

$$\frac{P_n}{Q_n} = \sum_{k=1}^{n} \frac{(-1)^{k+1}a_1 a_2 \ldots a_k}{Q_k Q_{k-1}}$$

$$= \frac{a_1}{Q_1 Q_0} - \frac{a_1 a_2}{Q_2 Q_1} + \frac{a_1 a_2 a_3}{Q_3 Q_2} - \cdots + \frac{(-1)^{n+1}a_1 a_2 a_3 \ldots a_n}{Q_n Q_{n-1}}$$

$$= \frac{2}{1} + \frac{4q}{1-q} - \frac{4q^3}{1+q^2} - \frac{4q^6}{1-q^3} + \frac{4q^{10}}{1+q^4} + \cdots + 4\frac{(-1)^{((n-1)(n-2))/2}q^{n(n-1)/2}}{1+(-q)^{n-1}}.$$

Letting $n \to \infty$, we obtain the required result.   ▯

THEOREM 3.   *Jacobi's two-square theorem.*

$$r_2(n) = 4\big[d_{1,4}(n) - d_{3,4}(n)\big].$$

PROOF: From (2.1) and (2.2), we have

$$\phi^2(q) = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}q^{n(n+1)/2}}{1+(-q)^n}$$

$$(2.3) \qquad\qquad = 1 + 4\sum_{n=1}^{\infty}\Big(\frac{q^{4n-3}}{1-q^{4n-3}} - \frac{q^{4n-1}}{1-q^{4n-1}}\Big)$$

on summing by the column-row method. Now, comparing the coefficients of q^n on both sides of (2.3), we obtain Jacobi's two-square Theorem. $\square$

THEOREM 4.   *If $|q| < 1$, then*

$$(2.4) \qquad \psi^2(q^2) = \frac{1}{1-q} + \frac{q(1-q)^2}{(1-q)(1+q^2)} + \frac{q(1-q^3)^2}{(1-q)(1+q^4)} + \frac{q(1-q^5)^2}{(1-q)(1+q^6)} + \cdots .$$

PROOF: From ([**2**, p. 17, Entry 12]), we have

$$\frac{(a^2q^3; q^4)_\infty (b^2q^3; q^4)_\infty}{(a^2q; q^4)_\infty (b^2q; q^4)_\infty} = \frac{1}{1-ab} + \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)} + \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} + \cdots .$$

Putting $a = b = \sqrt{q}$ in the above identity and using (1.5), we obtain (2.4). $\square$

THEOREM 5.   *If $|q| < 1$, then*

$$(2.5) \qquad \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1 - q^{2n+1}} = \frac{1}{1-q} + \frac{q(1-q)^2}{(1-q)(1+q^2)} + \frac{q(1-q^3)^2}{(1-q)(1+q^4)} + \cdots .$$

PROOF: Let P_n/Q_n denote the n-th convergent of the continued fraction (2.5). Then

$$Q_1 = 1 - q, Q_2 = (1-q)^2(1+q^2) + q(1-q)^2 = (1-q)^2(1+q+q^2) = (1-q)(1-q^3).$$

By induction on n, one can show that

$$Q_n = (1-q)(1-q^3)(1-q^5) \ldots (1-q^{2n-1}).$$

Then we have

$$\frac{P_n}{Q_n} = \sum_{k=1}^{n} \frac{(-1)^{k+1} a_1 a_2 \ldots a_k}{Q_k Q_{k-1}}$$

$$= \frac{a_1}{Q_1 Q_0} - \frac{a_1 a_2}{Q_2 Q_1} + \frac{a_1 a_2 a_3}{Q_3 Q_2} - \cdots + \frac{(-1)^{n+1} a_1 a_2 a_3 \ldots a_n}{Q_n Q_{n-1}}$$

$$= \frac{1}{1-q} - \frac{q}{1-q^3} + \frac{q^2}{1-q^5} - \cdots + \frac{(-1)^{n-1} q^{n-1}}{1-q^{2n-1}}.$$

Letting $n \to \infty$, we complete the proof of Theorem 5. $\square$

THEOREM 6.   *Jacobi's two-triangular number Theorem.*

$$t_2(n) = d_1(4n+1) - d_3(4n+1).$$

PROOF: From (2.4) and (2.5), we have

$$(2.6) \qquad \psi^2(q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1 - q^{2n+1}}.$$

Changing q to $-q$ in (2.6) and adding the resulting identity to (2.6), we obtain

$$2\psi^2(q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1-q^{2n+1}} + \sum_{n=0}^{\infty} \frac{q^n}{1+q^{2n+1}}$$

(2.7)
$$= 2\sum_{n=0}^{\infty} \frac{q^{2n}}{1-q^{8n+2}} - 2\sum_{n=0}^{\infty} \frac{q^{6n+4}}{1-q^{8n+6}}.$$

Changing q to $q^{1/2}$ in (2.7), we deduce

$$\psi^2(q) = \sum_{n=0}^{\infty} \frac{q^n}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{4n+3}}$$

$$= \sum_{n,m=0}^{\infty} q^{4mn+m+n} - \sum_{n,m=0}^{\infty} q^{4mn+3m+3n+2}$$

(2.8)
$$= \sum_{n,m=0}^{\infty} q^{((4m+1)(4n+1)-1)/4} - \sum_{n,m=0}^{\infty} q^{((4m+3)(4n+3)-1)/4}.$$

By comparing the coefficients of q^n on both sides of (2.8), we obtain the Jacobi's two-triangular number Theorem.   ❑

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