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RESEARCH ARTICLE

Closed approximate subgroups: compactness, amenability and approximate lattices

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Abstract
We investigate properties of closed approximate subgroups of locally compact groups, with a particular interest for approximate lattices (i.e., those approximate subgroups that are discrete and have finite co-volume).

We prove an approximate subgroup version of Cartan's closed-subgroup theorem and study some applications. We give a structure theorem for closed approximate subgroups of amenable groups in the spirit of the Breuillard–Green–Tao theorem. We then prove two results concerning approximate lattices: we extend to amenable groups a structure theorem for mathematical quasi-crystals due to Meyer; we prove results concerning intersections of radicals of Lie groups and discrete approximate subgroups generalising theorems due to Auslander, Bieberbach and Mostow. As an underlying theme, we exploit the notion of good models of approximate subgroups that stems from the work of Hrushovski, and Breuillard, Green and Tao. We show how one can draw information about a given approximate subgroup from a good model, when it exists.

Contents

1. Introduction

As defined by Tao [Tao08], a K-approximate subgroup of a group G is a subset A that is symmetric $A = A^{-1}$, contains the identity, and satisfies $AA \subset FA$ for some finite subset $F \subset G$ with $|F| \leq K$. In a seminal paper [Hru12], Hrushovski showed that approximate subgroups are closely related to neighbourhoods of the identity in locally compact groups. Breuillard–Green–Tao [BGT12] subsequently used his ideas in combination with techniques originating from Gleason's and Yamabe's work on Hilbert's 5th problem to prove a theorem describing the structure of arbitrary finite approximate subgroups. Their work was extended in Kreitlon Carolino's thesis [Car15] to handle approximate groups that are relatively compact neighbourhoods of the identity in an arbitrary locally compact group.

The goal of this paper is to further investigate properties of *infinite* approximate subgroups. We will prove several results in this direction. In particular, we will show (Theorem 4.1 below) that closed approximate subgroups of locally compact groups are commensurable to homomorphic images of open approximate groups in a possibly different locally compact ambient group. This can be seen as an analogue of the classical theorem of Cartan asserting that closed subgroups of Lie groups are Lie subgroups. In turn, this enables us to apply the main result of Kreitlon Carolino's thesis [Car15] to any compact approximate subgroup and remove the assumption in [Car15, Thm. 1.25] that the relatively compact approximate groups studied have nonempty interior (Theorem 4.6 below). We will also prove (Theorem 1.6) a structure theorem in the spirit of the Breuillard–Green–Tao theorem ([BGT12]) for *amenable* closed approximate subgroups of locally compact groups. This applies, in particular, to all closed approximate subgroups of amenable locally compact groups.

But most of the paper will be devoted to investigating a special class of approximate groups called *approximate lattices*. These were first systematically studied in recent work of Björklund and Hartnick [BH18]. The approximate lattices are those approximate subgroups Λ of an ambient locally compact group G that have finite co-volume (i.e., $G = \Lambda\mathcal{F}$ for some Borel subset $\mathcal{F}$ of G of finite Haar measure) and are uniformly discrete (or equivalently the identity is isolated in Λ^2). Note that this definition is due to Hrushovski [Hru22]; Björklund and Hartnick first defined *uniform* approximate lattices (when $\mathcal{F}$ is relatively compact) as well as a non-uniform version of this notion under the name of *strong approximate lattice*; see [BH18, Def. 4.9] and Subsection 2.3 below (the link between these three notions is investigated in [Mac22a]). When G is commutative, these sets had been defined and studied already in the 1970's by Yves Meyer [Mey72]; they are a model of the so-called *mathematical quasi-crystals* and have been much studied since, in particular in connection with mathematical physics [BG13].

A simple way to construct approximate lattices is via a cut-and-project scheme – namely, the datum (G, H, Γ) of two locally compact groups G and H, and a lattice Γ in $G \times H$, which projects injectively to G and densely to H. Given a symmetric relatively compact neighbourhood of the identity $W_0 \subset H$, one defines the *model set* $P_0(G, H, \Gamma, W_0) := p_G((G \times W_0) \cap \Gamma)$, where p_G is the projection to the first factor. It is easy to see that a model set, or even any set commensurable to a model set, must be an approximate lattice. For more on cut-and-project sets, we refer the reader to [BHP18, BHP22].

Central to our paper is an idea, exploited in [Hru12] and [BGT12] on the way to the structure theorems established there, which consists in defining a certain topology on the approximate group by means of what we will call here a *good model*:

Definition 1.1. Let Λ be an approximate subgroup of a group Γ. A group homomorphism $f : \Gamma \to H$ with target a locally compact group H is called a *good model (of* (Λ, Γ)) if:

1. $f(\Lambda)$ is relatively compact;
2. there is $U \subset H$ a neighbourhood of the identity such that $f^{-1}(U) \subset \Lambda$.

If Γ is generated by Λ, then we say that f is a good model of Λ. Any approximate subgroup commensurable to Λ will be called a *Meyer subset*.

This definition is reminiscent of the construction of the so-called *Schlichting completion* of a pair (Γ, Λ), where $\Lambda \leq \Gamma$ are discrete groups such that Γ commensurates Λ. Indeed, if U is also a compact subgroup, then $\Lambda := f^{-1}(U)$ is commensurated by Γ. See, for instance, the work of Tzanev [Tza03] on Hecke pairs or the works of Shalom and Willis [SW13] and of Caprace and Monod [CM09, §5D] on commensurators of discrete groups, where this construction plays a key role.

We will see that good models exist in many situations of interest. Indeed, our first observation is that for approximate lattices in an ambient group G, the existence of a good model is equivalent to being commensurable to a model set via a certain cut-and-project scheme (G, H, Γ).

Proposition 1.2. *Let Λ be an approximate lattice in a locally compact group G. Then Λ is a Meyer subset if and only if it is contained in and commensurable to a model set.*

Meyer [Mey72] (see also [Sch73]) showed that every approximate lattice in a locally compact commutative group G comes from a *cut-and-project* construction: in other words, it is commensurable to a model set, or, equivalently thanks to Proposition 1.2, it is a Meyer subset. The question of extending Meyer's theorem to other groups has been raised by Björklund and Hartnick in [BH18, Problem 1]. This has been achieved for nilpotent and solvable Lie groups in our previous works [Mac20] and [Mac22b] following a method close in spirit to Meyer's. A consequence of the tools developed in the present paper will be a new proof of this fact and indeed a generalization to all locally compact amenable ambient groups (Theorem 1.5 below). In a companion paper ([Mac23]), we show, using some key results of the present paper (notably Proposition 1.2 and Theorem 4.1) in combination with Zimmer's cocycle superrigidity theorem, that Meyer's theorem also holds for strong approximate lattices in semi-simple Lie groups without rank one factors (we also mention Hrushovski's [Hru22] which generalises Meyer's theorem to semi-simple groups via a different approach). Another consequence will be a proof of the analogue for approximate lattices of the classical facts about hereditary properties of lattices with respect to intersections with closed normal subgroups (Proposition 6.3) and, in particular, an Auslander-type theorem regarding the intersection with the amenable radical (Theorem 1.7).

We are now ready to state the main results of this paper. We consider first two interesting classes of approximate groups: compact approximate subgroups and amenable approximate subgroups (Definition 5.1). We will see that these types of approximate subgroups always have good models and thus are particularly regular types of approximate subgroups. In the case of compact approximate subgroups, this leads to a closed-subgroup theorem for approximate subgroups.

Theorem 1.3 (Closed-approximate-subgroup theorem). *Let Λ be a closed approximate subgroup of a locally compact group G. There are a locally compact group H, an injective continuous group homomorphism $\phi : H \to G$ and an open approximate subgroup Ξ of H such that for all $n \geq 0$, $\phi_{|\phi^{-1}(\Lambda^n)}$ is proper and a homeomorphism onto its image, and $\Lambda \subset \phi(\Xi) \subset \Lambda^3$. Furthermore, if G is a Lie group, then H is a Lie group.*

We say that a map between locally compact spaces is proper if the inverse image of any compact subset is also compact. Here, a good model of some compact approximate subgroup contained in Λ^2 appears implicitly as the inverse of the map ϕ. Theorem 1.3 and a theorem of Schreiber [Sch73] (which

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was recently given a new proof by Fish [Fis19]) show that, modulo a compact error term, the structure of closed approximate subgroups of Euclidean spaces is akin to the structure of closed subgroups. Recall that a subset Λ of an ambient group G is called *uniformly discrete* if e is isolated in $\Lambda^{-1}\Lambda$.

Proposition 1.4. *Let Λ be a closed approximate subgroup in $\mathbb{R}^d$. Then we can find a vector subspace $V_o \subset \mathbb{R}^k$, as well as a uniformly discrete approximate subgroup Λ_d and a compact approximate subgroup K_e both in a supplementary subspace V_d of V_o such that Λ is commensurable to $V_o + \Lambda_d + K_e$.*

Incidentally, Theorem 1.3 (in fact, the more general Theorem 4.1) enables us to remove an openness assumption from a result of Kreitlon Carolino's thesis ([Car15, Thm. 1.25]) which is an improvement of the Gleason–Yamabe theorem [Yam53, Thm. 3]. This yields a precise structure theorem for all compact approximate subgroups.

Likewise, we are able to prove that amenability assumptions force approximate subgroups to have good models. It is well known that in many situations, existence of some invariant finitely additive measures implies existence of a good model (see, for example, [Hru12, HKP22, San12, CS10, MW15]). We will say that a closed approximate subgroup of a locally compact group G is *amenable* if there exists an invariant finitely additive probability measure on Borel subsets of Λ (Section 5). This condition is in particular satisfied on any closed approximate subgroup close to an amenable normal subgroup (Proposition 5.13). This enables us to prove a generalisation of Meyer's seminal theorem [Mey72]:

Theorem 1.5 (Meyer theorem for amenable groups). *If Λ is an approximate lattice in an amenable locally compact second countable group G, then Λ is contained in and commensurable to a model set.*

Recall that we had already established this result in [Mac20, Mac22b] in the special case when G is a soluble Lie group. Our method here is very different, however, and is inspired from the work of Hrushovski [Hru12] and Breuillard–Green–Tao [BGT12].

It is interesting to show that an approximate subgroup is amenable beyond the realm of approximate lattices. By using the strong Tits' alternative due to Breuillard [Bre08] (and a consequence of it due to Breuillard, Green and Tao [BGT11]), we can prove a result reminiscent of the structure theorem of finite approximate subgroups [BGT12] (see also Proposition 5.6 for a more complete statement in the language of good models).

Theorem 1.6 (Structure of amenable approximate subgroups). *Let Λ be an amenable closed approximate subgroup of σ-compact locally compact group G. Then there is a closed approximate subgroup $\Lambda_{sol} \subset \overline{\Lambda^4}$ and a closed subgroup $N \subset \Lambda_{sol}$ such that*

1. *N is normal in $\langle \Lambda_{sol} \rangle$ and $\langle \Lambda_{sol} \rangle / N$ is a soluble group;*
2. *if $\langle \Lambda_{sol} \rangle$ is equipped with the topology given by Theorem 1.3, then $\langle \Lambda_{sol} \rangle / N$ is a Lie group;*
3. *there is a compact neighbourhood V of the identity in Λ^n (in the induced topology) for some $n \geq 0$ such that Λ is contained in $V\Lambda_{sol}$ and $V\Lambda_{sol} \cup \Lambda_{sol}V$ is an approximate subgroup commensurable to Λ.*

Subgroups, compact approximate subgroups (see Section 4) and approximate subgroups of soluble lie groups (see [Mac22b]) are natural and well-studied examples of amenable approximate subgroups. Theorem 1.6 asserts conversely that any amenable closed approximate subgroup of a locally compact group is built as a combination of these. We briefly mention two facts to illustrate the strength of Theorem 1.6: when G is supposed totally disconnected or Λ is supposed uniformly discrete (e.g., when Λ is an approximate lattice), we can choose Λ_{sol} commensurable to Λ (Corollaries 5.11 and 5.12). Then Λ is an extension of an amenable group by a soluble approximate subgroup.

Finally, we will use the ideas behind Theorem 1.6 to study generalisations of theorems due to Auslander [Aus63, Thm. 1] and Mostow [Mos71, Lem. 3.9] about intersections of lattices and radicals in Lie groups. Our main result in that direction follows:

Theorem 1.7 (Semi-simple + amenable decomposition). *Let Λ be a uniformly discrete approximate subgroup in a locally compact second countable group G. Suppose that there exists an amenable closed*

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normal subgroup A such that G/A is a finite direct product of simple algebraic groups over local fields as a topological group. If the projections of $\langle\Lambda\rangle$ to all simple factors of G/A are Zariski-dense, then the projection of Λ to G/A is uniformly discrete.

When specialized to approximate lattices, this answers a question of Hrushovski [Hru22, Question 7.11] and indeed generalises [Aus63, Thm. 1] and [Mos71, Lem. 3.9]. We point to Corollary 6.11 and the discussion immediately after for this and more.

Structure of the paper

In Section 2, we recall a few useful facts and definitions about approximate subgroups and approximate lattices. We then study general properties of good models in Section 3. Using these tools, we establish in Section 4 and Section 5 the structure theorems for, respectively, compact approximate subgroups and amenable approximate subgroups. In Section 6, we study intersections of approximate lattices with closed subgroups, eventually proving Theorem 1.7.

2. Preliminaries

2.1. Notation

In this paper, topological group is a group together with a topology making the inverse map and multiplication map continuous. A topological group is locally compact if it admits a compact Hausdorff neighbourhood of the identity. More generally, we follow throughout this paper Bourbaki's terminology. In particular, all compact sets are understood to be Hausdorff. Given a locally compact group G, a Haar measure of G refers to a left-Haar measure. We refer to [Bou89a, Ch. III], [Var84] for background on locally compact groups and Lie groups.

For a subset X of a group G and a non-negative integer n, define $X^{-1} = \{x^{-1}|x \in X\}$, $X^n := \{x_1 \cdots x_n | x_1, \dots, x_n \in X\}$ and $\langle X\rangle$ the group generated by X. Recall that an *approximate subgroup* is a subset Λ of a group that is symmetric (i.e., $\Lambda = \Lambda^{-1}$) and contains the identity, and such that there exists a finite subset $F \subset G$ with $\Lambda^2 := \{\lambda_1\lambda_2 \in G | \lambda_1, \lambda_2 \in \Lambda\} \subset F\Lambda$. A useful observation in the study of approximate subgroups is the so-called *modular law*. Namely, if $X, Y, Z \subset G$ are such that $X \subset YZ$, then $X \subset (Y \cap XZ^{-1})Z$.

2.2. Preliminaries on approximate subgroups and commensurability

We will say that two subsets $X, Y \subset G$ are *(left-)commensurable* if there exists a finite subset $F \subset G$ such that $X \subset FY$ and $Y \subset FX$. Note that commensurability is an equivalence relation between subsets of a group. An approximate subgroup is thus a symmetric subset Λ containing e such that Λ^2 is commensurable to Λ. By an easy induction, we see moreover that Λ^n is commensurable to Λ for all $n \geq 1$. When Λ_1, Λ_2 are two commensurable approximate subgroups, $\Lambda_1 \cup \Lambda_2$ is also an approximate subgroup.

We will denote by $\mathrm{Comm}_G(X)$ the subgroup of elements g of G such that gXg^{-1} is commensurable with X. We say that a subset $H \subset G$ *commensurates* X if $H \subset \mathrm{Comm}_G(X)$. If Λ is an approximate subgroup, then $\langle\Lambda\rangle$ commensurates Λ (i.e., $\langle\Lambda\rangle \subset \mathrm{Comm}_G(\Lambda)$) [Hru22, Lem. 5.1].

We collect here well-known facts about approximate subgroups and commensurability in a form and with hypotheses suitable to our discussion (See [Tao08, BGT12, Hru12, Toi20] for this and more background material).

Lemma 2.1 (Ruzsa's covering lemma). *Let X, Y be subsets of a group G and $F \subset X$ be maximal such that $(fY)_{f\in F}$ is a family of disjoint sets. Then $X \subset FYY^{-1}$.*

Proof. If $x \in X$, then there is $f \in F$ such that $xY \cap fY \neq \emptyset$. So $x \in FYY^{-1}$. □

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Lemma 2.2. *Let $X_0, X_1, \dots, X_r$ be subsets of a group G and $F_1, \dots, F_r \subset G$ be finite subsets such that $X_0 \subset F_i X_i$ for all integers $1 \leq i \leq r$. There is $F \subset G$ with $|F| \leq |F_1| \cdots |F_r|$ such that*

$$X_0 \subset F \cdot \bigcap_{1 \leq i \leq n} X_i^{-1} X_i.$$

Proof. Take $f := (f_i) \in F_1 \times \cdots \times F_r$, and whenever $\bigcap_{1\leq i\leq r} f_i X_i \neq \emptyset$, choose an element $x_f \in \bigcap_{1\leq i\leq r} f_i X_i$. If x is any element of X_0, then there must be some $f \in F_1 \times \cdots \times F_r$ such that $x \in \bigcap_{1\leq i\leq r} f_i X_i$. We thus have $x_f^{-1} x \in \bigcap_{1\leq i\leq n} X_i^{-1} X_i$. Defining $F := \{x_f | f \in F_1 \times \cdots \times F_n, X_0 \cap \bigcap_{1\leq i\leq r} f_i X_i \neq \emptyset\}$, we find

$$X_0 \subset F \cdot \bigcap_{1 \leq i \leq n} X_i^{-1} X_i. \qquad \square$$

Lemma 2.3. *Let $K_1, \dots, K_r$ be positive integers, and take a K_i-approximate subgroup Λ_i of G for all $1 \leq i \leq r$. We have*

1. *$\bigcap_{1\leq i\leq r} \Lambda_i^2$ is a $K_1^3 \cdots K_r^3$- approximate subgroup;*
2. *if $(\Xi_i)_{1\leq i\leq r}$ is a family of approximate subgroups with Ξ_i commensurable to Λ_i for all $1 \leq i \leq r$, then $\bigcap_{1\leq i\leq r} \Lambda_i^2$ and $\bigcap_{1\leq i\leq r} \Xi_i^2$ are commensurable.*

Proof. We know that Λ_i^4 is covered by K_i^3 left-translates of Λ_i for all $1 \leq i \leq r$. So (1) is a consequence of Lemma 2.2 applied to $X_0 = (\bigcap_{1\leq i\leq r} \Lambda_i)^4$ and $X_1 = \Lambda_1, \dots, X_r = \Lambda_r$. To prove (2), it suffices to show that $\bigcap_{1\leq i\leq r} \Lambda_i^2$ is covered by finitely many translates of $\bigcap_{1\leq i\leq r} \Xi_i^2$ by symmetry. Statement (2) is then a consequence of Lemma 2.2 applied to $X_0 = \bigcap_{1\leq i\leq r} \Lambda_i^2$ and $\Xi_1, \dots, \Xi_r$. □

Lemma 2.4. *Let Λ_1 and Λ_2 be two commensurable approximate subgroups of a group G. Let $\phi : H \to G$ be a group homomorphism. Then $\phi^{-1}(\Lambda_1^2)$ and $\phi^{-1}(\Lambda_2^2)$ are commensurable approximate subgroups of H.*

Proof. By Lemma 2.3, the subsets $\phi(H)\cap\Lambda_1^2$ and $\phi(H)\cap\Lambda_2^2$ are commensurable approximate subgroups. Taking $\{i, j\} \subset \{1, 2\}$, we can find a finite subset $F_{ij} \subset H$ such that $(\phi(H)\cap\Lambda_i^2)^2 \subset \phi(F_{ij})\left(\phi(H) \cap \Lambda_j^2\right)$. In other words,

$$\phi^{-1}(\Lambda_i^2)^2 \subset F_{ij}\phi^{-1}(\Lambda_j^2).$$

So $\phi^{-1}(\Lambda_1^2)$ and $\phi^{-1}(\Lambda_2^2)$ are commensurable approximate subgroups. □

2.3. Approximate lattices and cut-and-project schemes

First, recall the definition of approximate lattices:

Definition 2.5 (Approximate lattices, A.2, [Hru22]). Let Λ be an approximate subgroup of a locally compact group G. We say that Λ is an approximate lattice if

(i) Λ is uniformly discrete (i.e., $\Lambda^2 \cap W = \{e\}$ for some neighbourhood of the identity $W \subset G$);
(ii) there is $\mathcal{F} \subset G$ measurable of finite Haar measure such that $\Lambda\mathcal{F} = G$.

Recall that a *uniform approximate lattice* is a uniformly discrete approximate subgroup of G such that there exists a compact subset $K \subset G$ with $\Lambda K = G$. So uniform approximate lattices are approximate lattices, but there are approximate lattices that are not uniform.

In the above definition as well as in the rest of this paper, we use ‘uniformly discrete’ but are implicitly considering a notion that would be better coined as ‘left-uniform discreteness’. When considering symmetric subsets, however, the notions of ‘left-uniform discreteness’ and ‘right-uniform discreteness’ are equivalent. This applies in particular to approximate subgroups that are uniformly discrete.

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Remark moreover here that if Λ is a uniformly discrete approximate subgroup of some locally compact groups, all its powers Λ^n are also uniformly discrete approximate subgroups. As a useful consequence, the powers Λ^n of an approximate lattice Λ are also approximate lattices.

We recall now the definition of a cut-and-project scheme:

Definition 2.6 (Definitions 2.11 and 2.12, [BH18]). A *cut-and-project scheme* is a triple (G, H, Γ) consisting of two locally compact groups G and H and a lattice Γ in $G \times H$ which projects injectively to G and densely to H. For any symmetric relatively compact neighbourhood of the identity $W_0 \subset H$, we define the *model set*

$$P_0(G, H, \Gamma, W_0) := p_G((G \times W_0) \cap \Gamma) \subset G,$$

where $p_G : G \times H \to G$ denotes the natural projection.

It was shown in [BH18, Hru22] that model sets are approximate lattices and that they are uniform if and only if the lattice Γ they are associated with is uniform.

3. Good models: definition, first properties and examples

In this section, we investigate elementary properties of good models. We will prove in particular Proposition 1.2, Proposition 3.6 and Theorem 3.13.

3.1. About the definition of good models

Let us recall the definition of good models:

Definition. Let Λ be an approximate subgroup of a group Γ. A group homomorphism $f : \Gamma \to H$ with target a locally compact group H is called a *good model (of (Λ, Γ))* if

1. $f(\Lambda)$ is relatively compact;
2. there is $U \subset H$ a neighbourhood of the identity such that $f^{-1}(U) \subset \Lambda$.

Remark 3.1. Restricting the range of the good model f, we can always assume that f has dense image.

Definition 1.1 involves both the choice of a map f and an open subset U. However, up to commensurability, the choice of U does not matter as the following shows:

Lemma 3.2. *Let H be a locally compact group, Γ be a discrete group, V_1 and V_2 be symmetric relatively compact neighbourhoods of the identity in H and $f : \Gamma \to H$ be a group homomorphism. The subsets $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are commensurable approximate subgroups.*

Proof. Take $i, j \in \{1, 2\}$. The identity belongs to the interior $\operatorname{int}(V_j)$ of V_j so

$$V_i^2 \subset \bigcup_{h \in V_i^2} h \operatorname{int}(V_j).$$

But $\operatorname{int}(V_j)$ is open and V_i^2 is relatively compact, and thus, there is a finite subset $F_{ij} \subset V_i^2$ such that $V_i^2 \subset F_{ij} U$. Since V_1 and V_2 are moreover symmetric subsets, we have that V_1 and V_2 are commensurable approximate subgroups. Choose now a symmetric open neighbourhood of the identity W such that W^2 is contained in V_1 and V_2. Then $f^{-1}(W^2)$, $f^{-1}(V_1^2)$ and $f^{-1}(V_2^2)$ are commensurable approximate subgroups by Lemma 2.3. But for $i = 1, 2$, we have $f^{-1}(W^2) \subset f^{-1}(V_i) \subset f^{-1}(V_i^2)$. So $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are commensurable approximate subgroups. □

Corollary 3.3. *Let Λ be an approximate subgroup of a (discrete) group Γ and $f : \Gamma \to H$ be a good model of (Λ, Γ). If $U \subset H$ is a symmetric relatively compact neighbourhood of the identity, then $f^{-1}(U)$ is an approximate subgroup commensurable to Λ.*

Admitting a good model is a property that is stable under group homomorphisms.

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Lemma 3.4. *Let Λ be an approximate subgroup of a group Γ. Suppose that (Λ, Γ) has a good model. We have*

1. *if $\phi_1 : \Gamma_1 \to \Gamma$ is a group homomorphism, then $\phi_1^{-1}(\Lambda)$ is an approximate subgroup and $(\phi_1^{-1}(\Lambda), \Gamma_1)$ has a good model;*
2. *if $\phi_2 : \Gamma \to \Gamma_2$ is a group homomorphism, then $(\phi_2(\Lambda), \phi_2(\Gamma))$ has a good model.*

Proof. Let $f : \Gamma \to H$ be a good model of (Λ, Γ), and let $U \subset H$ be an open subset as in Definition 1.1. Set furthermore $\Lambda_1 := \phi_1^{-1}(\Lambda)$ and $f_1 := f \circ \phi_1$. Then $f_1(\Lambda_1) = f(\Lambda)$ is relatively compact and $f_1^{-1}(U) \subset \Lambda_1$. Hence, Λ_1 is an approximate subgroup by Lemma 3.2 and f_1 is a good model of (Λ_1, Γ_1). Let us now prove (2). Take a good model $f : \Gamma \to H$ of (Λ, Γ) with dense image and $U \subset H$ a symmetric neighbourhood of the identity such that $f^{-1}(U^2) \subset \Lambda$. Define $N := \overline{f(\ker(\phi_2))}$, which is a normal subgroup since $f(\Gamma)$ is dense. Now

$$(p_{H/N} \circ f)^{-1}(p_{H/N}(U)) \subset f^{-1}(U^2 f(\ker(\phi_2))) \subset f^{-1}(U^2)\ker(\phi_2) \subset \Lambda \ker(\phi_2),$$

where $p_{H/N} : H \to H/N$ denotes the natural projection. Therefore, the obvious map $\phi_2(\Gamma) \to H/N$ is a good model of $(\phi_2(\Lambda), \phi_2(\Gamma))$. □

3.2. Group-theoretic characterisation of good models

We will prove the following detailed version of Proposition 3.6:

Theorem 3.5. *Let Λ be an approximate subgroup of a group Γ. The following are equivalent:*

(1) *there is a good model $f : \Gamma \to H$ of (Λ, Γ);*
(2) *there exists a sequence $(\Lambda_n)_{n\geq 0}$ of approximate subgroups such that*
   (a) *$\Lambda_0 = \Lambda$;*
   (b) *for all integers $n \geq 0$ and all $\gamma \in \Gamma$, the approximate subgroups $\gamma\Lambda_n\gamma^{-1}$ and Λ are commensurable;*
   (c) *for all integers $n \geq 0$, we have $\Lambda_{n+1}^2 \subset \Lambda_n$;*
(3) *there exists a family of subsets $\mathcal{B}$ such that*
   (a) *there is $\Xi \in \mathcal{B}$ with $\Xi \subset \Lambda$;*
   (b) *all elements of $\mathcal{B}$ contain e and are commensurable to Λ;*
   (c) *for all $\Lambda_1 \in \mathcal{B}$ and $\gamma \in \Gamma$, there is $\Lambda_2 \in \mathcal{B}$ with $\gamma\Lambda_2^{-1}\Lambda_2\gamma^{-1} \subset \Lambda_1$.*

Moreover, when any of the three statements above is satisfied,

(4) *with $\mathcal{B}$ as in (3), we can choose a good model $f : \Gamma \to H$ such that f has dense image and $\mathcal{B}$ is a neighbourhood basis for the identity with respect to the initial topology on Γ given by f;*
(5) *there is a good model $f_0 : \Gamma \to H_0$ of (Λ, Γ) such that for any other good model $f : \Gamma \to H$ of (Λ, Γ), we have a continuous group homomorphism $\phi : H_0 \to H$ with compact kernel such that $f = \phi \circ f_0$;*
(6) *if Λ is a K-approximate subgroup, then there exists a sequence $(\Lambda_n)_{n\geq 0}$ with $\Lambda_0 = \Lambda^8$ and as in (2) such that Λ is covered by $C_{K,n}$ left-translates of Λ_n for all $n \geq 0$, where $C_{K,n}$ is an integer that depends on K and n only.*

Condition 2(b) above can be rephrased as saying that Λ_n and Λ are commensurable and that $\Gamma \subset \mathrm{Comm}_G(\Lambda)$. This reformulation makes clear that Theorem 3.5 shows that Λ has a good model if and only if $(\Lambda, \mathrm{Comm}_G(\Lambda))$ has a good model.

As mentioned in the introduction, Theorem 3.5 is folklore and well known to the expert. This is specially true in model theory, and the more common approach goes through elementary model-theoretic tools (see [Hru12, Lem. 3.3], [MW15] and [BGT12, Lem. 6.6] for a somewhat elementary approach). The first step is to embed Λ in a 'sufficiently saturated elementary extension' $\underline{\Gamma}$ – for instance, by means of an ultra-power of Λ over a sufficiently large ultra-filter. Then one can obtain a good model

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by quotienting out a normal subgroup naturally associated with $(\Lambda_n)_{n\neq 0}$ and equip this quotient with the logic topology. This provides a sleek construction and highlights how extra structure on Λ impacts the structure of the good model.

Our goal here is to provide an elementary proof of Theorem 3.5. Indeed, it is already essentially contained in the work of Weil on completion of uniform structures. Moreover, the form presented here will be more adapted to our discussion about regularity and continuity later on (Proposition 5.7). We also hope that this will be of use to mathematicians outside model theory.

Proof. (1) $\Rightarrow$ (2):

Choose a neighbourhood of the identity $U \subset H$ such that $f^{-1}(U) \subset \Lambda$. There exists a sequence $(U_n)_{n>0}$ of relatively compact symmetric neighbourhoods of the identity in H such that $U_0 = U$ and $U_{n+1}^2 \subset U_n$ for all integers $n \geq 0$. Define now $(\Lambda_n)_{n\geq 0}$ by $\Lambda_0 = \Lambda$ and $\Lambda_n = f^{-1}(U_n)$. We readily check that for all integers $n \geq 0$, we have $\Lambda_{n+1}^2 \subset \Lambda_n$. Furthermore, for all $\gamma \in \Gamma$, $\gamma\Lambda_n\gamma^{-1}$ is an approximate subgroup commensurable to Λ by Corollary 3.3. So (1) $\Rightarrow$ (2) is proved.

(2) $\Rightarrow$ (3):

Let $(\Lambda_n)_{n\geq 0}$ be as in (2). For any two subsets $X, Y \subset G$, define

$$X^Y := \bigcap_{y\in Y} yXy^{-1}.$$

Define now $\mathcal{B}$ by

$$\mathcal{B} := \left\{ \left(\Lambda_n^2\right)^F \mid n \in \mathbb{N}, F \subset \Gamma, |F| < \infty \right\}.$$

We know that $\Lambda_1^2 \subset \Lambda$ and that for all $\Xi \in \mathcal{B}$, we have $e \in \Xi$, and Ξ is an approximate subgroup commensurable to Λ (Lemma 2.3). Now, for all $n \in \mathbb{N}$ and $F \subset \Gamma$ finite, we have

$$\left(\left(\Lambda_{n+1}^2\right)^F\right)^2 \subset \left(\Lambda_{n+1}^4\right)^F \subset \left(\Lambda_n^2\right)^F,$$

and for $\gamma \in \Gamma$, we find

$$\gamma\left(\Lambda_n^2\right)^F\gamma^{-1} \subset \left(\Lambda_n^2\right)^{\gamma F}.$$

So $\mathcal{B}$ satisfies (3).

(3) $\Rightarrow$ (1):

Equip the group $\langle\Lambda\rangle$ with the topology defined by

$$\mathcal{T} = \{U \subset \Gamma | \forall \gamma \in U, \exists \Xi \in \mathcal{B}, \gamma\Xi \subset U\}.$$

By [Bou89b, Ch. III, §1.2, Proposition 1], the topology $\mathcal{T}$ is the unique topology that makes G into a topological group and such that $\mathcal{B}$ is a neighbourhood basis for e. Now, the closure $\overline{\{e\}}$ of the identity is a closed normal subgroup, and the group $\Gamma/\overline{\{e\}}$ equipped with the quotient topology is the maximal Hausdorff factor of Γ. Let $p : \Gamma \to \Gamma/\overline{\{e\}}$ be the natural map. Then $\{p(\Xi)|\Xi \in \mathcal{B}\}$ is a neighbourhood basis for the identity in $\Gamma/\overline{\{e\}}$. But the subsets that belong to $\mathcal{B}$ are pairwise commensurable, so the neighbourhoods $\{p(\Xi)|\Xi \in \mathcal{B}\}$ are pre-compact. Hence, the topological group $\Gamma/\overline{\{e\}}$ has a completion by [Bou89b, Ch. III, Ex. §3, Ex. 7,8]. In other words, there is a locally compact group H and a group homomorphism

$$i : \Gamma/\overline{\{e\}} \to H$$

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such that i has dense image and is a homeomorphism onto its image. Define the continuous map $f := i \circ p$. We will show that f is a good model. The group H is a complete space, and Λ is pre-compact in the topology $\mathcal{T}$ according to assumption (b). So $f(\Lambda)$ is a relatively compact subset of H. Recall that i is a homeomorphism onto its image, the map p is open and $\mathcal{B}$ is a neighbourhood basis for the identity. There is thus a neighbourhood of the identity $U \subset H$ such that $f^{-1}(U) \subset \Lambda$ according to assumption (a) and (c).

Statement (4) is straightforward from the proof of (3) $\Rightarrow$ (1). Let us prove (5). Let $\mathcal{T}_0$ denote the initial topology on Γ with respect to the class of all good models $f : \Gamma \to H$ of (Λ, Γ). In other words, the topology $\mathcal{T}_0$ is generated by the family of subsets $\{f^{-1}(U)\}_{f,U}$ where $f : \Gamma \to H$ and U run through all good models of (Λ, Γ) and all open subsets $U \subset H$. Define $\mathcal{B}_0$ as $\{U \in \mathcal{T}_0 | e \in U \subset \Lambda\}$. Since Λ is assumed to have a good model, we know that $\mathcal{B}_0$ is not empty. So take $\Xi \in \mathcal{B}_0$. Then there are good models $(f_i : \Gamma \to H_i)_{1 \le i \le r}$ of (Λ, Γ) and open relatively compact neighbourhoods of the identity $U_i \subset H_i$ for all $1 \le i \le r$ such that

$$\bigcap_{1 \le i \le r} f_i^{-1}(U_i) \subset \Xi \subset \Lambda.$$

But the map $f := (f_i)_{1 \le i \le r} : \Gamma \to \prod_{1 \le i \le r} H_i$ is a good model, so Corollary 3.3 implies that Λ is commensurable to $\bigcap_{1 \le i \le r} f_i^{-1}(U_i)$, and hence to Ξ. So $\mathcal{B}_0$ satisfies conditions (a) and (b) of (3). But condition (c) of (3) is also satisfied since $(G, \mathcal{T}_0)$ is a topological group. Indeed, a group with an initial topology given by group homomorphisms to topological groups is a topological group. Let now $f_0 : \Gamma \to H_0$ be as in the proof of (3) $\Rightarrow$ (1). Then every good model $f : \Gamma \to H$ of (Λ, Γ) is continuous with respect to $\mathcal{T}_0$. According to the universal properties of quotients and completions (see [Bou89b, Ch. III, §3.4, Prop. 8]), one can therefore find a continuous group homomorphism $\phi : H_0 \to H$ such that $\phi \circ f_0 = f$.

We now prove (6). Take a good model $f : \Gamma \to H$ of (Λ, Γ) with dense image. We can find a relatively compact open symmetric neighbourhood of the identity $U \subset H$ such that $\Lambda \subset f^{-1}(U) \subset \Lambda^2$. So $f^{-1}(U)$ is a K^3-approximate subgroup, and hence, U is a K^3-approximate subgroup as well. But by Lemma 5.8 we can find an symmetric open neighbourhood of the identity $S \subset U^4$ such that S^8 is contained in U^4 and C_K left-translates of S cover U for some constant C_K that depends on K only. We thus define $\Lambda_1 = f^{-1}(S)$ and $C_{K,1} = C_K$. A proof by induction on n then shows (6). □

One of the key takeaways can be summarized as follows:

Proposition 3.6. *Let Λ be an approximate subgroup of a group G. There exists a good model f of Λ if and only if there is a sequence of approximate subgroups $(\Lambda_n)_{n \ge 0}$ such that $\Lambda_0 = \Lambda$, and for all integers $n \ge 0$, we have $\Lambda_{n+1}^2 \subset \Lambda_n$ and Λ_n commensurable to Λ.*

Proof. Proposition 3.6 is the equivalence ‘(1) $\Leftrightarrow$ (2)’ in Theorem 3.5. □

Remark 3.7. We note here that, similarly, some results presented in Section 5 were foreshadowed by Weil’s work on group topologies. We point, for instance, to the striking similarities between the work of Weil on completions of measurable groups and the method of Sanders and Croot–Sisask presented below.

As a consequence of Theorem 3.5, we show that Meyer subsets almost have a good model:

Proposition 3.8. *Let Λ be an approximate subgroup of some group. If Λ is a Meyer subset, then there is a positive integer such that Λ^n has a good model.*

Proof. Note first that if $\Gamma = \langle \Lambda \rangle$, then Γ commensurates Λ. In this situation, 2.(b) of Theorem 3.5 becomes equivalent to the approximate subgroups Λ_n and Λ being commensurable for all $n \ge 0$; see the remark immediately after Theorem 3.5.

Let Λ be a Meyer subset. By Theorem 3.5, there is a sequence $(\Lambda_n)_{n \ge 0}$ of approximate subgroups commensurable to Λ such that $\Lambda_{n+1}^2 \subset \Lambda_n$ for all integers $n \ge 0$. The union of two approximate

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subgroups commensurable with Λ is also an approximate subgroup commensurable with Λ. Thus, applying Theorem 3.5 to the approximate subgroup $\Lambda \cup \Lambda_0$ together with the sequence $(\Lambda_n)_{n\geq 0}$ implies that $\Lambda \cup \Lambda_0$ has a good model, and so has $(\Lambda \cup \Lambda_0) \cap \langle \Lambda \rangle$ by Lemma 3.4. But $(\Lambda \cup \Lambda_0) \cap \langle \Lambda \rangle$ is commensurable to Λ, and $\langle \Lambda \rangle$ is generated by Λ. So there is a positive integer n such that $(\Lambda \cup \Lambda_0) \cap \langle \Lambda \rangle$ is contained in Λ^n. □

This last result prompts the following question:

Question 1. With notations as in Proposition 3.8, can n be chosen independently of Λ?

3.3. Universal properties and compatibility with limits

Approaching Theorem 3.5 via saturated elementary extensions highlights that, to some extent, good models correspond to a certain notion of quotients of groups by approximate subgroups (see the remark below Theorem 3.5). The elementary method presented above too enables us to build a good model that satisfies a quotient-like universal property. We think of this good model as a 'maximal' or 'initial' good model. We also identify a type of 'minimal' or 'final' good model.

Proposition 3.9. *Let Λ be an approximate subgroup of a group G, and let $\Gamma \subset G$ be a subgroup containing Λ such that (Λ, Γ) has a good model. We have*

1. *if $f_0 : \Gamma \to H_0$ is as in part (5) of Theorem 3.5, then any group homomorphism $g : \Gamma \to L$ with target a topological group and such that $g(\Lambda)$ is relatively compact factors through f_0 (i.e., there exists a continuous group homomorphism $h : H_0 \to L$ such that $h \circ f_0 = g$);*
2. *there is an approximate subgroup $\Xi \subset G$ commensurable to Λ and a good model $f : \langle \Xi \rangle \to H$ of Ξ with dense image and target a connected Lie group without nontrivial normal compact subgroups. Such a group H is unique up to continuous isomorphisms.*

Remark 3.10. The above constructions are not original and widely known in model theory; see, for instance, [HKP22] and references therein. One can note that f_0 enjoys a universal property similar to the Bohr compactification of a subgroup (i.e., the maximal group compactification of a given subgroup), and, indeed, if $\Lambda = \Gamma$, then H_0 is the Bohr compactification of Λ. The comparison with the Bohr compactification is mentioned in the introduction of [HKP22], and we mention here the work of Krupiński [Kru17]. Note finally that the model theoretic approach also offers universality statements stronger than (1).

The construction in (2) was also already thoroughly studied in [Hru12] where its canonicity was established – providing a much stronger uniqueness statement. We also note the related work of Fanlo [Fan23] and indicate that a polylogarithmic bound on the dimension is known by [JTZ23].

Proof. Take $\mathcal{B}_L$ a neighbourhood basis for the identity in L made of symmetric subsets, and take any $\mathcal{B}_\Lambda$ as in part (3) of Theorem 3.5. Then as a consequence of Lemma 2.2, we know that $\mathcal{B} := \{\Xi \cap g^{-1}(U) | \Xi \in \mathcal{B}_\Lambda, U \in \mathcal{B}_L\}$ satisfies the assumptions of part (3) of Theorem 3.5. Now by part (4) of Theorem 3.5 and by the universal properties of completions and quotients, we can build a good model $f : \Gamma \to H$ such that g factors through f. But according to part (5) of Theorem 3.5, we know that g factors through f_0 – proving (1).

Take $f : \langle \Lambda \rangle \to H$ a good model of Λ with dense image. By the Gleason–Yamabe theorem, there are $O \subset H$ an open subgroup and a normal compact subgroup K of O such that O/K is a connected Lie group without nontrivial normal compact subgroup; see, for instance, [Hru12, §4.1], for detail. Then $\Lambda' := f^{-1}(UK)$, where $U \subset O$ is any symmetric compact neighbourhood of the identity, is an approximate subgroup commensurable to Λ according to Corollary 3.3. But the composition of $f_{|\Lambda'}$ and the natural projection $O \to O/L$ is easily checked to be a good model of Λ'. Now take Ξ and Ξ' approximate subgroups commensurable to Λ, and let $f : \langle \Xi \rangle \to H$ and $f' : \langle \Xi' \rangle \to H'$ be good models of Ξ and Ξ', respectively. Suppose moreover that both satisfy (2). Let $D : \langle \Xi^2 \cap \Xi'^2 \rangle \to H \times H'$ be the diagonal map, and let Δ be the closure of its image. We want to show that $\Delta \cap (H \times \{e\})$ and

$\Delta \cap (\{e\} \times H')$ are compact subgroups. Let U denote a relatively compact open neighbourhood of the identity in Δ. Then $D(\langle \Xi^2 \cap \Xi'^2 \rangle) \cap U(\Delta \cap (\{e\} \times H'))$ is dense in $U(\Delta \cap (\{e\} \times H'))$. But the projection of $U(\Delta \cap (\{e\} \times H'))$ to H is a relatively compact set U_H. So $D^{-1}(U(\Delta \cap (\{e\} \times H')))$ is contained in $f^{-1}(U_H)$ which is covered by finitely many translates of Ξ (Corollary 3.3) and, by Lemma 2.2, is covered by finitely many translates of $\Xi^2 \cap \Xi'^2$. So $D(f^{-1}(U_H))$ is covered by finitely many translates of $D(\Xi^2 \cap \Xi'^2) \subset f(\Xi^2) \times f'(\Xi'^2)$ which is relatively compact. So $U(\Delta \cap (\{e\} \times H')) \subset \overline{D(f^{-1}(U_H))}$, which is compact, thus proving that $\Delta \cap (\{e\} \times H')$ is compact. A symmetric argument shows that $\Delta \cap (H \times \{e\})$ is compact. Moreover, $\Xi^2 \cap \Xi'^2$ is commensurable to both Ξ and Ξ' by Lemma 2.2. So $p_H(\overline{D(\Xi^2 \cap \Xi'^2)})$ contains $\overline{f(\Xi^2 \cap \Xi'^2)}$, which contains an open subset by the Baire category theorem. Therefore, Δ projects surjectively to H by connectedness. Likewise, the projection of Δ to H' is surjective. So $\Delta \cap (H \times \{e\})$ and $\Delta \cap (\{e\} \times H')$ are compact normal subgroups of H and H', respectively, which means that they are trivial. As a consequence, Δ is the graph of a continuous isomorphism $\phi : H \longrightarrow H'$ such that $f'_{|\langle \Xi^2 \cap \Xi'^2 \rangle} = \phi \circ f_{|\langle \Xi^2 \cap \Xi'^2 \rangle}$. This proves (2). □

We will show later on that, upon dropping the requirement that f_0 be a good model, part (1) of Proposition 3.9 can be extended to all approximate subgroups. See Proposition 4.7 below.

A key point in both [BGT12] and [Hru12] consists in utilising ultraproducts of approximate subgroups to study all members of a family of approximate subgroups at once. Notably, they showed that ultraproducts of finite approximate subgroups have a good model, as this enabled them to use features of Lie groups and locally compact groups (such as tools developed by Gleason, Yamabe and others in the resolution of Hilbert's fifth problem) to tackle problems about finite approximate subgroups. In a similar fashion, ultraproducts of locally compact approximate subgroups (i.e., compact symmetric neighbourhoods of the identity) were shown to have a good model in [Car15] in order to obtain uniform and quantitative – although non-effective – versions of the Gleason–Yamabe structure theorem for locally compact groups. We show that, in general, the property *to have a good model* is stable under ultrapoducts. The proof of this fact is not surprising and follows the idea of [BGT12] building upon the Sanders and Croot–Sisask arguments.

Proposition 3.11. *Let $(\Lambda_i)_{i\in I}$ be a family of K-approximate subgroups of the groups $(\Gamma_i)_{i\in I}$ for some fixed integer K.*

(1) *([BGT12, App. A]) the ultraproduct $\underline{\Lambda} := \prod_{i\in I} \Lambda_i/\mathcal{U}$ is an approximate subgroup for any ultrafilter $\mathcal{U}$ over I;*
(2) *if moreover Λ_i has a good model for all $i \in I$, then $\underline{\Lambda}^8$ has a good model.*

Suppose that there are a directed order $\leq$ on I and injective group homomorphisms $\psi_{ij} : \Gamma_i \longrightarrow \Gamma_j$ for all $i \leq j$ such that $\psi_{jk} \circ \psi_{ij} = \psi_{ik}$ for all $i \leq j \leq k$. And suppose moreover that $\psi_{ij}(\Lambda_i) \subset \Lambda_j$ whenever $i \leq j$. Then

(3) *the direct limit $\varinjlim_I \Lambda_i^2$ is an approximate subgroup;*
(4) *if moreover Λ_i has a good model for all $i \in I$, then $\varinjlim_I \Lambda_i^8$ has a good model.*

In the second part of Proposition 3.11, no commensurability assumption is required for the approximate subgroups Λ_j and $\psi_{ij}(\Lambda_i)$. The lack of such an assumption appears to be very surprising at first glance. We note moreover that Proposition 3.11 is original and has not been studied, to the knowledge of the author, from the model-theoretic point-of-view.

Proof. If Λ_i has a good model for all $i \in I$, then there are constants $(C_{K,n})_{n\geq 0}$ and sequences $(\Lambda_{i,n})_{n\geq 0}$ as in part (6) of Theorem 3.5. But then for any ultrafilter $\mathcal{U}$ over I, we know that $\underline{\Lambda}^8$ is covered by $C_{K,n}$ left-translates of $\underline{\Lambda_n} := \prod_{i\in I} \Lambda_{i,n}/\mathcal{U}$ for all $n \geq 0$ (see, for example, [BGT12, App. A] for background material on ultraproducts of groups). So $(\underline{\Lambda_n})_{n\geq 0}$ satisfies part (2) of Theorem 3.5, and $\underline{\Lambda_0} = \underline{\Lambda}^8$ has a good model.

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Let $\mathcal{U}$ be an ultrafilter on I that contains the subsets $\{i \in I | j \leq i\}$ for all $j \in I$. Such an ultrafilter exists because this family has the finite intersection property (note moreover that $\mathcal{U}$ may be principal if I contains a final element). Write $\underline{\Lambda}$ and $\underline{\Gamma}$ for the ultraproducts of $(\Lambda_i)_{i \in I}$ and $(\Gamma_i)_{i \in I}$ over $\mathcal{U}$, respectively. By the universal property of direct limits, there is a natural map $\phi : \varinjlim_I \Gamma_i \to \underline{\Gamma}$, and we compute that $\phi^{-1}(\underline{\Lambda}) = \varinjlim_I \Lambda_i$. So $\varinjlim_I \Lambda_i^2$ is an approximate subgroup by Lemma 2.4. If every Λ_i has a good model, then the approximate subgroup $\underline{\Lambda}^8$ has a good model by (2). So $\varinjlim_I \Lambda_i^8$ has a good model by Lemma 3.4. □

One may wonder if a converse to part (2) (and (4)) of Proposition 3.11 holds. The next section will give rise to an interesting counterexample.

Lemma 3.12. *There is a sequence $(\Lambda_n)_{n \geq 0}$ of approximate subgroups such that $\prod_{n \geq 0} \Lambda_n / \mathcal{U}$ has a good model, but for all $n \geq 0$, Λ_n is not a Meyer subset, where $\mathcal{U}$ is any non-principal ultrafilter on $\mathbb{N}$.*

Lemma 3.12 is certainly well known – or, at the very least, not surprising – to the expert, but we could not locate a reference. We delay the proof to the end of the next subsection.

3.4. An approximate subgroup without a good model

The main goal of this section is to prove the existence of approximate subgroups that are not Meyer subsets:

Theorem 3.13. *Let F_2 be the free group over two generators a and b. For any two reduced words $w, x \in F_2$, define $o(x, w)$ as the number of occurrences of w in x. Then for any $w \in F_2 \backslash \{a, b, a^{-1}, b^{-1}, e\}$ of length l, the set*

$$\{g \in F_2 : |o(g, w) - o(g, w^{-1})| \leq 3l\}$$

is an approximate subgroup but not a Meyer subset.

Recall that a *quasi-morphism* of a group G is a map $f : G \to \mathbb{R}$ such that

$$C(f) := \sup_{g_1, g_2 \in G} |f(g_1 g_2) - f(g_1) - f(g_2)| < \infty.$$

We say that f is *symmetric* if for all $g \in G$, we have $f(g^{-1}) = -f(g)$, and that it is *homogeneous* if for all $n \in \mathbb{Z}$ and $g \in G$, we have $f(g^n) = nf(g)$; see, for instance, [Kot04] for background on quasi-morphisms. Just like group homomorphisms, quasi-morphisms give rise to families of approximate subgroups. The approximate subgroups produced that way are often called *quasi-kernels*.

Lemma 3.14. *Let G be a group and $f : G \to \mathbb{R}$ be a symmetric quasi-morphism. Then for all $R > C(f)$, the set $f^{-1}([-R; R])$ is a $2\frac{2R+C(f)}{R-C(f)} + 1$ -approximate subgroup.*

Proof. Let Λ denote the set $f^{-1}([-R; R])$. Then Λ is symmetric since f is symmetric and the set $f(\Lambda^2)$ is contained in $[-2R - C(f); 2R + C(f)]$. Set $\delta := R - C(f) > 0$, and choose a finite subset $F \subset \Lambda^2$ with $|F| = |f(F)|$ such that $f(F)$ is a maximal δ-separated subset of $f(\Lambda^2)$. We know that $|F| \leq 2\frac{2R+C(f)}{\delta} + 1$, and, in addition, we have

$$f(\Lambda^2) \subset \bigcup_{\gamma \in F} f(\gamma) + [-\delta; \delta].$$

Take $\lambda \in \Lambda^2$ and $\gamma \in F$ such that $|f(\lambda) - f(\gamma)| \leq \delta$. We have

$$|f(\gamma^{-1}\lambda) - (f(\lambda) - f(\gamma))| \leq C(f),$$

so

$$|f(\gamma^{-1}\lambda)| \leq C(f) + \delta = R.$$

Hence, $\gamma^{-1}\lambda \in \Lambda$ and $\Lambda^2 \subset F\Lambda$. □

Since all bounded maps are quasi-morphisms, quasi-morphisms are often studied up to a bounded error. This gives an equivalence relation between quasi-morphisms that can be translated as a commensurability condition on quasi-kernels.

Lemma 3.15. *Let G be a group, $f_1, f_2 : G \to \mathbb{R}$ be two symmetric quasi-morphisms and $\Lambda_1 := f_1^{-1}([-R_1; R_1])$ and $\Lambda_2 := f_2^{-1}([-R_2; R_2])$ for $R_1 > C(f_1)$ and $R_2 > C(f_2)$. Then, if $\eta := \sup_{g\in G} |f_1(g) - f_2(g)| < \infty$, the approximate subgroups Λ_1 and Λ_2 are commensurable. More precisely, there is $F \subset G$ with $|F| \leq \max(\frac{2(R_1+\eta)}{R_2-C(f_2)}+1, \frac{2(R_2+\eta)}{R_1-C(f_1)}+1)$ such that $\Lambda_1 \subset F\Lambda_2$ and $\Lambda_2 \subset F\Lambda_1$.*

Conversely, if Λ_1 and Λ_2 are commensurable, then there is $\alpha \in \mathbb{R}$ nontrivial such that $f_1 - \alpha f_2$ is bounded. Furthermore, if f_1 and f_2 are homogeneous, then $f_1 = f_2\alpha$.

Proof. Write $\delta_1 := R_1 - C(f_1)$. Choose $F_1 \subset \Lambda_2$ with $|F_1| = |f_1(F_1)|$ such that $f_1(F_1)$ is a maximal δ_1-separated subset of $f_1(\Lambda_2)$. Since $f_1(\Lambda_2) \subset [-(R_2+\eta); R_2+\eta]$, we know that $|F_1| \leq \frac{2(R_2+\eta)}{\delta_1} + 1$. As in the proof of Lemma 3.14, we find that $\Lambda_2 \subset F_1\Lambda_1$. By symmetry, there is $F_2 \subset G$ with $|F_2| \leq \frac{2(R_1+\eta)}{R_2-C(f_2)}+1$ such that $\Lambda_1 \subset F_2\Lambda_2$.

Let us now prove the converse statement. Both f_1 and f_2 are within bounded distance of an homogeneous quasi-morphism [Kot04]. Therefore, by the first part of Lemma 3.15, we only have to prove the converse assuming that f_1 and f_2 are homogeneous. First of all, if $f_1 = 0$, then $\Lambda_2 = f_2^{-1}([-R_2; R_2])$ is commensurable to G. So $f_2(G)$ is bounded. But f_2 is homogeneous, so $f_2 = 0$. Suppose now that f_1 is nontrivial, and take $g_0 \in G$ such that $f_1(g_0) > 0$. Define that map

$$\hat{f} : g \mapsto f_1(g_0)f_2(g) - f_2(g_0)f_1(g).$$

It is a homogeneous quasi-morphism with $\hat{f}(g_0) = 0$. Moreover, any set commensurable to Λ_1 (equivalently, to Λ_2) has bounded image by $\hat{f}$. For all $g \in G$, there is $n \in \mathbb{Z}$ such that $|f_1(g) - nf_1(g_0)| \leq |f_1(g_0)|$. Thus,

$$G = \langle g_0 \rangle f_1^{-1}([-C(f_1) - f_1(g_0); f_1(g_0) + C(f_1)]).$$

But $f_1^{-1}([-C(f_1) - f_1(g_0); f_1(g_0) + C(f_1)])$ is commensurable to Λ_1 according to the first part of the proof and, hence, is mapped to a bounded set by $\hat{f}$. So $\hat{f}$ must have bounded image and, therefore, must be trivial. In other words, $f_2 = \frac{f_2(g_0)}{f_1(g_0)} f_1$. □

Our main result links properties of quasi-morphisms to whether the quasi-kernel is a Meyer subset or not.

Proposition 3.16. *Let G be a finitely generated group, and let $f : G \to \mathbb{R}$ be a homogeneous quasi-morphism. Choose a real number $R > C(f)$. If the approximate subgroup $f^{-1}([-R; R])$ is a Meyer subset, then f is a group homomorphism.*

Proof. If f is bounded, then $f = 0$. So assume that f is unbounded. Take $R' > C(f)$ such that $f^{-1}([-R'; R'])$ generates G. We know that $f^{-1}([-R'; R'])$ is an approximate subgroup (Lemma 3.14) and a Meyer subset (Lemma 3.15). By Proposition 3.8, there is an integer $n \geq 1$ such that there are a good model $f_0 : G \to H$ of some power, say n, of $f^{-1}([-R'; R'])$ with dense image. In particular, f_0 is a good model of $\Lambda := f^{-1}([-n(R' + C(f)); n(R' + C(f))])$ according to Lemma 3.15. Since $f_0(G)$ is dense in H, we have that $\overline{f_0(\Lambda)}$ is a neighbourhood of the identity. So the subgroup generated by the

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compact set $\overline{f_0(\Lambda)}$ is open (so clopen) and contains $f_0(G)$, and hence equals H. The group H is thus compactly generated. We will now show that $f = cf_0$ for some real number $c > 0$. We start with two claims:

Claim 3.4.1. The set of commutators $\{h_1h_2h_1^{-1}h_2^{-1} | h_1, h_2 \in H\}$ is relatively compact.

Proof. By Lemma 3.15, the approximate subgroup $f_0(f^{-1}([-3C(f); 3C(f)]))$ is commensurable to $f_0(\Lambda)$. So $K := \overline{f_0(f^{-1}([-3C(f); 3C(f)]))}$ is compact. Take now $\gamma_1, \gamma_2 \in G$. We have $|f(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1})| \leq 3C(f)$; hence, $f_0(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}) \in K$. But $f_0(G)$ is dense in H, so we find $\{h_1h_2h_1^{-1}h_2^{-1} | h_1, h_2 \in H\} \subset K$. □

Claim 3.4.2. Let $\gamma_0 \in G$ be such that $f(\gamma_0) > 0$. Then there is a compact subset $K \subset H$ such that for all $\gamma \in G$, we have $f_0(\gamma) \in \langle f_0(\gamma_0)\rangle K$.

Proof. Write $R_0 := f(\gamma_0)$. Then the subset $f^{-1}([-R_0 - C(f); R_0 + C(f)])$ is commensurable to Λ by Lemma 3.15. So the subset

$$K := \overline{f_0(f^{-1}([-R_0 - C(f); R_0 + C(f)]))}$$

is compact. But for any $\gamma \in G$, there is an integer l such that $|f(\gamma) - lf(\gamma_0)| \leq R_0$ so $f_0(\gamma_0)^{-l}f_0(\gamma) = f_0(\gamma_0^{-l}\gamma) \in K$. □

Now all conjugacy classes of H are relatively compact according to Claim 3.4.1. So we can find a compact normal subgroup $K \subset H$ and non-negative integers k, l such that $H/K \simeq \mathbb{R}^k \times \mathbb{Z}^l$ (see [GM71, Thm. 3.20]). We will show that $k + l \leq 1$. Let $p : H \to H/K$ denote the natural projection. Then the group homomorphism $p \circ f_0$ has dense image, and $p \circ f_0(\Lambda)$ is relatively compact. If H/K is compact, then $H/K \simeq \{e\}$. Otherwise, we can find $\gamma_0 \in G \setminus \Lambda$ so $f(\gamma_0) > 0$. According to Claim 3.4.2, every $\gamma \in G$ satisfies $p \circ f_0(\gamma) \in \langle p \circ f_0(\gamma_0)\rangle L$ where L is some compact subset of H/K. Therefore, $\langle p \circ f_0(\gamma_0)\rangle$ is an infinite cyclic co-compact subgroup, and hence, $k + l \leq 1$. Choose a neighbourhood $U \subset H$ of the identity such that $f_0^{-1}(U) \subset \Lambda$. Since K is a compact subgroup of H and the subgroup $f_0(G)$ is dense, we can find an integer $m \geq 0$ such that $K \subset f(\Lambda^m)U$. Then $V := p(U) \subset H/K$ is such that

$$f_0^{-1}(p^{-1}(V)) \subset f_0^{-1}(UK) \subset \Lambda^{m+2}.$$

So $p \circ f_0$ is a good model of Λ^{m+2} with image dense in $\mathbb{R}$ or $\mathbb{Z}$. By the converse of Lemma 3.15 $f = \alpha \cdot p \circ f_0$ for some $\alpha \in \mathbb{R}$, so f is a group homomorphism. □

Proof of Theorem 3.13. Recall that w is a reduced word of length l in F_2 the free group over $\{a, b\}$ and that $o(g, w)$ counts the occurrence of w as a reduced sub-word of g with overlap. Suppose that $w \notin \{a, b, a^{-1}, b^{-1}, e\}$. According to [Bro81, §3. (a)], the map

$$\begin{aligned} f_w : & F_2 \longrightarrow \mathbb{R} \\ & g \longmapsto o(g, w) - o(g, w^{-1}) \end{aligned}$$

is a symmetric quasi-morphism with $C(f_w) \leq 3(l - 1)$. Moreover, f_w is within distance δ of a unique homogeneous quasi-morphism, $\tilde{f}_w$ say, that is not a group homomorphism. But according to Lemma 3.14, the set

$$\left\{g \in F_2 | |o(g, w) - o(g, w^{-1})| \leq 3l\right\} = f_w^{-1}([-3l; 3l)])$$

is an approximate subgroup. Moreover, by Lemma 3.15, it is commensurable to $\tilde{f}_w^{-1}([-C(\tilde{f}_w) - \delta; C(\tilde{f}_w) + \delta])$. So if $\left\{g \in F_2 \middle| |o(g, w) - o(g, w^{-1})| \leq 3l\right\}$ is a Meyer subset, then $\tilde{f}_w^{-1}([-C(\tilde{f}_w) - \delta; C(\tilde{f}_w) + \delta])$ is a Meyer subset, and hence, $\tilde{f}_w$ is a group homomorphism according to Proposition 3.16 – contradiction. □

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At first glance, Theorem 3.13 seems to contradict the conjecture from [MW15, p. 57] stating 'even without the definable amenability assumption a suitable Lie model exists'. It is interesting to note that another example going in that direction is given in [HKP22] and that it is also built thanks to quasi-morphisms. Theorem 3.13, however, only refutes a naive interpretation of the conjecture from [MW15, p. 57]. Indeed, Hrushovski shows in [Hru22] that one can always construct such a model using a combination of both homomorphisms and quasi-homomorphisms. This provides a positive answer to the conjecture from [MW15, p. 57] and indicates that the 'suitable Lie model' should be interpreted in the sense of quasi-homomorphisms.

Finally, we give a proof of Lemma 3.12:

Proof of Lemma 3.12. Let w be a reduced word of length l in F_2 the free group over $\{a, b\}$, and suppose that $w \notin \{a, b, a^{-1}, b^{-1}, e\}$. Let f_w be as in the proof of Theorem 3.13. Define $\Lambda_n := f_w^{-1}([-3^n l; 3^n l])$ for all $n > 0$. Then $\Lambda_n^2 \subset \Lambda_{n+1}$, and K left-translates of Λ_n cover Λ_{n+1} for some K independent of n (Lemma 3.15). So the sequence of subsets $(\underline{\Lambda}_k)_{k\geq 0}$ defined by $\underline{\Lambda}_k := \prod_{n\geq 0} f_w^{-1}([-3^{n-k}l; 3^{n-k}l])/\mathcal{U}$ is made of well-defined approximate subgroups commensurable to $\prod_{n\geq 0} \Lambda_n/\mathcal{U}$ since $\mathcal{U}$ is non-principal (see, for example, [BGT12, App. A]). Besides, $\underline{\Lambda}_k^2 \subset \underline{\Lambda}_{k-1}$ for all $k \geq 1$ since $\mathcal{U}$ is non-principal. So $\prod_{n\geq 0} \Lambda_n/\mathcal{U}$ has a good model according to Theorem 3.5. However, for all $n \geq 0$, Λ_n is not a Meyer subset according to Theorem 3.13. □

3.5. Good models and cut-and-project schemes

In this section, we relate good models (Definition 1.1) to the non-commutative cut-and-project schemes (Definition 2.6). Note that when the ambient group is abelian, Meyer was the first to notice the striking link between cut-and-project schemes and some large approximate subgroups (see [Mey72] and [Sch73] for this and more).

Lemma 3.17. *Let Λ be a discrete approximate subgroup of a locally compact group G, and let Γ be a group that contains it. If (Λ, Γ) has a good model $f : \Gamma \to H$, then the graph of f defined by $\Gamma_f := \{(\gamma, f(\gamma)) | \gamma \in \Gamma\}$ is a discrete subgroup of $G \times H$.*

Proof. Choose a neighbourhood of the identity $U \subset H$ such that $f^{-1}(U) \subset \Lambda$ (Definition 1.1) and an open subset $V \subset G$ such that $V \cap \Lambda = \{e\}$. For $\gamma \in \Gamma$, we know that $(\gamma, f(\gamma)) \in V \times U$ implies $f(\gamma) \in U$; hence, $\gamma \in \Lambda$. But $\gamma \in V$, so $\gamma = e$, and we find $\Gamma_f \cap (V \times U) = \{e\}$. □

An easy consequence of Lemma 3.17, in the spirit of [BH18, Prop. 2.13, (iv)], asserts that the graph of a good model of an approximate subgroup Λ has finite co-volume as soon as Λ itself has finite co-volume.

Proposition 3.18. *Let Λ be an approximate lattice of a locally compact group G. Suppose that Λ has a good model $f : \Gamma \to H$ with dense image. The graph Γ_f of f is a lattice in $G \times H$, and Λ is contained in and commensurable to a model set contained in Λ^2.*

Proof. Let $\mathcal{F} \subset G$ be a measurable subset of finite Haar measure such that $\mathcal{F}\Lambda = G$ ([Hru22, Prop. A.2]). Define $W_0 := \overline{f(\Lambda)}$. Then Γ_f is discrete by Lemma 3.17. Moreover, we know that for all $(g, h) \in G \times H$, there is $\gamma_1 \in \Gamma_f$ such that $(g, h)\gamma_1^{-1} \in G \times W_0$. By assumption, we have $\gamma_2 \in \Gamma_f \cap (G \times W_0)$ such that $p_G((g, h)\gamma_1^{-1}\gamma_2^{-1}) \in \mathcal{F}$. Therefore, we know that $(g, h) \in \left(\mathcal{F} \times W_0 W_0^{-1}\right)\Gamma_f$. So Γ_f is a lattice in $G \times H$. Now, $\Lambda \subset P_0(G, H, \Gamma_f, W_0) \subset \Lambda^2$, which concludes. □

Proof of Proposition 1.2. Assume that Λ is a model set. Let (G, H, Γ) be a cut-and-project scheme, and let W_0 be a neighbourhood of the identity $W_0 \subset H$ such that $P_0(G, H, \Gamma, W_0) = \Lambda$ (Definition 2.6). Denote by $p_G : G \times H \to G$ and $p_H : G \times H \to H$ the natural projections. The map $(p_G)_{|\Gamma}$ is injective and

$$\Lambda = P_0(G, H, \Gamma, W_0) = p_G((G \times W_0) \cap \Gamma).$$

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But

$$p_G((G \times W_0) \cap \Gamma) = \left(p_H \circ (p_G)_{|\Gamma}^{-1}\right)^{-1}(W_0),$$

where we think of $(p_G)_{|\Gamma}$ as a bijective map from Γ to $p_G(\Gamma)$. We know that $\langle \Lambda \rangle \subset p_G(\Gamma)$, so set

$$\begin{aligned} \tau : \langle \Lambda \rangle &\longrightarrow H \\ \gamma &\longmapsto p_H \circ (p_G)_{|\Gamma}^{-1}(\gamma). \end{aligned}$$

Then τ is a group homomorphism, W_0 is a symmetric relatively compact neighbourhood of the identity and $\tau^{-1}(W_0) = \Lambda$. So τ is a good model of Λ.

Conversely, Λ is a Meyer subset. So there is $n \geq 1$ such that Λ^n has a good model by Proposition 3.8. Therefore, Λ^n – hence, Λ – is contained in and commensurable to a model set by Proposition 3.18. □

Remark 3.19. Note that the map τ introduced in the first part of the proof of Proposition 1.2 is well known in the abelian setting and is called the *star-map* (see, for instance, [BG13, §7.2]).

4. A closed-approximate-subgroup theorem

We give in this section a proof of Theorem 1.3 and investigate some applications.

4.1. Globalisation in Hausdorff topological groups

We start by proving a general form of Theorem 1.3.

Theorem 4.1. *Let Λ be a compact approximate subgroup of a Hausdorff topological group G and Γ a subgroup that contains Λ and commensurates it. There is a locally compact group H, an injective continuous group homomorphism $\phi : H \to G$ and a compact symmetric neighbourhood V of the identity in H such that $\phi(V) = \Lambda^2$ and $\phi(H) = \Gamma$.*

The key observation needed to prove Theorem 4.1 is the fact that locally a closed approximate subgroup behaves like a group.

Lemma 4.2. *Let Λ be a closed approximate subgroup of a locally compact group G, and let Ξ be a subset covered by finitely many left-translates of Λ. There is an open neighbourhood of the identity $U(\Xi) \subset G$ such that*

$$\Xi \cap U(\Xi) \subset \Lambda^2 \cap U(\Xi).$$

Proof. Choose a finite subset $F \subset G$ such that $\Xi \subset F\Lambda$. Define the open subset

$$U(\Xi) := G \setminus \left(\bigcup_{f \in F, f \notin \Lambda} f\Lambda \right).$$

Since $e \in f\Lambda$ implies $f \in \Lambda^{-1} = \Lambda$, the subset $U(\Xi)$ contains the identity. We thus have

$$\begin{aligned} U(\Xi) \cap \Xi &\subset U(\Xi) \cap F\Lambda \\ &\subset \bigcup_{f \in F, f \in \Lambda} f\Lambda \\ &\subset \Lambda^2. \end{aligned}$$ □

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Lemma 4.2 asserts that the restriction of the group operations of G to some neighbourhood of the identity in Λ gives rise to a structure of a *local topological group*. But it is well known that a local topological group Ω with a continuous embedding into a global group H can be globalised, and that the procedure gives rise to a topological group structure on the subgroup of H generated by the image of Ω (see [Gol10] for this and more on local groups).

Proof of Theorem 4.1. For all $\gamma \in \Gamma$, let $U(\gamma)$ be the neighbourhood of the identity such that $\gamma\Lambda^4\gamma^{-1} \cap U(\gamma) \subset \Lambda^2$ (Lemma 4.2). Choose a neighbourhood basis for the identity $\mathcal{B}$ made of closed subsets in G and define $\mathcal{B}_\Lambda$ as the family of subsets $\{\Lambda^2 \cap U^{-1}U | U \in \mathcal{B}\}$. The subsets in $\mathcal{B}_\Lambda$ are all contained in and commensurable to Λ^2 by Lemma 2.2. Take any $U \in \mathcal{B}$ and any $\gamma \in \Gamma$ and choose $V \in \mathcal{B}$ such that $\gamma(V^{-1}V)^2\gamma^{-1} \subset U(\gamma) \cap U$. Then

$$\begin{aligned}\gamma\left(V^{-1}V \cap \Lambda^2\right)^2\gamma^{-1} &\subset \gamma(V^{-1}V)^2\gamma^{-1} \cap \gamma\Lambda^4\gamma^{-1}\\ &\subset U(\gamma) \cap U \cap \gamma\Lambda^4\gamma^{-1}\\ &\subset U \cap \Lambda^2.\end{aligned}$$

So $\mathcal{B}_\Lambda$ checks all conditions of a neighbourhood basis, so there is a topology $\mathcal{T}$ on Γ making Γ into a topological group and for which $\mathcal{B}_\Lambda$ is neighbourhood basis about e [Bou89b, Ch. III, §1.2, Proposition 1]. We can moreover prove that the inclusion map $\Gamma \to G$ is continuous. We will achieve this in a number of steps.

First, take $\lambda \in \Lambda^2$ and take any $U \in \mathcal{B}$ with $U^{-1}U \subset U(e)$. Then

$$\Lambda^2 \cap \lambda U^{-1}U = \lambda(\lambda^{-1}\Lambda^2 \cap U^{-1}U) \subset \lambda(\Lambda^4 \cap U^{-1}U) = \lambda(\Lambda^2 \cap U^{-1}U).$$

Remark that $\Lambda^2 \cap \lambda U^{-1}U$ is a neighbourhood of λ in the subspace topology of $\Lambda^2 \subset G$ and, because $\Lambda^2 \cap U^{-1}U \in \mathcal{B}_\Lambda$, $\lambda(\Lambda^2 \cap U^{-1}U)$ is a neighbourhood of λ in Γ equipped with $\mathcal{T}$. We have therefore showed that the identity map $id : \Lambda^2 \to \Lambda^2$ is continuous, where the source space is equipped with the subspace topology from G and the target space is equipped with the subspace topology from Γ equipped with $\mathcal{T}$.

Remark now that the inverse map of id_{Λ^2} is simply the restriction to Λ^2 of the inclusion map $\Gamma \to G$. So we have proved continuity in the wrong direction. To reverse it, we will use the fact that a bijective continuous map from a compact space to a Hausdorff space has a continuous inverse. Since $\bigcap_{\Xi \in \mathcal{B}_\Lambda} \Xi \subset \bigcap_{U \in \mathcal{B}} U^{-1}U = \{e\}$, Γ equipped with $\mathcal{T}$ is indeed Hausdorff. Therefore, id_{Λ^2} has a continuous inverse, and Λ^2 is a compact subset of e in Γ equipped with $\mathcal{T}$. In particular, Γ is locally compact, and the restriction of the inclusion map $\Gamma \to G$ to Λ^2 is continuous. Since Λ^2 is a neighbourhood of e, $\Gamma \to G$ is a continuous group homomorphism. □

We can now turn to the proof of Theorem 1.3. This result is akin to Cartan's closed-subgroup theorem as it shows that closed approximate subgroups of Lie groups have a Lie group structure, at least locally.

Proof of Theorem 1.3. Apply Theorem 4.1 to $(\Lambda^2 \cap V^2, \langle\Lambda\rangle)$, where V is any symmetric compact neighbourhood of the identity in G. Note that $\Lambda^2 \cap V^2$ is indeed an approximate subgroup by Lemma 2.3. This yields an injective continuous group homomorphism $\phi : H \to G$ with image $\langle\Lambda\rangle$ and such that $\phi^{-1}(\Lambda^2 \cap V^2)$ is a compact neighbourhood of the identity. Since finitely many translates of Λ cover $\Lambda^2 \cap V^2$, $\phi^{-1}(\Lambda)$ has nonempty interior as well. The approximate subgroup $\phi^{-1}(\Lambda)$ is therefore contained in the interior of $\phi^{-1}(\Lambda^3)$. We have, moreover, that for any $K \subset G$ compact, the subset $K \cap \Lambda$ is covered by finitely many left-translates of $\Lambda^2 \cap V^2$. So $\phi^{-1}(K \cap \Lambda)$ is compact and $\phi_{|\phi^{-1}(\Lambda)}$ is proper.

If, moreover, G is a Lie group, then H is a Lie group as a consequence of [Bou89c, Ch. III, §8, Corollary 1]. □

In particular, it enables one to define unambiguously the Lie algebra associated to a closed approximate subgroup of a Lie group. Note that this last fact could also be proved as a consequence of Lemma 4.2 and [Bou89c, Chapter III, §8, Prop. 2].

Corollary 4.3. *Let Λ be a compact approximate subgroup of a Lie group G. Suppose that Λ is contained in a subgroup Γ of G that commensurates Λ. Let $\mathfrak{g}$ denote the Lie algebra of G. Then there is a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, invariant under the adjoint action of Γ and such that for any compact symmetric neighbourhood $V \subset \mathfrak{h}$ of 0, Λ is commensurable with the approximate subgroup* $\exp(V)$.

Proof. This result is essentially an application of the above and [Bou89c, Ch. III, §8, Prop. 2]. We sketch a proof and refer the reader to the relevant part of [Bou89c, Ch. III, §8] for background. Apply Theorem 4.1 to (Λ, Γ). There is an injective continuous group homomorphism $\phi : H \to G$ with image Γ and such that $\phi^{-1}(\Lambda^2)$ is a compact neighbourhood of the identity. Since G is a Lie group and ϕ is injective, H is a Lie group [Bou89c, Ch. III, §8, Cor. 1]. Let $\mathfrak{h}'$ denote the Lie algebra of H and $\mathfrak{h}$ its image through the differential $d\phi$ of ϕ. The map $d\phi$ yields a linear isomorphism between $\mathfrak{h}'$ and $\mathfrak{h}$. Then $\mathfrak{h}$ is obviously invariant under the adjoint action of $\phi(H) = \Gamma$. Moreover, for any compact symmetric neighbourhood V of 0 in $\mathfrak{h}'$, $\exp(V)$ is a compact symmetric neighbourhood of e in H, and hence commensurable to $\phi^{-1}(\Lambda^2)$. Thus, $\phi(\exp(V)) = \exp(d\phi(V))$ is commensurable to Λ as claimed. □

We will use Lemma 4.2 and the point of view of local groups once more later on in Section 5 to define – at least locally – the quotient of an ambient group by a closed approximate subgroup. This will then enable us to build local Borel sections of closed approximate subgroups (see Lemma 5.16). We will also make use of this point of view when proving the structure theorem for amenable closed approximate subgroups (Theorem 1.6).

4.2. Closed approximate subgroups of Euclidean spaces

As a first consequence, we investigate the structure of closed approximate subgroups of Euclidean spaces. A key ingredient is a result due to Schreiber concerning the coarse structure of approximate subgroups of Euclidean spaces ([Sch73]). A new proof of this result was recently given by Fish [Fis19] (see also the generalisation to linear real soluble groups [Mac22b]).

Theorem Schreiber, [Sch73, Fis19]. *For any approximate subgroup Λ in a Euclidean space V, there is a vector subspace $V' \subset V$ and a compact neighbourhood of the identity $K \subset V$ such that $\Lambda \subset V' + K$ and $V' \subset \Lambda + K$*

We sketch here a proof making use of the structure of amenable approximate subgroups that we will establish below (Section 5). Identify V with $\mathbb{R}^n$, and let $\|\cdot\|_\infty$ be the sup norm on $\mathbb{R}^n$. Define $L := \{x \in \mathbb{Z}^n | \exists \lambda \in \Lambda, \|x - \lambda\|_\infty \leq 1\}$. We have $\Lambda \subset [-1;1]^n + L$ and $L \subset [-1;1]^n + \Lambda$. Then L is an approximate subgroup. By Proposition 5.7, $L + L + L + L$ has a good model. So by part (2) of Proposition 3.9, there is an approximate subgroup L' commensurable to L that has a good model $f : \langle L' \rangle \to \mathbb{R}^n$. So f extends to an $\mathbb{R}$-linear map f' from the $\mathbb{R}$-span of L' to $\mathbb{R}^n$. The vector subspace we are looking for is the kernel of f'. We leave verification of the above details to the reader.

Let us now state the main result of this section:

Proposition 4.4. *Let Λ be a closed approximate subgroup of $\mathbb{R}^n$. There are two vector subspaces V_o and V_d of $\mathbb{R}^n$, a uniformly discrete approximate subgroup $\Lambda_d \subset V_d$ and a compact approximate subgroup $K \subset V_d$ such that $V_o \oplus V_d = \mathbb{R}^n$ and Λ is commensurable to $V_o + \Lambda_d + K$. Furthermore,*

1. *there is a vector subspace $V_e \subset V_d$ such that we can choose K to be any compact neighbourhood of the identity in V_e and $V_e \cap \Lambda_d^2 = \{0\}$;*
2. *there are a non-negative integer m, a linear map $\phi : \mathbb{R}^m \to V_d$, a subspace $V_d' \subset \mathbb{R}^m$ with $V_d' \cap \ker(\phi) = \{0\}$ such that Λ_d is contained in and commensurable to $\phi\big(\mathbb{Z}^m \cap \big(V_d' + [-a;a]^m\big)\big)$ for some real $a > 0$.*

Proof. While we have used additive notations in the statement, we will stick with multiplicative notations in the proof for the sake of consistency. We will use several times the following claim:

Claim 4.2.1. Let Λ be a closed approximate subgroup in $\mathbb{R}^n$; there exists a relatively compact subset $K' \subset \Lambda^2$ with $\overline{K'} \subset \Lambda^4$ that generates $\langle \Lambda \rangle$.

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This is a consequence of Schreiber's theorem and was already worked out in [Mac22b, Prop. 3] in a broader context.

Proof of Claim 4.2.1. Take $\Lambda \subset HK$ and $H \subset \Lambda K$ where K is symmetric compact and H is a vector subspace, as given by Schreiber's Theorem. Let B be a closed ball in H, so H is generated by B as a group and B is compact. Take $\lambda \in \Lambda$ arbitrary and $h \in H$ such that $\lambda \in hK$. Take $h_1, \ldots, h_r \in B$ with $h = h_1 \cdots h_r$. Since $H \subset \Lambda K$, pick $\lambda_i \in \Lambda$ such that $h_1 \cdots h_i \in \lambda_i K$ for $1 \leq i < r$ and $\lambda = \lambda_r$ and $\lambda_0 = 0$. Then, $\lambda_{i-1}^{-1}\lambda_i \in h_i K^2 \subset BK^2$, so

$$\lambda = \lambda_r = \prod_{i=1}^{r} \lambda_{i-1}^{-1}\lambda_i \in (\Lambda^2 \cap (BK^2))^r.$$

Thus, $\langle \Lambda \rangle$ is generated by $K' = \Lambda^2 \cap (BK^2)$, which is relatively compact and contained in Λ^2. Since Λ is an approximate subgroup, $K' \subset F\Lambda$ with $F \subset \Lambda^3$ finite, so $\overline{K'} \subset F\Lambda \subset \Lambda^4$ as Λ is closed. □

We start with two sub-cases. Suppose first that Λ is uniformly discrete. Then $\langle \Lambda \rangle$ is finitely generated (Claim 4.2.1), so $\langle \Lambda \rangle \simeq \mathbb{Z}^m$. Take a linear map $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that the restriction of ϕ to $\mathbb{Z}^m$ yields an isomorphism $\langle \Lambda \rangle \simeq \mathbb{Z}^m$. Apply now Schreiber's theorem to $\Lambda' := \phi^{-1}(\Lambda) \cap \mathbb{Z}^m$. We get a vector subspace $V_d' \subset \mathbb{R}^m$ and a box $B := [-a; a]^m$ for some real $a > 1$ such that $\Lambda' \subset V_d' B$ and $V_d' \subset \Lambda' B$. Thus, Λ' is contained in and commensurable with $\mathbb{Z}^m \cap V_d' B$, so $\phi(\mathbb{Z}^m \cap V_d' B)$ indeed contains and is commensurable with $\phi(\Lambda') = \Lambda$. It remains to prove $V_d' \cap \ker \phi = \{0\}$. Otherwise, $D := \left(V_d' \cap \ker \phi\right) B \cap \mathbb{Z}^m$ is infinite. But $\phi(D) \subset \phi(B)$, which is bounded. In addition, $\phi(D) \subset \phi(\mathbb{Z}^m \cap V_d' B)$ and $\phi(\mathbb{Z}^m \cap V_d' B)$ is covered by finitely many translates of Λ, and hence is uniformly discrete. So $\phi(D)$ is finite. Since ϕ is injective on $\mathbb{Z}^m$, we reach a contradiction.

Suppose now that Λ has nonempty interior. Then the interior of Λ^2 is symmetric, contains the identity and is commensurable to Λ. Hence, it is an open approximate subgroup commensurable to Λ. Take V and K as in Schreiber's theorem. We know that K^2 is covered by finitely many translates of Λ. So $VK \subset \Lambda K^2$ is covered by finitely many translates of Λ. But $\Lambda \subset VK$, so Λ and VK are commensurable. So $V_o := V$ and $\Lambda_d := \{e\}$ work.

Let us go back to the general case. Let V_o be the largest vector subspace covered by finitely many left-translates of Λ, and take V_d any supplementary space. Now, Λ is contained in $V_o\left(V_d \cap \Lambda V_o^{-1}\right)$. But $V_o \cap \Lambda^2$ is commensurable to V_o and finitely many translates of $V_d \cap \Lambda^2$ cover $V_d \cap \Lambda V_o$ by Lemma 2.3. So $V_d \cap \Lambda V_o$ and $(V_d \cap \Lambda^2)$ are commensurable. So Λ is covered by finitely many translates of (and, thus, commensurable to) $(V_o \cap \Lambda^2)(V_d \cap \Lambda^2)$. In turn, Λ is commensurable to $V_o(V_d \cap \Lambda^2)$.

Now, let $i : L \longrightarrow \mathbb{R}^n$ be the injective Lie group homomorphism given by Theorem 1.3 applied to $V_d \cap \overline{\Lambda^2}$ (recall that $\overline{\Lambda^2} \subset \Lambda^4$ so is commensurable with Λ), and let Λ' denote the inverse image of $V_d \cap \overline{\Lambda^2}$. Assume as we may that $L = \langle \Lambda' \rangle$. Let L^0 denote the connected component of the identity of L. Then L^0 is a connected torsion free abelian lie group, and hence, $L^0 \simeq \mathbb{R}^k$. Note, moreover, that $i_{|L^0}$ is an injective continuous group homomorphism, and hence an $\mathbb{R}$-linear map and a homeomorphism onto its image. We have that $(\Lambda')^2 \cap L^0$ is an approximate subgroup with nonempty interior so it is commensurable to $V'W$ (by the above paragraph), where $V' \subset L^0$ is a vector subspace and W is a compact neighbourhood of the identity in L^0. By construction of V_d, we know that $V' = \{0\}$. So $(\Lambda')^2 \cap L^0$ is a relatively compact neighbourhood of the identity in L^0. Recall now that $i_{|\Lambda'}$ is proper, and hence, $i_{|\Lambda'^4}$ is proper as well. But L is a torsion-free abelian Lie group, and there is a compact subset C contained in $(V_d \cap \Lambda^2)^4 = i(\Lambda'^4)$ that generates $i(\langle \Lambda' \rangle)$ (Claim 4.2.1). So L is generated by the compact subset $i^{-1}(C)$. Thus, L/L^0 is a finitely generated abelian group, concluding $L \simeq \mathbb{R}^k \oplus \mathbb{Z}^l$ for some non-negative integers k, l. So we can identify L with a closed subgroup of $\mathbb{R}^k \times \mathbb{R}^l$ with $L^0 = \mathbb{R}^k \times \{0\}$.

According to Schreiber's theorem, there is vector subspace $V \subset \mathbb{R}^k \times \mathbb{R}^l$ and a compact neighbourhood of the identity $K \subset \mathbb{R}^k \times \mathbb{R}^l$ such that $\Lambda' \subset VK$ and $V \subset \Lambda' K$. We claim that $L^0 \cap V = \{0\}$. Indeed, $L^0 \cap V \subset \Lambda' K$ so $L^0 \cap V \subset \Lambda' K_0$, where $K_0 := (L^0 \cap V)\Lambda' \cap K$. Now, K_0 is a relatively compact subset of L, and since Λ' has nonempty interior in L, finitely many translates of Λ' cover K_0. So finitely

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many translates of $i(\Lambda')$ cover $i(V \cap L^0)$. Thus, $i(V \cap L^0)$ is a subspace of V_o. But $i(V \cap L^0) \subset V_d$, so $i(V \cap L^0) \subset V_d \cap V_o = \{0\}$. In turn, $V \cap L^0 = \{0\}$ by injectivity of i.

So choose a vector subspace V' such that $L^0 \oplus V \oplus V' = \mathbb{R}^k \times \mathbb{R}^l$. The projection of L to $V \oplus V'$ parallel to L^0 is then a discrete subgroup $\Gamma \subset V \oplus V'$, and we find $L = L^0 \oplus \Gamma$. Moreover, the projection of Λ' to L^0 is contained in K, so it is a bounded subset with nonempty interior. Since $\Lambda'^2 \cap L^0$ is a neighbourhood of $\{0\}$ and L^0 is connected, the projection of Λ' to L^0 is contained in $(\Lambda'^2 \cap L^0)^m$ for some $m > 0$. So for every $\lambda \in \Lambda'$, there is $\lambda_0 \in (\Lambda'^2 \cap L^0)^m$ such that $\lambda\lambda_0^{-1} \in \Gamma$ (i.e., $\Lambda' \subset (\Lambda'^2 \cap L^0)^m(\Lambda'^{2m+1} \cap \Gamma)$). So Λ' is commensurable to $\left(\Lambda'^2 \cap \Gamma\right)\left(\Lambda'^2 \cap L^0\right)$ by Lemma 2.3. As $i_{|\overline{\Lambda'^2}}$ is proper, we can set $\Lambda_d = i(\overline{\Lambda'^2} \cap \Gamma)$, $K = i(\overline{\Lambda'^2} \cap L^0)$ and $V_e := i(L^0)$. Note that indeed Λ_d is discrete because Γ is, K is compact because $\Lambda'^2 \cap L^0$ is relatively compact and $\overline{\Lambda'^2} \cap L^0$ is an approximate subgroup commensurable to $\Lambda'^2 \cap L^0$ because $\overline{\Lambda'^2} \cap L^0 \subset \Lambda'^4 \cap L^0$ and Lemma 2.3. Remark finally that since K has nonempty interior in V_e, it must be commensurable with any compact neighbourhood of the origin in V_e. This concludes the proof of (1) and of the result. □

The situation becomes even more striking when Λ is a closed approximate subgroup in a one dimensional Euclidean space:

Corollary 4.5. *Let Λ be a closed approximate subgroup of $\mathbb{R}$. Then one and only one of the following is true:*

1. *Λ is finite;*
2. *there are real numbers $0 < a < b < \infty$ such that $[-a; a] \subset \Lambda^2 \subset [-b; b]$;*
3. *Λ is a uniform approximate lattice (i.e., uniformly discrete and relatively dense);*
4. *there is $n \in \mathbb{N}$ such that $\Lambda^n = \mathbb{R}$.*

In particular, Λ is uniformly discrete, or Λ has nonempty interior.

Proof. By Proposition 4.4, there are three cases: $V_o = \mathbb{R}$ and $V_e = V_d = \{0\}$, $V_d = V_e = \mathbb{R}$ and $V_o = \{0\}$, or $V_d = \mathbb{R}$ and $V_e = V_o = \{0\}$. The first case corresponds to (4). In the second case, Λ is commensurable to a closed interval, so (2). In the third case, $\Lambda = \Lambda_d$ is uniformly discrete. If Λ is finite, then we get (1). If Λ is infinite, then we get (3) by [Fis19, Prop. 3.1]. □

4.3. Structure of compact approximate subgroups

We prove the following now:

Theorem 4.6 (Structure of compact approximate subgroups). *Let Λ be a compact approximate subgroup of a Hausdorff topological group G, and let $\langle\Lambda\rangle$ be the subgroup it generates. Then $\langle\Lambda\rangle$ admits a structure of locally compact group such that the inclusion $\langle\Lambda\rangle \subset G$ is continuous and $\Lambda \subset \langle\Lambda\rangle$ is a compact subset with nonempty interior. Moreover, for every $\varepsilon > 0$, there is an approximate subgroup $\Lambda' \subset \Lambda^{16}$ that generates a subgroup open in the topology of $\langle\Lambda\rangle$ and a compact subgroup $H \subset \Lambda'$ normalised by Λ' such that*

(i) *Λ can be covered by $O_{K,\varepsilon}(1)$ translates of Λ';*
(ii) *$\langle\Lambda'\rangle/H$ is a Lie group of dimension $O_K(1)$.*

If $\mathfrak{l}'$ denotes the Lie algebra of $\langle\Lambda'\rangle/H$ and Λ'' the image of Λ' in $\langle\Lambda'\rangle/H$, then there exists a norm $|\cdot|$ on $\mathfrak{l}'$ such that

(iii) *for $X, Y \in \mathfrak{l}'$, we have $|[X, Y]| \leq O_K(|X||Y|)$;*
(iv) *for $g \in \Lambda''$, the operator norm (induced by $|\cdot|$) of $\mathrm{Ad}(g) - \mathrm{Id}$ is $O_K(\varepsilon)$;*
(v) *there is a convex set $B \subset \mathfrak{l}'$ such that $\Lambda''/\exp(B)$ is a finite $O_{K,\varepsilon}(1)$-approximate local group.*

Proof. Recall that Kreitlon-Carolino proved the statement of Theorem 4.6 with the additional assumption that Λ is open ([Car15, Thm. 1.25]). We will show how Theorem 4.6 reduces to this situation. Let

L and $f : L \to G$ be the locally compact group and the homomorphism given by Theorem 4.1 and $V \subset L$ be a neighbourhood such that $f(V) = \Lambda^2$. Then V is a compact neighbourhood of the identity, so there is an open symmetric subset $\tilde{V}$ such that $V \subset \tilde{V} \subset V^2$. The subset $\tilde{V}$ is thus an open relatively compact K^6-approximate subgroup. But f is an injective continuous homomorphism, so [Car15, Thm. 1.25] applied to $\tilde{V}$ yields Theorem 4.6. □

4.4. Bohr-type compactification

We mention yet another application of Theorem 4.1 that generalises further the Bohr compactification of a discrete group.

Proposition 4.7. *Let Γ be a group that commensurate an approximate subgroup $\Lambda \subset \Gamma$. Then there is a group homomorphism $f_0 : \Gamma \to H_0$ (unique up to continuous group isomorphism $H_0 \to H_0'$) with dense range and $f_0(\Lambda)$ relatively compact that satisfies the following universal property:*

(∗) *if $f : \Gamma \to H$ is a group homomorphism with H locally compact and $f(\Lambda)$ relatively compact, then there is a continuous group homomorphism $\phi : H_0 \to H$ such that $f = \phi \circ f_0$.*

Proof. Let $\mathcal{R}$ be a set of representatives of group homomorphisms $f : \Gamma \to H$ with H locally compact, dense image and $f(\Lambda)$ relatively compact, up to the following equivalence: $f_1 : \Gamma \to H_1$ and $f_2 : \Gamma \to H_2$ are equivalent if there is a continuous group isomorphism $\phi : H_1 \to H_2$ such that $\phi \circ f_1 = f_2$. Notice that $\mathcal{R}$ is not empty as it contains the map to the trivial group. Then the group $H_{\mathcal{R}} := \prod_{f:\Gamma \to H_1 \in \mathcal{R}} H_2$ equipped with the product topology is a Hausdorff topological group, and $f_{\mathcal{R}}(\Lambda)$ is relatively compact where $f_{\mathcal{R}} : \Gamma \to H_{\mathcal{R}}$ is the diagonal map. One readily sees that $f_{\mathcal{R}}$ satisfies the universal property (∗). The topological group $H_{\mathcal{R}}$ need not be locally compact however. By Theorem 4.1, there are a locally compact group H_0, a group homomorphism $f_0 : \Gamma \to H_0$ and a continuous group homomorphism $\phi : H_0 \to H_{\mathcal{R}}$ such that $f_{\mathcal{R}} = \phi \circ f_0$. □

The above proposition can be interpreted as existence of a smallest Meyer subset containing a given approximate subgroup Λ. Given $f_0 : \Gamma \to H_0$ as in Proposition 4.7, set $\Lambda_0 := f_0^{-1}(\overline{f_0(\Lambda^2)})$. Note that by construction, $\overline{f_0(\Lambda^2)}$ is a compact neighbourhood of the identity in H_0. So Λ_0 is an approximate subgroup and has a good model. We see now that if Λ' contains Λ and has a good model f, then f must factor through f_0. Therefore, $f(\Lambda_0)$ is relatively compact and finitely many left-translates of Λ' cover Λ_0. This discussion yields the following:

Lemma 4.8. *Let Γ be a group that commensurates an approximate subgroup $\Lambda \subset \Gamma$. Let $f_0 : \Gamma \to H_0$ be as in Proposition 4.7. Then*

1. *Λ has a good model if and only if f_0 is a good model;*
2. *Λ is Meyer subset if and only if it is commensurable to $f_0^{-1}(\overline{f_0(\Lambda)})$.*

Remark 4.9. Hrushovski's [Hru22, §5.8] yields moreover the following fascinating result: if $\Lambda \subset \Lambda_0$ are as above, then the numbers $n \geq 0$ such that there are pairwise non-commensurable approximate subgroups $\Lambda \subset \Lambda_n \subset \ldots \subset \Lambda_1 \subset \Lambda_0$ are bounded.

5. Amenable approximate subgroups

5.1. Amenable approximate subgroups: definition

Definition 5.1. Let Λ be a closed approximate subgroup of a locally compact group G. Define $\mathcal{B}(\Lambda)$ as the set of those Borel subsets of G that are covered by finitely many left-translates of Λ. We say that Λ is *amenable* if there exists a finitely additive measure m defined on $\mathcal{B}(\Lambda)$ such that

1. (finiteness) $0 < m(\Lambda) < \infty$;
2. (left-invariance) for all $g \in G$ and $X \in \mathcal{B}(\Lambda)$, we have $m(gX) = m(X)$.

According to Theorem 4.1, compact approximate subgroups of topological groups are amenable, and m is easily obtained from a Haar measure (and Lemma 5.2 below). We will see in Subsection 5.4 below that any closed approximate subgroup of an amenable locally compact group is amenable.

The definition above is extremely close to the definition of *definably amenable approximate subgroups* introduced by Massicot and Wagner in [MW15]. There they study finitely additive measures defined on definable subsets in some structure. When Λ is discrete, our definition is, in fact, a special case of definably amenable approximate subgroup (with all subsets of $\langle\Lambda\rangle$ being definable). However, it does not seem obvious how to present Definition 5.1 as a special case of definably amenable approximate subgroups when Λ is not discrete. Indeed, algebras of Borel subsets and of definable subsets behave differently with respect to set operations such as projections.

We note now that Definition 5.1 is, in fact, *local*:

Lemma 5.2. *Let m be a finitely additive measure defined on the Borel subsets of Λ and such that*

1. *(finiteness)* $m(\Lambda) = 1$*;*
2. *(local left-invariance)* $m(gX) = m(X)$ *whenever* $g \in G, X \subset \Lambda$ *and* $gX \subset \Lambda$.

Then Λ is amenable, and m can be extended to a finitely additive measure as in Definition 5.1.

Proof. Consider $X \in \mathcal{B}(\Lambda)$ and $X_1, \dots, X_r$ a Borel partition of X such that there are $f_1, \dots, f_r \in G$ with $f_iX_i \subset \Lambda$. We will prove that the quantity $\sum_{i=1}^r m(f_iX_i)$ depends only on X. Defining $\tilde{m}(X) = \sum_{i=1}^r m(f_iX_i)$ then yields the extension we are looking for. Take $Y_1, \dots, Y_s$ a second partition with $g_1, \dots, g_s \in G$ as above. We have

$$\begin{aligned}
\sum_{i=1}^r m(f_iX_i) &= \sum_{i=1}^r \sum_{j=1}^s m(f_i(X_i \cap Y_j)) \\
&= \sum_{j=1}^s \sum_{i=1}^r m(f_ig_j^{-1}g_j(X_i \cap Y_j)) \\
&= \sum_{j=1}^s \sum_{i=1}^r m(g_j(X_i \cap Y_j)) \\
&= \sum_{j=1}^s m(g_jY_j),
\end{aligned}$$

where we have used local left-invariance to go from the second to the third line. □

As an immediate corollary we find the following:

Lemma 5.3. *Let Λ and Ξ be closed approximate subgroups of some locally compact group. If Λ and Ξ are commensurable and Ξ is amenable, then Λ is amenable.*

Note that a careful study of elementary properties of invariant finitely additive measures was carried out in [HKP22].

5.2. Amenable approximate subgroups of linear groups

We will exploit the strong Tits' alternative, following an idea from [BGT11], to prove the following:

Lemma 5.4. *Let k be any field. Let Λ be an amenable closed K-approximate subgroup of some locally compact group. Let $\psi : \overline{\Lambda^5} \to \mathrm{GL}_d(k)$ be a local group homomorphism (i.e., $\psi(xy) = \psi(x)\psi(y)$ whenever $x, y, xy \in \overline{\Lambda^5}$). Assume that ψ is continuous and has countable image. Then there is $\Lambda' \subset \Lambda^2$ commensurable with Λ such that every finite subset of $\psi(\Lambda')$ generates a virtually soluble subgroup.*

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Proof. Fix m an invariant finitely additive measure as in Definition 5.1. By the strong Tits alternative [Bre08, Thm. 1.1], there is an integer $N := N(d)$ such that for every subset $F \subset \mathrm{GL}_d(k)$ finite, either F generates a virtually soluble subgroup or F^N contains two elements generating a free group. We will apply this result in combination with the following:

Claim 5.2.1. Let $X \subset \overline{\Lambda^5}$ be a Borel subset and $x, y \in \overline{\Lambda^5}$ be such that $\psi(x)$ and $\psi(y)$ generate a free group, and $\{x, y, x^{-1}, y^{-1}\}X \subset \overline{\Lambda^5}$. Then

$$m(\{x, y, x^{-1}, y^{-1}\}X) \geq 3m(X).$$

Suppose first that the claim is true. Since $\overline{\Lambda^4}$ has a good model (Proposition 5.7), there is $S \subset \overline{\Lambda^4}$ an approximate subgroup commensurable with Λ such that $S^l \subset \overline{\Lambda^4}$, where $l \geq N\big(4\log_3(K) + 1\big)$. Take a finite symmetric subset $F \subset S$ and assume for a contradiction that $\psi(F)$ does not generate a virtually soluble subgroup. According to the strong Tits alternative and the claim,

$$3^{l/N} m(\Lambda) \leq m(F^l\Lambda) \leq m(\overline{\Lambda^5}) \leq K^4 m(\Lambda).$$

Hence, $3^{l/N} = 3K^4 \leq K^4$: a contradiction. So $\Lambda' := S$ works. It remains only to prove the claim.

Proof of Claim 5.2.1. Let $x, y \in X$ be two elements such that $F_2 := \langle\psi(x), \psi(y)\rangle$ is free. Choose R a system of right representatives of $\psi(\Lambda)$ over F_2. For every reduced word $w \in F_2 \setminus \{e\}$ (in the letters $\psi(x), \psi(y)$), define the subset Λ_w as the subset of those elements λ of Λ such that $\psi(\lambda) = vr$ with $r \in R$ and $v \in F_2$, where v starts with w when written as a reduced word. In other words, $v = wv'$ for some $v' \in F_2$ and the last letter of the reduced word w is not equal to the inverse of the first letter of the reduced word v'. Since R and F_2 are countable, Λ_w is Borel. We define, moreover, $\Lambda_e := \Lambda \cap \psi^{-1}(R)$. We have the disjoint union decomposition

$$\Lambda = \bigsqcup_{w \in \{e\} \cup \psi\{x,y,x^{-1},y^{-1}\}} \Lambda_w. \tag{5.1}$$

Furthermore, we have

$$\{x, y, x^{-1}, y^{-1}\}\Lambda \supset \bigsqcup_{\alpha \in \{x,y,x^{-1},y^{-1}\}} \alpha\Lambda_e \sqcup \bigsqcup_{\substack{\alpha,\beta \in \{x,y,x^{-1},y^{-1}\} \\ \alpha \neq \beta^{-1}}} \alpha\Lambda_{\psi(\beta)}. \tag{5.2}$$

Therefore, a combination of (5.1) and (5.2) yields

$$m(\{x, y, x^{-1}, y^{-1}\}\Lambda) \geq 3m(\Lambda). \qquad \square$$

The proof is now complete. □

In the case of characteristic 0 fields, we obtain a stronger result thanks to the Tits' alternative ([Tit72, Thm. 1]) in characteristic 0.

Corollary 5.5. *With notations as in Lemma 5.4. If k has characteristic 0, then $\psi(\Lambda')$ generates a virtually soluble subgroup.*

We will also invoke Lemma 5.4 in the case of a positive characteristic field. In that situation as well, we will be able to draw strong information by combining it with well-known results of Tits' [Tit72, Prop. 2.8].

5.3. Structure of amenable approximate subgroups

The main result of this subsection is the following:

Proposition 5.6. *Let Λ be an amenable closed approximate subgroup of a σ-compact locally compact group G. There is $\Lambda' \subset \overline{\Lambda^4}$ a closed approximate subgroup commensurable to Λ that has a good model $f : \langle\Lambda'\rangle \to H$ such that*

1. *H is a connected Lie group;*
2. *$f_{|\overline{\Lambda'^2}}$ is continuous;*
3. *if $p : H \to S$ denotes the projection to the quotient S of H by its maximal solvable normal subgroup, then $(p \circ f)(\Lambda')$ is a neighbourhood of the identity.*

The first step towards Proposition 5.6 is to prove a result in the spirit of Hrushovski's stabilizer theorem from [Hru12]:

Proposition 5.7. *Let Λ be a closed approximate subgroup of a σ-compact locally compact group G. If Λ is amenable, then $\overline{\Lambda^4}$ has a good model f such that $f_{|\overline{\Lambda^4}}$ is continuous.*

Hrushovski's study of *near-subgroups* ([Hru12]) immediately implies the above Proposition 5.7 when Λ is discrete. To deal with the general case, we rely on a variation of an argument due to Massicot–Wagner in [MW15] about a definably amenable approximate subgroup inspired by Sanders' [San12] and Croot–Sisask's [CS10], who proved it for finite abelian approximate groups.

Lemma 5.8. *Let Λ be an amenable closed approximate subgroup of a σ-compact locally compact group G, and let k be a positive integer. There is an approximate subgroup $S \subset \Lambda^2$ commensurable to Λ such that $S^k \subset \Lambda^4$.*

Proof. Let $\Xi \subset \Lambda$ be Haar measurable such that $m(\Xi) \geq tm(\Lambda)$ for some $t \in (0;1]$. Set $X(\Xi) := \{g \in \Lambda^2 | m(g\Xi \cap \Xi) \geq stm(\Lambda)\}$, where $s = \frac{t}{2K}$. By the proof of [MW15, Thm. 12], the approximate subgroup Λ is covered by at most $N := \lfloor \frac{1}{s} \rfloor$ left-translates of $X(\Xi)$.

Define now

$$f(t) := \inf\left\{ \frac{m(\Xi\Lambda)}{m(\Lambda)} \middle| \Xi \subset \Lambda \text{ closed, } m(\Xi) \geq tm(\Lambda)\right\}.$$

Note that f is well defined since the product of two σ-compact subsets is a σ-compact subset, and hence Borel. We also know that $f(t) \in [1;K]$ for all $t \leq 1$. Take $t \geq c_{K,k}$ such that $f(\frac{t^2}{2K}) \geq (1 - \frac{1}{4k})f(t)$ (where we can choose $c_{K,k} = \frac{1}{(2K)^{2^n-1}}$ with $n = \left\lceil \frac{\log(K)}{\log((1-\frac{1}{4k})^{-1})} \right\rceil$; see [MW15, Lem. 11]) and choose $\Xi \subset \Lambda$ closed such that $m(\Xi) \geq tm(\Lambda)$ and $\frac{m(\Lambda\Xi)}{m(\Lambda)} \leq (1 + \frac{1}{4k})f(t)$.

If $g \in X(\Xi)$, we have

$$\begin{aligned} m(g\Xi\Lambda \cap \Xi\Lambda) &\geq m((g\Xi \cap \Xi)\Lambda) \\ &\geq f\left(\frac{t^2}{2K}\right)m(\Lambda) \\ &\geq \left(1 - \frac{1}{4k}\right)f(t)m(\Lambda) \\ &\geq \frac{1 - \frac{1}{4k}}{1 + \frac{1}{4k}}m(\Xi\Lambda). \end{aligned}$$

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Hence,

$$m((g\Xi\Lambda)\Delta(\Xi\Lambda)) \leq 2\left(1 - \frac{1-\frac{1}{4k}}{1+\frac{1}{4k}}\right)m(\Xi\Lambda)$$
$$< \frac{1}{k}m(\Xi\Lambda).$$

Thus, by the telescopic formula for symmetric differences, we have

$$m((g_1\cdots g_k\Xi\Lambda)\Delta(\Xi\Lambda)) < m(\Xi\Lambda). \tag{*}$$

As a consequence, $X(\Xi)^k \subset \Lambda^4$ and $\lfloor \frac{2K}{t} \rfloor \leq \frac{2K}{c_{K,k}} \leq (2K)^{2^{4\log(K)k+1}}$ translates of $X(\Xi)$ cover Λ. □

Remark 5.9. For the proof of Lemma 5.8 to work, the finitely additive measure m need not be defined on all Borel subsets, but only on a certain lattice of subsets generated by Λ, finite intersections of translates of Λ and certain products of such subsets. We point to [HKP22] where ideas of that nature are studied in detail.

Proof of Proposition 5.7. Let $\phi : \tilde{G} \to G$ be the map given by the closed approximate subgroup theorem (Theorem 1.3). We know that $\phi_{|\phi^{-1}(\Lambda)}$ is an homeomorphism onto its image. So $\phi^{-1}(\Lambda)$ is amenable as well. From now on, we will therefore assume that Λ has nonempty interior in G. By Lemma 5.8, there is a closed approximate subgroup $\Lambda_1 \subset \overline{\Lambda^2}$ commensurable to Λ and such that $\overline{\Lambda_1^8} \subset \overline{\Lambda^4}$. According to Lemma 5.3, the closed approximate subgroup Λ_1 is amenable. We can thus build inductively a sequence of closed approximate subgroups $(\Lambda_n)_{n\geq 0}$ commensurable to Λ such that $\Lambda_0 = \Lambda$ and $(\overline{\Lambda_{n+1}^4})^2 \subset \overline{\Lambda_{n+1}^8} \subset \overline{\Lambda_n^4}$ for all integers $n \geq 0$. By Theorem 3.5 applied to the sequence $(\overline{\Lambda_n^4})_{n\geq 0}$, we obtain that $\overline{\Lambda^4}$ has a good model. But now, for all $n \geq 0$, the approximate subgroup Λ_n is commensurable to Λ. According to the Baire category theorem, we have that the interior of Λ_n is not empty. Hence, $\overline{\Lambda_n^4}$ is a neighbourhood of the identity. So we see that the restriction to $\overline{\Lambda^4}$ of the good model built in the proof of Theorem 3.5 – see (4) of Theorem 3.5 – is, in fact, continuous. □

Proof of Proposition 5.6. In the proof that follows, we will repeatedly use a number of fundamental results from Lie theory, including Lie's theorem, which provides a correspondence between Lie groups and Lie algebras [Var84, Thm. 2.8.2]. We refer the interested reader more generally to [Var84] where this relationship is explored in detail. In this proof, we provide pointers based on this reference.

Let us assume – as in the proof of Proposition 5.7 – that the interior of Λ is not empty. Note that, then, $\langle\Lambda\rangle$ is an open subgroup. According to Proposition 5.7, $\overline{\Lambda^4}$ has a continuous good model $f_0 : \langle\Lambda\rangle \to H_0$ with range dense in H_0. Take $W_0 \subset H_0$ a relatively compact neighbourhood of the identity such that $f_0^{-1}(W_0) \subset \overline{\Lambda^4}$. According to the Gleason–Yamabe theorem, there are a symmetric compact neighbourhood of the identity $W_1 \subset W_0$ and a compact subgroup $K \subset W_1$ normal in the group H_1 generated by W_1 such that $H := H_1/K$ is a connected Lie group. Write $\Lambda_1 := f_0^{-1}(W_1)$, and let $f : \langle\Lambda_1\rangle \to H$ denote the map obtained that way. The Baire category theorem therefore implies that $\langle\Lambda_1\rangle$ is an open subgroup in G. Let W_2 be a neighbourhood of the identity in H – to be chosen later – contained in the projection of W_1, and write $\Lambda_2 := f^{-1}(W_2)$. By Lemma 3.2, the closed approximate subgroup Λ_2 is commensurable to Λ. So f restricted to $\langle\Lambda_2\rangle$ satisfies (1) and (2).

Let V be a symmetric compact neighbourhood of the identity in G. Consider the subset $A := f(\overline{\Lambda_2^2} \cap V^2)$. According to Lemma 2.3, $\overline{\Lambda_2^2} \cap V^2$ – and, hence, A – is an approximate subgroup. By Lemma 2.3 again, $\overline{\Lambda_2^2} \cap V^2$ is commensurated by $\langle\Lambda_2\rangle$. So A is commensurated by $f(\langle\Lambda_2\rangle)$. Since f is continuous, A is compact. Let $\mathfrak{h}$ denote the Lie algebra of H and $\mathfrak{a}$ denote the Lie algebra of A (Corollary 4.3). Note first that since $\mathfrak{a}$ depends only on the commensurability class of A, $\mathfrak{a}$ is independent of the choices of V and W_2. Furthermore, since A is commensurated by $f(\langle\Lambda_2\rangle)$, the Lie algebra $\mathfrak{a}$ is normalised by the dense subgroup $f(\langle\Lambda_2\rangle)$. So $\mathfrak{a}$ is an ideal.

To prove (3), it suffices now to show that $\mathfrak{h}/\mathfrak{a}$ is soluble. By Ado's theorem, there is a Lie subgroup L of some $\mathrm{GL}_n(\mathbb{R})$ with Lie algebra isomorphic to $\mathfrak{h}/\mathfrak{a}$ ([Var84, Thm. 3.17.8]) together with a local Lie group homomorphism $\phi : H \to L$ such that the differential $d\phi$ can be identified to the quotient map $\mathfrak{h} \to \mathfrak{h}/\mathfrak{a}$. When H is simply connected, existence of the homomorphism is ensured by [Var84, Thm. 2.7.5], and existence of a local homomorphism in general is then provided by [Var84, Thm. 2.8.2]. Assume W_2 chosen sufficiently small for ϕ to be defined over $\overline{W_2^{10}}$. Notice that the Lie algebra of the compact approximate subgroup $\phi(A)$ is $d\phi(\mathfrak{a}) = \{0\}$. In other words, $\phi(A)$ is finite (Theorem 1.3 and [Var84, Thm. 2.7.3. (i) and 2.8.2]). But, the group G is σ-finite, so Λ_2 is covered by countably many left-translates of V. Hence, Λ_2 is covered by countably many translates of $\Lambda_2^2 \cap V^2$. This implies that $f(\Lambda_2)$ is covered by countably many left-translates of A and that $\phi \circ f(\Lambda_2)$ is countable. Since $\overline{\Lambda_2^5} \subset \Lambda_2^7$, $\phi \circ f(\overline{\Lambda_2^5})$ is countable as well. The map $\psi := \phi \circ f_{|\overline{\Lambda_2^5}}$ is therefore a well-defined continuous map that has countable image. According to Corollary 5.5, there is X commensurable to Λ_2 such that $\psi(X)$ generates a virtually soluble subgroup. Note that $\overline{f(\Lambda_2)}$ has nonempty interior in H and ϕ is open (because $d\phi$ is surjective and [Var84, Thm. 2.7.3. (i), 2.8.2 and Lem. 2.5.3]). So $\psi(\Lambda_2)$ is dense in a subset with nonempty interior of L. Now, this means that $\psi(X)$ is dense in a subset with nonempty interior. So L is a virtually soluble connected Lie group, and hence, its Lie algebra $\mathfrak{h}/\mathfrak{a}$ is soluble [Var84, Cor. 3.18.9].

In particular, $\mathfrak{a}$ contains all the semi-simple Lie sub-algebras in $\mathfrak{h}$. Write $\mathfrak{s}$ the Lie algebra of S and $d\pi$ the differential of the natural projection $\pi : H_0 \to S$. The kernel of the differential $d\pi$ is the maximal solvable Lie algebra of $\mathfrak{h}$. Indeed, the kernel of $d\pi$ is a soluble ideal by definition so contained in the maximal soluble Lie algebra $\mathfrak{r}$. Conversely, the group corresponding to $\mathfrak{r}$ ([Var84, Thm. 2.5.2]) is a normal connected soluble subgroup [Var84, Cor. 3.18.9], thus contained in the radical. As $d\pi$ is surjective, $d\pi(\mathfrak{a}) = \mathfrak{s}$ by the Levi decomposition [Var84, Thm. 3.14.1]. So $\pi(A)$ has Lie algebra $\mathfrak{s}$, which means that A has nonempty interior (Theorem 1.3 and [Var84, Thm. 2.7.3. (i) and 2.8.2]). So f satisfies (1) – (3). □

We can now derive Theorem 1.6:

Proof of Theorem 1.6. Apply Proposition 5.6 to find $\Lambda' \subset \Lambda^4$ approximate subgroup commensurable to Λ and $f : \langle \Lambda' \rangle \to H$ good model with dense image in a connected Lie group H, $f_{|\Lambda'}$ continuous and $p \circ f(\Lambda')$ a neighbourhood of the identity in H/R, where $p : H \to H/R$ is the quotient homomorphism by the radical. First, using that f is a good model, pick a compact symmetric neighbourhood W of the identity of H such that $f^{-1}(W^2) \subset \Lambda'$. Since $f^{-1}(W)$ is commensurable to Λ' (Corollary 3.3), we get that $p \circ f(f^{-1}(W))$ is commensurable to $p \circ f(\Lambda')$, so it has nonempty interior. Also, as $f_{|\Lambda'}$ is continuous, $f^{-1}(W)$ has nonempty interior in Λ'. Now, there is a compact subset A of $f^{-1}(W)$ with nonempty interior in Λ' such that $p \circ f(A)$ has nonempty interior. Indeed, $f^{-1}(W)$ is a closed subset of a σ-compact locally compact group, so there is a countable family $(A_n)_{n \geq 0}$ of compact subsets such that $\bigcup_{n \geq 0} A_n = f^{-1}(W)$. Thus, $p \circ f(f^{-1}(W)) \subset \bigcup_{n \geq 0} p \circ f(A_n)$. Since $p \circ f(f^{-1}(W))$ has nonempty interior, $p \circ f(A_n)$ has nonempty interior for some n by the Baire category theorem, and we can take $A = A_n$. Since p is continuous, there is a compact symmetric subset $W_1 \subset W$ with nonempty interior such that $p(W_1) \subset p \circ f(A)$. Then, $f^{-1}(W_1) \subset \Lambda^4$ is commensurable to Λ by Corollary 3.3. For any $\lambda \in f^{-1}(W_1)$, there is $a \in A$ such that $f(a^{-1}\lambda) \in R$. Now, $f(a^{-1}\lambda) \in W^2 \cap R$. Taking $\Lambda_{sol} = f^{-1}(W^2 \cap R) \subset \Lambda' \subset \Lambda^4$, we conclude that $f^{-1}(W_1) \subset A\Lambda_{sol}$, so $\Lambda \subset FA\Lambda_{sol}$ with F finite subset of Λ^{11}. It follows that $V := FA \cup \{e\} \cup (FA)^{-1}$ is a compact symmetric subset of Λ^{12} with nonempty interior in Λ^{12} such that $\Lambda \subset V\Lambda_{sol}$. Obviously, $\Lambda \subset V\Lambda_{sol} \cup \Lambda_{sol}V \subset \Lambda^{18}$, so $V\Lambda_{sol} \cup \Lambda_{sol}V$ is an approximate subgroup commensurable to Λ, establishing (3). Consider $\langle \Lambda_{sol} \rangle / \ker f$ with the quotient topology from the induced topology given by Theorem 1.3. By the Isomorphism Theorem (for groups and topological spaces), the induced map $\langle \Lambda_{sol} \rangle / \ker f \to R$ is a continuous 1-to-1 group homomorphism. By [Bou89c, Ch. III, §8, Cor. 1], $\langle \Lambda_{sol} \rangle / \ker f$ is a Lie group getting (2). As R is a soluble group, $\langle \Lambda_{sol} \rangle / \ker f$ is soluble, getting (1). □

Remark 5.10. We can furthermore obtain dimensional bounds by applying the proof strategy of [BGT12, Lem. 10.4] (with additional input the recent [AJTZ21]). One can obtain that way $N' \subset \Lambda'_{sol} \subset \Lambda^{20}$

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satisfying (1), (2) and (3) of Theorem 1.6 and such that $\langle \Lambda'_{sol} \rangle / N'$ has dimension $O(K^{20})$. We believe that this bound is far from sharp and could be replaced by a logarithmic one. Thus, we refrain exploring that direction.

Under additional assumptions, the structure of amenable approximate subgroups appears even more strikingly:

Corollary 5.11. *Let Λ be an amenable uniformly discrete approximate subgroup of a σ-compact locally compact group G. Then there is $\Lambda' \subset \Lambda^4$ commensurable to Λ and a closed subgroup $N \subset \Lambda'$ normalised by Λ' such that $\langle \Lambda' \rangle / N$ is soluble.*

Corollary 5.12. *Let Λ be an amenable closed approximate subgroup of a totally disconnected σ-compact locally compact group G. Then there is $\Lambda' \subset \Lambda^4$ and a closed subgroup $N \subset \Lambda'$ normalised by Λ' such that $\langle \Lambda' \rangle / N$ is soluble.*

Sketch of the proof. Apply Proposition 5.6 to obtain the good model $f : \langle \Lambda' \rangle \to H$. Recall that f has dense range and H is a connected Lie group, and write $p : H \to H/R$ the quotient by the radical. Then $p \circ f(\Lambda')$ contains a neighbourhood of the identity, and $p \circ f_{|\Lambda'}$ is continuous with respect to the topology inherited from Theorem 1.3. But this topology is totally disconnected, and Λ' contains a neighbourhood of the identity. By the van Dantzig theorem [Tao14, Thm. 1.6.7], there is a compact subgroup $K \subset \Lambda'$ which is open in the topology from Theorem 1.3. Since H is a Lie group, $p \circ f(K)$ is finite. But Λ' is covered by countably many translates of K, so $p \circ f(\Lambda')$ is countable. Hence, H/R is discrete (i.e., trivial).

5.4. Approximate Subgroups in Amenable Groups

Our goal in this section is to exhibit a natural family of amenable approximate subgroups – that is, to prove the following:

Proposition 5.13. *Let Λ be a closed approximate subgroup in a second countable locally compact group G. Let H be an amenable closed normal subgroup of G, and suppose that the projection of Λ to G/H is relatively compact. Then Λ is amenable.*

The proof consists of two steps. We first show that any neighbourhood of a normal amenable subgroup is an amenable approximate subgroup (Proposition 5.14). We then prove heredity of amenability for closed approximate subgroups (Proposition 5.15).

Proposition 5.14. *Let G be a locally compact group, let H be a closed amenable normal subgroup and let $W \subset G$ be a compact symmetric neighbourhood of the identity. Then WH is an amenable closed approximate subgroup of G.*

Proof. Let us first recall some notations and definitions (see [Gre69] for more details). Fix a left-Haar measure μ_G on G. Define the left- and right-translates of a function $f : G \to \mathbb{R}$ by $g \in G$ as the maps ${}_gf : x \mapsto f(g^{-1}x)$ and $f_g : x \mapsto f(xg)$, respectively. A function $f : G \to \mathbb{R}$ is *right-uniformly continuous* if it is continuous for the right uniformity – that is, for any real number $\epsilon > 0$, there is a neighbourhood $U(\epsilon) \subset G$ of the identity such that for all $g \in U(\epsilon)$ and $x \in G$, we have $|f(x) - {}_gf(x)| < \epsilon$. The set of right-uniformly continuous bounded functions on G will be denoted by $C_{b,ru}(G)$. Likewise, the set of continuous bounded functions (resp. continuous functions with compact support) on G will be denoted by $C_b(G)$ (resp. $C_c(G)$). We have $C_c(G) \subset C_{b,ru}(G) \subset C_b(G)$. One readily checks that G acts continuously by left-translations on the normed vector space $C_{b,ru}(G)$ equipped with the norm $||\cdot||_\infty$. A linear map $F : X \to \mathbb{R}$ is said *left-invariant* if the subspace $X \subset C_{b,ru}(G)$ is stable by the G-action and if for every $g \in G$ and $f \in X$, we have $F({}_gf) = F(f)$. It is said *positive* if for all $f \in X$ with $f \geq 0$, we have $F(f) \geq 0$.

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The vector subspace of $C^0_{b,ru}(G)$ we want to consider is

$$X := \{f \in C_{b,ru}(G) |\ p(\mathrm{supp}(f)) \text{ is relatively compact.}\},$$

where $p : G \to G/H$ is the natural projection. We will prove the following claim.

Claim 5.4.1. There exists a nontrivial left-invariant positive linear map $m : X \to \mathbb{R}$.

Proof. Fix $\mu_{G/H}$ a right-Haar measure on G/H. First of all, note that X is stable under the action of G. Since H is an amenable locally compact group, there is a left-invariant mean $m_H : C_b(H) \to \mathbb{R}$ according to [Gre69, Thm. 2.2.1]. This means that m_H is a left-invariant positive linear functional such that for any $f \in C_b(H)$, we have $m_H(f) \leq ||f||_\infty$ and $m_H(\mathbb{1}_H) = 1$. Take $f \in X$ and consider the map

$$\begin{aligned} \tilde{f} : G &\to \mathbb{R} \\ x &\mapsto m_H\left({}_x f_{|H}\right), \end{aligned}$$

where ${}_x f_{|H} := ({}_x f)_{|H}$. We will show that $\tilde{f}$ is continuous and invariant under left-translation by elements of H. Indeed, if $h, x \in H$ and $g \in G$, then ${}_{hg} f(x) = ({}_g f)(h^{-1}x)$. But $h^{-1}x \in H$ if and only if $x \in H$. So ${}_{hg} f_{|H} = {}_h\left({}_g f_{|H}\right)$, and hence, for $x \in G$, we have

$${}_h\tilde{f}(x) = \tilde{f}(h^{-1}x) = m_H\left({}_{h^{-1}x} f_{|H}\right) = m_H\left({}_{h^{-1}}\left({}_x f_{|H}\right)\right) = m_H\left({}_x f_{|H}\right) = \tilde{f}(x).$$

Moreover, for any $x_1, x_2 \in G$, we have

$$|\tilde{f}(x_1) - \tilde{f}(x_2)| = |m_H({}_{x_1} f_{|H}) - m_H({}_{x_2} f_{|H})| = |m_H({}_{x_1} f_{|H} - {}_{x_2} f_{|H})| \leq ||\,{}_{x_1} f - {}_{x_2} f||_\infty.$$

But f is right-uniformly continuous, so $\tilde{f}$ is continuous. Therefore, there exists a unique continuous function $f_{G/H} : G/H \to \mathbb{R}$ such that $\left(f_{G/H} \circ p\right)(x) = m_H({}_x f_{|H})$ (recall that p denotes the natural projection). The map $f \mapsto f_{G/H}$ is linear, sends non-negative functions to non-negative functions and $||f_{G/H}||_\infty \leq ||f||_\infty$ for all $f \in C_{b,ru}(G)$. Furthermore, we have $\mathrm{supp}(f_{G/H}) \subset \overline{p(\mathrm{supp}(f))}$, so $f_{G/H}$ is a continuous function with compact support. We are thus able to define

$$\begin{aligned} m : X &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_{G/H} f_{G/H}(t) d\mu_{G/H}(t). \end{aligned}$$

The map m is a positive linear map with $|m(f)| \leq ||f||_\infty$. Choose a compact neighbourhood of the identity $U \subset G$ and $f \in X$ such that $f(x) = 1$ for all $x \in UH$. To ensure such a map exists, we proceed as follows. There exists by Urysohn's lemma a compactly supported continuous function f_0 defined on G/H such that $f_0(x) = 1$ for all $x \in p(U)$. The map f_0 is in particular right uniformly continuous. Since p is moreover right-uniformly continuous on G and G/H, $f_0 \circ p$ is thus right uniformly continuous. So $f := f_0 \circ p$ works. Then for all $x \in UH$, we have ${}_x f_{|H} = 1$, so $f_{G/H}(p(x)) = 1$. This implies $m(f) \geq \mu_{G/H}(p(U)) > 0$, so m is nontrivial. It only remains to check that m is left-invariant. Take $g, x \in G$ and $f \in X$. Then

$$\begin{aligned} \left(({}_g f)_{G/H} \circ p\right)(x) &= m_H(({}_x({}_g f))_{|H}) \\ &= m_H(({}_{xg} f)_{|H}) \\ &= (f_{G/H} \circ p)(xg) \\ &= f_{G/H}(p(x)p(g)). \end{aligned}$$

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Therefore, $({}_g f)_{G/H} = (f_{G/H})_{p(g)}$. But $\mu_{G/H}$ is right-invariant, so

$$\begin{aligned} m({}_g f) &= \int_{G/H} ({}_g f)_{G/H}(t) d\mu_{G/H}(t) \\ &= \int_{G/H} \left(f_{G/H}\right)_{p(g)}(t) d\mu_{G/H}(t) \\ &= \int_{G/H} f_{G/H}(t) d\mu_{G/H}(t) \\ &= m(f). \end{aligned}$$

So Claim 5.4.1 is proved. □

Write $Y := \{f \in L^\infty(G) : p(\mathrm{supp}(f)) \text{ rel. compact}\}$. Following the proof of [Gre69, Lem. 2.2.2], one now sees that for any $\phi \in C_c(G)$ taking non-negative values and such that $\int_G \phi(t) d\mu_G(t) = 1$, the map

$$\begin{aligned} m_\phi : Y &\longrightarrow \mathbb{R} \\ f &\longmapsto m(\phi * f) \end{aligned}$$

is independent of ϕ and m_ϕ is nontrivial, left-invariant and positive. Let us recall the argument here. The map m_ϕ is positive because m is positive and because $\phi * f$ takes non-negative values as soon as f and ϕ take non-negative values. That m_ϕ takes finite values is due to the fact that for ϕ continuous compactly supported and $f \in Y$, we have $\phi * f \in X$ – indeed, $p(\mathrm{supp}\, \phi * f)$ is clearly relatively compact and $\phi * f$ is right uniformly continuous by [Gre69, Lem. 2.1.2]. Furthermore, let K denote the support of ϕ, and let $f \in X$ be a non-negative function with $f(x) = 1$ for all $x \in K^{-1}KH$ (we have explained how to construct such a function in the proof of Claim 5.4.1). Then $\phi * f(x) = 1$ for all $x \in KH$. So $m(\phi * f) \geq \mu_{G/H}(p(K)) > 0$ by the proof of Claim 5.4.1. It remains to prove the independence on ϕ. Take $f \in Y$ and write

$$\begin{aligned} m^f : C_c(G) &\longrightarrow \mathbb{R} \\ \phi &\longmapsto m(\phi * f). \end{aligned}$$

We will show that m^f is a Haar measure. By using linearity of m and bilinearity of convolution, we may reduce to the case where f takes non-negative values. In addition, for $g \in G$, $f \in Y$ and $\phi \in C_c(G)$, we have ${}_g(\phi * f) = ({}_g\phi) * f$. Therefore, m^f is a left-invariant, positive linear functional because m is. By the Riesz–Markov representation theorem, there is a Radon measure λ such that $m^f(\phi) = \int_G \phi d\lambda$. Since m^f is left-invariant, λ is a Haar measure – that is, there is a constant $k(f)$ such that

$$m_\phi(f) = m^f(\phi) = k(f) \int_G \phi d\mu_G.$$

Hence, if $\phi_1, \phi_2 \in C_c(G)$ with $\int_G \phi_i d\mu_G = 1$ for $i = 1, 2$, then $m_{\phi_1}(f) = m_{\phi_2}(f)$, which proves independence on ϕ. That m_ϕ is left-invariant is now a consequence of the independence on ϕ; see the argument in [Gre69, Prop. 2.1.3] which holds verbatim. So the map defined over Borel subsets $B \in \mathcal{B}(WH)$ – where $W \subset G$ is a symmetric compact neighbourhood of the identity – by $B \mapsto m_\phi(\mathbb{1}_B)$ is as in Definition 5.1. So WH is amenable. □

We will now prove the second step.

Proposition 5.15. *Let Λ and Ξ be two closed approximate subgroups of a second countable locally compact group G, and suppose that $\Lambda \subset \Xi$. If Ξ is amenable, then Λ is amenable.*

We start with a lemma that is essentially about building a local section of approximate subgroups.

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Lemma 5.16. *Let Λ be a closed approximate subgroup in a first-countable locally compact group G. For any neighbourhood of the identity W of G, there is a Borel subset $S \subset W$ such that $S\Lambda^2$ has nonempty interior and $S^{-1}S \cap \Lambda^2 = \{e\}$. Suppose moreover that G is second-countable, and take $X \subset G$. Then there is a Borel subset $S' \subset XW$ with $(S')^{-1}S' \cap \overline{\Lambda^2} = \{e\}$ and $X \subset S'\overline{\Lambda^4}$.*

Proof. Let V be a symmetric compact neighbourhood of the identity with V^6 contained in the intersection of W and an open neigbourhood U of e given by Lemma 4.2 such that $\Lambda^8 \cap U \subset \Lambda^2 \cap U$. Define the relation $\sim$ on V by $g \sim h$ if and only if $g \in h\Lambda^2$. We have that $\sim$ is reflexive and symmetric because $e \in \Lambda^2$ and Λ^2 is symmetric. In addition, given $g_1, g_2, g_3 \in V$ such that $g_1 \sim g_2$ and $g_2 \sim g_3$, we have $g_1 \in g_2\Lambda^2 \subset g_3\Lambda^4$. So $g_3^{-1}g_1 \in \Lambda^4 \cap V^2 \subset \Lambda^2$, which yields $g_1 \sim g_3$. Hence, $\sim$ is an equivalence relation. Write Y the quotient space $V/\sim$ endowed with the quotient topology, and let $q : V \to Y$ be the quotient map. Notice that $\overline{\Lambda^2} \cap U = \Lambda^4 \cap U = \Lambda^2 \cap U$. This means that $g \sim h$ if and only if $g \in h\overline{\Lambda^2}$. Take any subset $X \subset V$; then $q^{-1}(q(X)) = X\Lambda^2 \cap V = X\overline{\Lambda^2} \cap V$. This implies that q is a continuous surjective map. Since V is compact, q is furthermore a perfect proper map. As the image of a compact space by a continuous map, Y is thus compact. Since q is a continuous closed map and V is normal, Y is a normal space [Eng89, Thm. 1.5.20]. Moreover, since G is first-countable and $V \subset G$ is compact, V is second countable. Now, the image of a second countable space by a perfect map is second countable [Eng89, Thm. 3.7.19], so Y is second countable. So Y is metrizable by [Eng89, Thm. 4.2.8]. By a theorem of Federer and Morse [FM43, Thm. 5.1], there is a Borel subset $S \subset V$ such that $q_{|S}$ is bijective (i.e., $S^{-1}S \cap \Lambda^2 = \{e\}$ and $V \subset S\Lambda^2$).

Now, suppose G is second countable. Since $\overline{X}$ is second countable, there is a sequence $(g_n)_{n\geq 0}$ of elements of X with $g_0 = e$ such that $(g_nS\overline{\Lambda^2})_{n\geq 0}$ covers $\overline{X}$. Define recursively $S_0 = S$ and

$$S_{n+1} = g_{n+1}S \setminus \bigcup_{m\leq n} S_m\overline{\Lambda^2} \subset XW$$

for all $n \geq 0$. We claim that S_n is a Borel subset for all $n \geq 0$. If $n = 0$, the result is clear. We proceed now by induction on n. By assumption, S_m is a Borel subset for all $m < n$. Moreover,

$$S_m^{-1}S_m \cap \overline{\Lambda^2}^2 \subset S^{-1}S \cap \overline{\Lambda^2}^2 \subset S^{-1}S \cap \Lambda^2 = \{e\},$$

where the second inclusion is a consequence of $S^{-1}S \subset U$ and $\overline{\Lambda^2} \subset \Lambda^4$. Therefore, the multiplication map $G \times G \to G$ is injective when restricted to the Borel subset $S_m \times \overline{\Lambda^2}$. Since G and $G \times G$ are second countable locally compact groups – hence, Polish spaces – we thus have by the Lusin–Souslin theorem [Kec95, Thm. 15.1] that $S_m\overline{\Lambda^2}$ is Borel, and so is S_n. Define now $S_\infty := \bigcup_{n\geq 0} S_n$. We have that $S_\infty^{-1}S_\infty \cap \overline{\Lambda^2} = \bigcup_{n\geq 0}\left(S_n^{-1}S_n \cap \overline{\Lambda^2}\right) = \{e\}$ as $S_n \cap S_m\overline{\Lambda^2} = \emptyset$ for all $n > m$. Also, by induction on n, $g_nS \subset S_\infty\overline{\Lambda^2}$ for all $n \geq 0$. So $X \subset S_\infty\overline{\Lambda^2}^2 \subset S_\infty\overline{\Lambda^4}$ (in fact, we find $\overline{X} \subset S_\infty\overline{\Lambda^4}$). □

Proof of Proposition 5.15. By Theorem 1.3, we can assume that Ξ has nonempty interior in G. Take a Borel section S given by Lemma 5.16 applied to the approximate subgroup Λ, the subset $X = \Xi$ and $W \subset \Xi^2$. Set $S' := S^{-1}$. We have $\Xi \subset \overline{\Lambda^4}S'$ and $\Lambda S' \subset \overline{\Xi^4}$. In addition, the multiplication map $\Lambda \times S' \to \Lambda S'$ is a bijective measurable map. Hence, by the Lusin–Souslin theorem [Kec95, Thm. 15.1] (note that the hypotheses are verified since G and $G \times G$ are second countable locally compact groups, hence Polish), if $B \subset \Lambda$ is any Borel subset, then BS' is Borel. For all Borel subsets $B \subset \Lambda$, define therefore $m'(B) := m(BS')$. We have that m' is a locally left-invariant finitely additive measure with $m'(\Lambda) < \infty$. And we claim that $0 < m'(\Lambda)$. There is indeed a finite subset F of G such that $\Xi \subset F\Lambda S'$. By left-invariance of m, we find that $m'(\Lambda) \geq \frac{1}{|F|}m(\Xi) > 0$. We conclude by invoking Lemma 5.2. □

Proof of Proposition 5.13. We have that Λ is contained in WH with W a symmetric compact neighbourhood of the identity. But WH is amenable by Proposition 5.14. So Λ is amenable by Proposition 5.15. □

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Corollary 5.17. *Let G be an amenable second countable locally compact group. If $\Lambda \subset G$ is a closed approximate subgroup, then Λ is amenable.*

Proof. Corollary 5.17 is a consequence of Proposition 5.13 applied to G, and $H = G$. □

Corollary 5.18. *If Λ is an approximate subgroup of a countable discrete amenable group G, then Λ is amenable.*

Proof. The group G equipped with the discrete topology is an amenable second countable locally compact group. Moreover, Λ is obviously a closed approximate subgroup in this topology. So Corollary 5.18 is a consequence of Corollary 5.17. □

5.5. Structure of approximate subgroups in amenable groups

The Meyer-type theorem (Theorem 1.5) is the most immediate consequence of the above results:

Proof of Theorem 1.5. Since Λ is uniformly discrete and G is Hausdorff, Λ is closed. Also, as Λ is uniformly discrete, it is amenable by Corollary 5.17, and so the approximate subgroup $\overline{\Lambda^4} = \Lambda^4$ has a good model, f say, by Proposition 5.7. Thus, the approximate subgroup Λ^4 is contained in and commensurable to a model set by Proposition 1.2. (In fact, the proof of Proposition 1.2 reveals that $\Lambda^4 \ker(f) \subset \Lambda^8$ is a model set). □

Another key consequence is the fact that approximate lattices are uniform, when considered in the right ambient group:

Proposition 5.19. *Let Λ be an approximate lattice in an amenable locally compact second countable group G.*

1. *there is $L \subset G$ a closed subgroup such that Λ is covered by finitely many cosets of L, and $\Lambda^2 \cap L$ is a uniform approximate lattice in L;*
2. *if G is an S-adic Lie group (i.e., locally a finite product of real and p-adic Lie groups; see, for example, [BQ14]), then Λ is a uniform approximate lattice.*

Proof. We know that Λ is amenable (Proposition 5.15). Let Λ' and $f : \langle\Lambda'\rangle \to H$ be given by Proposition 5.6. Since Λ' is uniformly discrete and G is second countable, Λ' is countable. Write $p : H \to H/R$ the projection to the quotient of H by the radical. We know that $p \circ f(\Lambda')$ contains a neighbourhood of the identity. Since H/R is connected and Λ' is countable, H/R must be trivial. So H is soluble. Hence, $G \times H$ is amenable (for soluble groups are amenable and products of amenable groups are amenable). By Proposition 3.18, the graph Γ_f of f is a lattice in $G \times H$. According to [BQ14, Prop. 3.7], Γ_f is a uniform lattice in $\overline{\Gamma_f(G^0 \times H)}$, where G^0 denotes the connected component of the identity in G. But $\overline{\Gamma_f(G^0 \times H)} = \overline{\langle\Lambda'\rangle G^0} \times H$. Write $L := \overline{\langle\Lambda'\rangle G^0}$; then L satisfies the conclusions of (1) by [BH18, Prop. 2.13].

Let us prove (2). There are Λ' commensurable to Λ and $f : \langle\Lambda'\rangle \to H$ a good model of Λ' with dense image and target a connected Lie group without nontrivial normal compact subgroups by Corollary 5.17, Proposition 5.7 and Proposition 3.9.(2). As in the first paragraph, we also have that H is soluble. So the graph $\Gamma_f \subset G \times H$ of f is a lattice by Proposition 3.18, and there is a symmetric compact neighbourhood of the identity $W_0 \subset H$ such that $\Lambda' \subset P_0(G, H, \Gamma_f, W_0) \subset \Lambda'^2$. Since $G \times H$ is amenable, we know by [BQ14, Prop. 5.1] that Γ_f is a uniform lattice in $G \times H'$. So Λ' – hence, Λ – is uniform by [BH18, Prop. 2.13]. □

We are also able to prove finite generation of discrete approximate subgroups of soluble Lie groups – extending a classical result concerning discrete subgroups (see [Rag72, Prop. 3.8]):

Proposition 5.20. *Let R be a connected soluble Lie group. If $\Lambda \subset R$ is a uniformly discrete approximate subgroup, then $\langle\Lambda\rangle$ is finitely generated.*

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Proof. According to Corollary 5.17 and Proposition 5.7, the approximate subgroup Λ^4 has a good model. According to Lemma 3.4, there is an approximate subgroup $\Lambda' \subset \Lambda^4$ that has a good model $f : \langle\Lambda'\rangle \to H$ with dense image and target a connected Lie group. Since $\langle\Lambda'\rangle$ is soluble, we obtain that H is soluble. Now, the graph of f, denoted by Γ_f, is a discrete subgroup of the connected soluble Lie group $R \times H$ (Lemma 3.17), and therefore, Γ_f is finitely generated by [Rag72, Prop. 3.8]. Let $F_1 \subset \Lambda'$ be a finite set of generators of $\langle\Lambda'\rangle$ and $F_2 \subset \langle\Lambda\rangle$ be a finite subset such that $\Lambda \subset F_2\Lambda'$. Then $F_1 \cup F_2$ is a finite set that generates $\langle\Lambda\rangle$. □

6. Generalisation of theorems of Mostow and Auslander

6.1. Strong approximate lattices

In this section, we will also consider objects related to approximate lattices called 'strong approximate lattices'. They are defined as follows.

Let G be a locally compact second countable group, and let $\mathcal{C}(G)$ be the set of closed subsets of G. The *Chabauty-Fell* topology on $\mathcal{C}(G)$ is defined by the subbase of open subsets

$$U^V = \{F \in \mathcal{C}(G) | F \cap V \neq \emptyset\} \text{ and } U_K = \{F \in \mathcal{C}(G) | F \cap K = \emptyset\}$$

for all $V \subset G$ open and $K \subset G$ compact. One can check that the map

$$\begin{aligned} G \times \mathcal{C}(G) &\to \mathcal{C}(G) \\ (g, F) &\mapsto gF \end{aligned}$$

defines a continuous action of the group G on $\mathcal{C}(G)$ and that $\mathcal{C}(G)$ is a compact second countable set (see [Fel62]). Convergence in the Chabauty-Fell topology can also be characterised in the following way: a sequence $(F_i)_{i\geq 0}$ converges to $F \in \mathcal{C}(G)$ if and only if (1) for every $x \in F$, there are $x_i \in F_i$ for all $i \in \mathbb{N}$ such that $x_i \to x$ as $i \to \infty$; (2) If $x_i \in F_i$ for all $i \in \mathbb{N}$, then every accumulation point of $(x_i)_{i\geq 0}$ lies in F (see [BHS19, §2.2]). Given a closed subset F of G, we define the *invariant hull* Ω_F of F as the closure of the G-orbit of F (i.e., $\overline{G \cdot F}$) equipped with the induced continuous G-action. Note that if H is a closed subgroup, then Ω_H is isomorphic as a compact G-space to the one-point compactification of G/H ([BHS19, Lemm. 6]).

Definition 6.1 [BH18]. Let Λ be an approximate subgroup of a locally compact second countable group G. We say that Λ is a *strong approximate lattice* if

1. Λ is uniformly discrete – that is, $\Lambda^2 \cap V = \{e\}$ for some neighbourhood of the identity V;
2. there is a G-invariant Borel probability measure ν on Ω_Λ with $\nu(\{\emptyset\}) = 0$ (we say that ν is *proper*).

In particular, a subgroup is a lattice if and only if it is a strong approximate lattice.

We proved in [Mac22a, §2.2.8] that all strong or uniform approximate lattices are approximate lattices, but it is not known whether the converse holds or not. Nonetheless, when the ambient group is amenable, the notions of strong approximate lattices and approximate lattices are equivalent ([Mac22a, Lem. 2.2.32]).

Finally, it was shown in [BHP18, Cor. 3.5] that model sets built with windows satisfying mild regularity assumptions are strong approximate lattices.

6.2. Intersections of approximate lattices and closed subgroups

We will show a general theorem about intersections of approximate lattices and closed subgroups. Proposition 6.3 is close in spirit to a classical fact about lattices (see, for instance, [Rag72, Thm. 1.13]). See also [BH22] for other results around this topic in the framework of strong (and) uniform approximate lattices.

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We start with a related property concerning measures on the hull:

Proposition 6.2. *Let G be a second countable locally compact group, $X_0 \subset G$ be a uniformly discrete and $H \subset G$ be a closed normal subgroup. Write $p : G \to G/H$ be the natural projection, and suppose that Ω_{X_0} admits a proper G-invariant Borel probability measure ν_0. If $p(X_0)$ is uniformly discrete, then there is $X \in \Omega_{X_0}$ such that $\Omega_{X \cap H}$ admits a proper H-invariant Borel probability measure.*

Proof. Let $\mathcal{P}(X_0, H)$ denote the set of H-invariant Borel probability measures. Since $\nu_0 \in \mathcal{P}(X_0, H)$ and is proper, take ν_1 that appears in the ergodic decomposition ([EW11, Thm. 8.20]) of ν_0. Then $\nu_1 \in \mathcal{P}(X_0, H)$ is ergodic and $\nu_1(\{\emptyset\}) = 0$. Then ν_1 is a Borel probability measure on the compact metrizable space X_0, so $\nu_1(X_0 \setminus K) = 0$, where K denotes the support of ν_1. The compact subset K is H-invariant, and for any open subset $U \subset \Omega_{X_0}$, we have $\nu_1(U \cap K) = 0$ if and only if $U \cap K = \emptyset$. Thus, according to, for example, [Zim84, Prop. 2.1.7], there is $X_1 \in K$ such that $K = \overline{H \cdot X_1}$. Furthermore, $X_1 \neq \emptyset$ since $\overline{H \cdot \emptyset} = \{\emptyset\}$ and $\nu_1(\{\emptyset\}) = 0$. Choose now $x_1 \in X_1$. Then $e \in x_1^{-1}X_1 \subset \overline{X_0^{-1}X_0}$ by [BH18, Lem. 4.6]. Moreover, the subgroup H is normal so the pull-back $\nu_2 := \left(x_1^{-1}\right)_* \nu_1$ is an H-invariant ergodic Borel probability measure with $\nu_2(\{\emptyset\}) = 0$ and support $x_1^{-1}K = \overline{H \cdot X_2}$, where $X_2 = x_1^{-1}X_1$. Define the map

$$\begin{aligned} \pi : \overline{H \cdot X_2} &\longrightarrow \mathcal{C}(H) \\ X &\longmapsto X \cap H. \end{aligned}$$

We see that π is H-equivariant. We claim moreover that π is continuous. Since $p(X_0)$ is uniformly discrete, there is an open subset $U \subset G$ such that $\overline{X_0^{-1}X_0H} \cap U = H$. Note that for all $X \in \overline{H \cdot X_2}$, we have $X \subset \overline{X_2^{-1}X_2} = \overline{X_1^{-1}X_1} \subset \overline{X_0^{-1}X_0}$ by [BH18, Lem. 4.6]. So $p(X) \subset p(\overline{X_0^{-1}X_0})$ for all $X \in \overline{H \cdot X_2}$. So for any open subset $V \in H$, we have

$$\begin{aligned} \pi^{-1}(U^V) &= \pi^{-1}(\{Y \in \mathcal{C}(H) | Y \cap V \neq \emptyset\}) \\ &= \{X \in \overline{H \cdot X_2} | X \cap (U \cap W) \neq \emptyset\} \\ &= U^{U \cap W} \cap \overline{H \cdot X_2}, \end{aligned}$$

where $W \subset G$ is any open subset such that $H \cap W = V$. Likewise for any compact subset $L \subset H$, we have

$$\begin{aligned} \pi^{-1}(U_L) &= \pi^{-1}(\{Y \in \mathcal{C}(H) | Y \cap L = \emptyset\}) \\ &= \{X \in \overline{H \cdot X_2} | X \cap L = \emptyset\} \\ &= U_L \cap \overline{H \cdot X_2}, \end{aligned}$$

where we consider $L \subset H \subset G$. So π is indeed a continuous map. Thus, $\pi(\overline{H \cdot X_2})$ is a compact subset of $\mathcal{C}(H)$, and $\pi(X_2) = X_2 \cap H$ has dense orbit in $\pi(\overline{H \cdot X_2})$. So $\pi(\overline{H \cdot X_2}) = \Omega_{X_2 \cap H}$. Set $\nu_3 := \pi_*\left((\nu_2)_{|\overline{H \cdot X_2}}\right)$, where $(\nu_2)_{|\overline{H \cdot X_2}}$ is the restriction of the measure ν_2 to its support $\overline{H \cdot X_2}$, which is a well-defined H-invariant ergodic Borel probability measure since Ω_{X_0} is metric compact. Then ν_3 is a H-invariant ergodic Borel probability measure on $\Omega_{X_2 \cap H}$. Suppose now that $\nu_3(\{\emptyset\}) > 0$; then $\nu_3(\{\emptyset\}) = 1$ by ergodicity. Thus, $\pi^{-1}(\{\emptyset\})$ is an H-invariant compact co-null subset of $\overline{H \cdot X_2}$, which means $\pi^{-1}(\{\emptyset\}) = \overline{H \cdot X_2}$ because $\overline{H \cdot X_2}$ is the support of ν_2. Therefore, $\pi(X_2) = \emptyset$: a contradiction. Hence, $\nu_3(\{\emptyset\}) = 0$, so ν_3 is a proper H-invariant Borel probability measure on $\Omega_{X_2 \cap H}$. So $X := X_2$ works. □

Proposition 6.3. *Let Λ be a uniformly discrete approximate subgroup of a locally compact group G. Assume that H is a closed subgroup of G such that $p(\Lambda)$ is locally finite where $p : G \to G/H$ is the natural map. We have*

1. *if* Λ *is a uniform approximate lattice, then* $\Lambda^2 \cap H$ *is a uniform approximate lattice in* H*;*
2. *if* Λ *is a strong approximate lattice,* G *is second countable and* H *is normal and amenable, then* $\Lambda^2 \cap H$ *is a strong approximate lattice in* H*;*
3. *if* Λ *is an approximate lattice,* G *is second countable and* H *is normal, then* $\Lambda^2 \cap H$ *is an approximate lattice in* H*.*

Proof. We will first prove (1). We know that $\Lambda^2 \cap H$ is an approximate subgroup according to Lemma 2.3. Moreover, since Λ is uniformly discrete, so is $\Lambda^2 \cap H$. We must prove that $\Lambda^2 \cap H$ is co-compact in H. Let $K \subset G$ be a compact subset such that $K\Lambda = G$. Since $p(\Lambda)$ is locally finite, there is $F \subset \Lambda$ finite such that $K^{-1}H \cap \Lambda \subset FH$. Take any $h \in H$; then there are $\lambda \in \Lambda$ and $k \in K$ such that $k\lambda = h$, which implies $\lambda \in K^{-1}H \cap \Lambda$, and we can find $f \in F$ such that $f^{-1}\lambda \in H \cap \Lambda^2$. Therefore, $h \in KF\left(H \cap \Lambda^2\right)$.

Let us move on to the proof of (2). Again, note that $\Lambda^2 \cap H$ is uniformly discrete and is an approximate subgroup according to Lemma 2.3. By Proposition 6.2, there is $P \subset \Lambda^2$ such that $\Omega_{P\cap H}$ admits a proper H-invariant Borel probability measure. If H is amenable, there is by [Ros81, Thm. 1.10] an admissible probability measure μ on H such that any μ-stationary Borel probability measure on $\Omega_{\Lambda^2\cap H}$ is H-invariant (see [BH18, Rem. 4.14] for the necessary definitions). Any H-invariant measure is μ-stationary, so $\Omega_{P\cap H}$ admits a proper μ-stationary Borel probability measure. Since $\Lambda^2 \cap H \supset P \cap H$, we know according to [BHS19, Lem. 31 and Cor. 10] that $\Omega_{\Lambda^2\cap H}$ admits a proper μ-stationary Borel probability measure. Hence, $\Omega_{\Lambda^2\cap H}$ admits a proper H-invariant Borel probability measure, meaning that the approximate subgroup $\Lambda^2 \cap H \supset P \cap H$ is a strong approximate lattice in H.

Let us show (3). We will rely on an equivalent definition of approximate lattices established in [Hru22, App. A] and the quotient formula for the Haar measure. Since $p(\Lambda)$ is locally finite and is an approximate subgroup, it is uniformly discrete. Therefore, we may choose $F_{G/H}$ a measurable subset of positive Haar measure (possibly infinite) such that the multiplication map $p(\Lambda) \times F_{G/H} \to G/H$ is one-to-one. Take a measurable section of $F_{G/H}$ in G – that is, a Borel subset $\tilde{F}_{G/H} \subset G$ such that the projection from $\tilde{F}_{G/H} \subset G$ to $F_{G/H}$ is bijective (such a subset exists by [Keh84, Thm., p156]). Let now $F_H \subset H$ be any Borel subset of positive measure such that the multiplication $(\Lambda^2 \cap H) \times F_H \to H$ is one-to-one. We first notice that the multiplication map $\Lambda \times F_H\tilde{F}_{G/H} \to G$ is one-to-one. Indeed, take $\lambda_1, \lambda_2 \in \Lambda$, $\tilde{f}_1, \tilde{f}_2 \in \tilde{F}_{G/H}$ and $f_1, f_2 \in F_H$ such that $\lambda_1 f_1 \tilde{f}_1 = \lambda_2 f_2 \tilde{f}_2$. Projecting to G/H, we see that $\tilde{f}_1 = \tilde{f}_2$ and $p(\lambda_1) = p(\lambda_2)$. So $\lambda_2^{-1}\lambda_1 f_1 = f_2$. But $\lambda_2^{-1}\lambda_1 \in \Lambda^2 \cap H$, so $f_1 = f_2$ and $\lambda_1 = \lambda_2$. By the Lusin–Souslin theorem [Kec95, Thm. 15.1], $F_H\tilde{F}_{G/H}$ is measurable. And by [Hru22, Prop. A.2], we see that $F_H\tilde{F}_{G/H}$ has finite Haar measure. According to the quotient formula [Bou04, Ch. VIII, §2.7, Prop. 10], there are left-Haar measures μ_H and $\mu_{G/H}$ on H and G/H, respectively, such that

$$\int_G f(g)d\mu_G(g) = \int_{G/H}\int_H f(hq(g))\Delta(q(g))d\mu_H(h)d\mu_{G/H}(g),$$

where $q : G/H \to G$ denotes a Borel section and $\Delta : G \to \mathbb{R}_{>0}$ is the unique scalar such that $\Delta(g)\mu_H$ is the push-forward measure of μ_H under conjugation by $g \in G$. By uniqueness of Haar measures up to scalars, Δ is a well-defined group homomorphism, and testing the equality of measures against continuous functions shows that Δ is continuous. Since $F_H\tilde{F}_{G/H}$ has finite Haar measure, we deduce from the quotient formula that F_H has finite measure. So by [Hru22, Prop. A.2], again $\Lambda^2 \cap H$ is an approximate lattice in H.

To fully exploit the quotient formula, we show now that $\Delta(g) = 1$ for all $g \in G$ (mimicking the proof of unimodularity from [BH18]). Let $U \subset H$ be a symmetric neighbourhood of the identity such that $U^2 \cap \Lambda^6 = \{e\}$. For all $\lambda \in \Lambda$, we have $\lambda^{-1}U^2\lambda \cap (\Lambda^2 \cap H)^2 = \{e\}$. By [Hru22, Prop. A.2],

$$\Delta(\lambda)\mu_H(U) \leq \mu_H(\lambda^{-1}U\lambda) \leq C,$$

where C is a real number independent of λ. Hence, $\Delta(\Lambda) \subset \mathbb{R}_{>0}$ is bounded away from $+\infty$ (and 0 by symmetry). But Λ^2 has property (S) (see Definition 6.7 and Proposition 6.8). So one sees that $\Delta(G)$ is bounded (i.e., $\Delta(G) = \{1\}$). By the quotient formula once more, both F_H and $F_{G/H}$ have finite

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Haar measure in H and G/H, respectively. So by [Hru22, Prop. A.2] again, both $\Lambda^2 \cap H$ and $p(\Lambda)$ are approximate lattices. □

Remark 6.4. We use Proposition 6.3 in the companion paper [Mac23] to define and study a notion of irreducibility for approximate lattices. In the language of [Mac23], part (2) reduces to showing that $\Lambda^2 \cap H$ is a $\star$-approximate lattice, a notion close to the one of strong approximate lattice.

We prove next a converse of sorts of Proposition 6.3:

Lemma 6.5. *Let Λ be a uniformly discrete approximate subgroup of a locally compact group G. Assume that H is a closed subgroup of G such that $\Lambda^2 \cap H$ is a uniform approximate lattice in H. Then $p(\Lambda)$ is a locally finite subset of G/H where $p : G \to G/H$ is the natural map.*

Proof. Let $K \subset G/H$ be a compact subset. Then there is a compact subset $L \subset G$ such that $p(L) = K$. Since $\Lambda^2 \cap H$ is relatively dense in H, there is a compact subset $L' \subset G$ such that $LH \subset L'(\Lambda^2 \cap H)$. Take $\lambda \in \Lambda \cap LH$. Since $\lambda \in (L'(\Lambda^2 \cap H))$, we have $\lambda \in ((L' \cap \Lambda^3)(\Lambda^2 \cap H))$. So $p(\Lambda) \cap K \subset p(L' \cap \Lambda^3)$, which is indeed finite. □

As a first application, we investigate the intersections of uniform approximate lattices with centralisers:

Corollary 6.6. *Let Λ be a uniform approximate lattice in a locally compact group G. Then for all $\gamma \in \langle\Lambda\rangle$, the approximate subgroup $\Lambda^2 \cap C(\gamma)$ is a uniform approximate lattice in $C(\gamma)$ the centraliser of γ. Moreover, if G is a Lie group and $\langle\Lambda\rangle$ is dense in G, then $\Lambda^2 \cap Z(G)$ is a uniform approximate lattice in $Z(G)$ the centre of G.*

Proof. Let $n \geq 0$ be an integer such that $\gamma \in \Lambda^n$, and consider the map

$$\begin{aligned} \varphi : G &\longrightarrow G \\ g &\longmapsto g\gamma g^{-1}. \end{aligned}$$

Then φ factors as $\varphi = \psi \circ p$, where $\psi : G/C(\gamma) \to G$ is a continuous injective map and $p : G \to G/C(\gamma)$ is the natural map. But $\varphi(\Lambda) \subset \Lambda^{n+2}$ and so is locally finite. Hence, $p(\Lambda)$ is locally finite as well. By part (1) of Proposition 6.3, we deduce that $\Lambda^2 \cap C(\gamma)$ is a uniform approximate lattice in $C(\gamma)$.

Now if G is a Lie group and $\langle\Lambda\rangle$ is dense in G, then $Z(G) = \bigcap_{\gamma \in \langle\Lambda\rangle} C(\gamma)$. But $Z(G)$ is a Lie group and so are the $C(\gamma)$'s. Thus, there are $\gamma_1, \dots, \gamma_r \in \langle\Lambda\rangle$ such that $\dim(Z(G)) = \dim(\bigcap_i C(\gamma_i))$. Consider now the map

$$\begin{aligned} \varphi : G &\longrightarrow G^r \\ g &\longmapsto (g\gamma_1 g^{-1}, \dots, g\gamma_r g^{-1}). \end{aligned}$$

As above, φ factors as $\varphi = \psi \circ p$ with $\psi : G/(\bigcap_i C(\gamma_i)) \to G^r$ an injective and continuous map and $p : G \to G/(\bigcap_i C(\gamma_i))$ the natural map. But $\varphi(\Lambda) \subset \prod_{1 \leq i \leq r} \Lambda^{n+2}$, where n is a positive integer such that $\{\gamma_1, \dots, \gamma_r\} \subset \Lambda^n$. Thus, $\varphi(\Lambda)$ is locally finite and so is $p(\Lambda)$. By part (1) of Proposition 6.3, we deduce that $\Lambda^2 \cap \bigcap_i C(\gamma_i)$ is a uniform approximate lattice in $\bigcap_i C(\gamma_i)$. But $Z(G)$ is an open subgroup of $\bigcap_i C(\gamma_i)$, so $p'(\Lambda^2 \cap \bigcap_i C(\gamma_i))$ is obviously locally finite where $p' : \bigcap_i C(\gamma_i) \to (\bigcap_i C(\gamma_i))/Z(G)$ is the natural map. By part (1) of Proposition 6.3, once again we have that

$$\left(\Lambda^2 \cap \bigcap_i C(\gamma_i)\right)^2 \cap Z(G) \subset \Lambda^4 \cap Z(G)$$

is a uniform approximate lattice in $Z(G)$. By Lemma 2.3, we find that $\Lambda^2 \cap Z(G)$ is a uniform approximate lattice in $Z(G)$. □

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6.3. Borel density for approximate lattices

The Borel density theorem asserts that lattices in simple algebraic groups are Zariski-dense. The usual route to show Borel density-type theorems for groups with finite co-volume is to start by proving that the subgroup considered has *property (S)* (see, for example, [Bor60]).

Definition 6.7 (Definition 1.1, [Bor60]). Let G be a locally compact group. A closed subset $X \subset G$ has *property (S)* if for all neighbourhoods $W \subset G$ of the identity and all $g \in G$, there is $n \in \mathbb{N}$ such that $g^n \in WXW$.

Approximate lattices have property (S). Hence, they exhibit similar density properties.

Proposition 6.8. *Let G be a locally compact second countable group. We have*

1. *if* Λ *is an approximate lattice in G, then* Λ^2 *has property* (S) *([Hru22, A.11]) ;*
2. *if X is a closed subset and* Ω_X *has a proper G-invariant Borel probability measure, then* $\overline{X^{-1}X}$ *has property* (S).

While the (2) will not be needed, it serves as an illustration of the method to prove both (1) and (2).

Proof. Let us prove (2). By assumption, there is a proper G-invariant Borel probability measure ν on Ω_X. If W is any symmetric neighbourhood of the identity, then the open subset U^W satisfies $\nu(U^W) > 0$. Indeed, U^W is open and

$$\Omega_X \setminus \{\emptyset\} = \bigcup_{g \in G} U^{gW} = \bigcup_{g \in G} gU^W.$$

Since G is second countable, we can find $D \subset G$ countable such that

$$\Omega_X \setminus \{\emptyset\} = \bigcup_{d \in D} dU^W.$$

But $\nu(\Omega_X \setminus \{\emptyset\}) = 1$, so there is $d \in D$ such that $0 < \nu(dU^W) = \nu(U^W)$. Therefore, for any $g \in G$, there is an integer $1 \leq n < \left(\nu(U^W)\right)^{-1}$ such that $\nu(U^W \cap g^n U^W) > 0$. So we can find $P \in U^W \cap g^n U^W$. Thus, $P \cap W \neq \emptyset$ and $P \cap g^n W \neq \emptyset$. That implies that $P^{-1}P \cap Wg^nW \neq \emptyset$. But $P^{-1}P \subset \overline{X^{-1}X}$ ([BH18, Lem. 4.6]), so $g^n \in W\overline{X^{-1}X}W$. □

Remark 6.9. Borel density for approximate lattices was first investigated in [BHS19]. Their method was completely different, however. The proof of Proposition 6.8 has the advantage of yielding a short proof that can be directly applied to cases not covered by [BHS19] – for instance, approximate lattices in S-adic Lie groups.

6.4. Proof of Theorem 1.7

We will make use of the Tits alternative over local fields (see [Tit72]). For lack of an exact reference, we include a proof relying on [Tit72]:

Lemma 6.10. *Let k be a local field. Let* Γ *be a subgroup in* $\mathbb{G}(k)$ *the k-points of a simple algebraic group* $\mathbb{G}$ *defined over k (i.e.,* $\mathbb{G}$ *admits no nontrivial connected normal subgroup defined over k). Suppose that all finitely generated subgroups of* Γ *are virtually soluble; then* Γ *is not Zariski-dense.*

Proof. When k has characteristic 0, the Tits' alternative [Tit72, Thm. 1] implies that Γ is virtually soluble. So it cannot be Zariski-dense. Suppose that k has positive characteristic. Suppose that Γ is Zariski-dense. If Γ contains a finite subset X that generates an infinite subgroup, then choose one such that the Zariski-closure H_X of $\langle X \rangle$ has maximal dimension. By maximality, one finds that the Zariski-connected component of the identity of H_X is normalised by Γ. Therefore, $H_X = \mathbb{G}(k)$. But $\langle X \rangle$ is

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virtually soluble: a contradiction. So every finite subset generates a finite subgroup. Notice now that the prime field of k is a finite subfield. So any element algebraic over the prime field of k is contained in a finite subfield of k and, thus, is a root of unity. By the structure of the unit group of local fields [Neu99, Prop. 5.7(ii)], we find that there are only finitely many elements that are algebraic over the prime field of k (i.e., the intersection k_a of k with the algebraic closure of its prime field is a finite subfield). Apply now [Tit72, Thm. 4] to $\Gamma \subset \mathbb{G}(k)$. Since every finite subset of Γ generates a finite subgroup, the subgroup $\mathbb{G}_1$ is the trivial subgroup ([Tit72, Thm. 4. (i)]). By [Tit72, Thm. 4.(iv)] now, Γ is isomorphic to a subgroup of $\mathbb{G}_a(k_a)$, where $\mathbb{G}_a$ is an algebraic semi-simple group defined over the finite field k_a. Hence, Γ is finite. So Γ is not Zariski-dense. □

Proof of Theorem 1.7. Let W be a relatively compact neighbourhood of the identity in G. Define the uniformly discrete subset $\Lambda_A := \Lambda^2 \cap W^{-1}WA$. The subset Λ_A is an approximate subgroup that is commensurated by $\langle\Lambda\rangle$. Indeed, first notice that both Λ^2 and $W^{-1}WA$ are commensurated by $\langle\Lambda\rangle$, and then apply Lemma 2.2. We know in addition that Λ_A is an amenable discrete approximate subgroup (Proposition 5.15). Let $p_S : G \to S$ be the natural projection to any simple factor S of G/A. Now Lemma 5.4 provides an approximate subgroup Λ'_A commensurable with Λ_A such that every finite subset of $p_S(\Lambda'_A)$ generates a virtually soluble subgroup. Since G is second countable and Λ_A uniformly discrete, Λ_A is countable. Moreover, S is the group of k-points of a simple algebraic group defined over a local field, so we can apply Lemma 5.4. By Lemma 6.10, the group generated by $p_S(\Lambda'_A)$ is not Zariski-dense. In particular, the Zariski-closed Zariski-connected subgroup H provided by [BHS19, Thm. 17] applied to $p_S(\Lambda'_A)$ is a proper subgroup of S. Indeed, recall that [BHS19, Thm. 17] implies that there is $g \in S$ normalising H and $F \subset S$ finite such that

$$gH \subset \overline{p_S(\Lambda'_A)}^Z \subset FH,$$

where $\overline{p_S(\Lambda'_A)}^Z$ denotes the Zariski-closure of $p_S(\Lambda'_A)$. Notice now that the commensurator of $p_S(\Lambda'_A)$ contains $p_S(\langle\Lambda\rangle)$ and so is Zariski-dense. If $s \in S$ is any element commensurating $p_S(\Lambda'_A)$, then there is $F' \subset S$ finite such that $sp_S(\Lambda'_A)s^{-1} \subset F'p_S(\Lambda'_A) \subset F'FH$. Hence, $sgs^{-1}(sHs^{-1}) \subset F'FH$, which implies that $sHs^{-1} \subset H$ since H is Zariski-connected. So H is normalised by the commensurator of $p_S(\Lambda'_A)$ – which is Zariski-dense – and is a proper subgroup of S. Since S is almost simple, H must be finite. We conclude that $p_S(\Lambda'_A)$ – hence, $p_S(\Lambda_A)$ – is finite. As a conclusion, the projection of Λ_A to G/A must be finite (i.e., the projection of Λ to G/A is uniformly discrete). □

Theorem 1.7 now implies a crucial decomposition theorem for approximate lattices answering a question raised in [Hru22, Question 7.11]:

Corollary 6.11 (Auslander–Mostow-type theorem for approximate lattices). *Let Λ be an approximate lattice in a locally compact second countable group G. Let A be an amenable closed normal subgroup, and suppose that G/A is a finite direct product of simple algebraic groups over local fields as a topological group. Suppose also that the projections of $\langle\Lambda\rangle$ to all compact simple factors of G/A are Zariski-dense. Then*

1. *$\Lambda^2 \cap A$ is an approximate lattice in A;*
2. *the projection of Λ to G/A is an approximate lattice in G/A.*

Note that the assumption on compact factors is not part of [Hru22, Question 7.11]. Without this additional assumption, however, [Hru22, Question 7.11] admits a negative answer as can already be seen from [Gen15, §3]. This assumption is moreover easy to impose, as it is in particular satisfied when $\langle\Lambda\rangle$ projects to a Zariski-dense subgroup of G/A.

Proof. We have that $\langle\Lambda\rangle$ has property (S) (Proposition 6.8). By the Borel density theorem [Wan71, Cor. 1.4] (and [Mar91, Ch. I, Prop. 2.3.6], see more generally [Mar91, Ch. I, §2] for relevant backkground), we have that the projection of $\langle\Lambda\rangle$ to any non-compact simple factor is Zariski-dense. By our assumption, the projection of $\langle\Lambda\rangle$ to any compact simple factor is also Zariski-dense. So we can invoke Theorem 1.7 and conclude that the projection of Λ to G/A is uniformly discrete, and hence locally finite. By Proposition 6.3, we obtain the desired result. □

Corollary 6.11 enables us to decompose many approximate lattices into a semi-simple part and an amenable part. In light of Theorem 1.5 and [Hru22, Mac23], both parts are known to be Meyer subsets. This invites us to wonder the following:

Question 2. With the notations of Corollary 6.11. Let $p : G \to G/A$ denote the natural projection. We know that both $p(\Lambda)$ and $\Lambda^2 \cap A$ are contained in model sets. Is Λ contained in a model set?

In [Hru22] is provided an example of an approximate lattice in a central extension of $SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p)$ by $\mathbb{Q}_p$ with no good model, showing that the answer to Question 2 can be negative in some instances. A general answer to Question 2 should therefore take the (Lie, algebraic, connected, etc.) structure of the ambient group G into account. See [Hru22, Question 7.12] for a related question and discussions around this topic.

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