COMPOSITIO MATHEMATICA

Fano 3-folds in codimension 4, Tom and Jerry. Part I

Gavin Brown, Michael Kerber and Miles Reid

Compositio Math. **148** (2012), 1171–1194.

doi:10.1112/S0010437X11007226

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

Compositio Math. 148 (2012) 1171–1194
doi:10.1112/S0010437X11007226

Fano 3-folds in codimension 4, Tom and Jerry. Part I

Gavin Brown, Michael Kerber and Miles Reid

Abstract

We introduce a strategy based on Kustin–Miller unprojection that allows us to construct many hundreds of Gorenstein codimension 4 ideals with 9×16 resolutions (that is, nine equations and sixteen first syzygies). Our two basic games are called Tom and Jerry; the main application is the biregular construction of most of the anticanonically polarised Mori Fano 3-folds of Altınok's thesis. There are 115 cases whose numerical data (in effect, the Hilbert series) allow a Type I projection. In every case, at least one Tom and one Jerry construction works, providing at least two deformation families of quasismooth Fano 3-folds having the same numerics but different topology.

1. Introduction and the classification of Fano 3-folds

A *Fano 3-fold* X is a normal projective 3-fold whose anticanonical divisor $-K_X = A$ is $\mathbb{Q}$-Cartier and ample. We eventually impose additional conditions on the singularities and class group of X, such as terminal, $\mathbb{Q}$-factorial, quasismooth, prime (that is, class group $\mathrm{Cl}\, X$ of rank one) or $\mathrm{Cl}\, X = \mathbb{Z} \cdot A$, but more general cases occur in the course of our arguments.

This work is part of the Graded ring database project [BK], and is a sequel to Altınok's thesis [Alt05] and [ABR02]. We study X via its anticanonical graded ring

$$R(X, A) = \bigoplus_{m \in \mathbb{N}} H^0(X, mA).$$

Choosing generators of $R(X, A)$ embeds X as a projectively normal subvariety $X \subset \mathbb{P}(a_1, \ldots, a_n)$ in weighted projective space. The anticanonical ring $R(X, A)$ is known to be Gorenstein, and we say that $X \subset \mathbb{P}(a_1, \ldots, a_n)$ is projectively Gorenstein. The *codimension* of X refers to this anticanonical embedding. The discrete invariants of a Fano 3-fold X are its *genus* g (defined by $g + 2 = h^0(X, -K_X)$) together with a basket of terminal cyclic quotient singularities; for details see 3.1 and [ABR02].

In small codimension we can write down hypersurfaces, codimension 2 complete intersections and codimension 3 Pfaffian varieties fluently. This underlies the classification of Fano 3-folds in codimension $\leqslant 3$ (see [Alt98, Ian00, Rei80]): the famous 95 weighted hypersurfaces, 85 codimension 2 families and 70 families in codimension 3, of which 69 are 5×5 Pfaffians. Gorenstein in codimension 4 remains one of the frontiers of science: there is no automatic structure theory, and deformations are almost always obstructed. Type I projection and Kustin–Miller unprojection (see [KM83, PR04, Rei]) is a substitute that is sometimes adequate. This paper addresses codimension 4 Fano 3-folds in this vein.

Received 22 September 2010, accepted in final form 21 June 2011, published online 14 May 2012.
2010 Mathematics Subject Classification 14J45 (primary), 13D40 14J28 14J30 14Q15 (secondary).
Keywords: Mori theory, Fano 3-fold, unprojection, Sarkisov program.

The analysis of [Alt98, Alt05, ABR02, BK] provides 145 numerical candidates for codimension 4 Fano 3-folds. This paper isolates 115 of these that can be studied using Type I projections, hence as Kustin–Miller unprojections. Our main result is Theorem 3.2: each of these 115 numerical candidates occurs in at least two ways (the Tom and Jerry of the title), that give rise to topologically distinct varieties X.

The reducibility of the Hilbert scheme of Fano 3-folds is a systematic feature of our results, that goes back to Takagi's study of prime Fano 3-folds with basket of $\frac{1}{2}(1,1,1)$ points [Tak02, Theorem 0.3]. He describes families of varieties having the same invariants, but arising from different 'Takeuchi programs', that is, different Sarkisov links. Four of his numerical cases have codimension 4. The first, No. 1.4 in the tables of [Tak02], is our initial case $X \subset \mathbb{P}^7(1^7,2)$; it projects to the $(2,2,2)$ complete intersection, so has 7×12 resolution and is unrelated to Tom and Jerry. Takagi's three other pairs of codimension 4 cases correspond to our Tom and Jerry families as follows:

$$\begin{aligned} X \subset \mathbb{P}^7(1^4,2^4):&\quad \mathrm{Tom}_1 = \text{No. 2.2 (8 nodes)}, \quad \mathrm{Jer}_{45} = \text{No. 3.3 (9 nodes)},\\ X \subset \mathbb{P}^7(1^5,2^3):&\quad \mathrm{Tom}_1 = \text{No. 5.4 (7 nodes)}, \quad \mathrm{Jer}_{23} = \text{No. 4.1 (8 nodes)},\\ X \subset \mathbb{P}^7(1^6,2^2):&\quad \mathrm{Tom}_1 = \text{No. 4.4 (6 nodes)}, \quad \mathrm{Jer}_{15} = \text{No. 1.1 (7 nodes)}.\end{aligned}$$

Each of these is prime. In our treatment, each of these numerical cases admits one further Jerry family consisting of Fano 3-folds of Picard rank $\geqslant 2$.

Section 2 traces the origin of Tom and Jerry back to the geometry of linear subspaces of $\mathrm{Grass}(2,5)$ and associated unprojections to twisted forms of $\mathbb{P}^2\times\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$; for more on this, see Section 9. Section 3 is a detailed discussion of our main Theorem 3.2, whose proof occupies the rest of the paper. Flowchart 3.5 maps out the proof, which involves many thousands of computer algebra calculations. Section 9 discusses the wider issue of codimension 4 formats, and serves as a mathematical counterpart to the computer algebra of Sections 5–8. We do not elaborate on this point, but Tom and Jerry star in many other parallel or serial unprojection stories beyond Fano 3-folds of codimension 4, notably our work in progress on diptych varieties.

We are indebted to a referee for several pertinent remarks that led to improvements, and to a second referee who verified our computer algebra calculations independently.

2. Ancestral examples

2.1 Linear subspaces of Grass(2, 5)

A del Pezzo variety of degree 5 is an n-fold $Y_5^n \subset \mathbb{P}^{n+3}$ of codimension 3, defined by five quadrics that are Pfaffians of a 5×5 skew matrix of linear forms. Thus, Y is a linear section of Plücker $\mathrm{Grass}(2,5)\subset\mathbb{P}(\bigwedge^2 V)$ (here $V=\mathbb{C}^5$). We want to unproject a projective linear subspace $\mathbb{P}^{n-1}$ contained as a divisor in Y to construct a degree 6 del Pezzo variety $X_6^n\subset\mathbb{P}^{n+4}$. The crucial point is the following.

Lemma 2.1. *The Plücker embedding* $\mathrm{Grass}(2,5)$ *contains two families of maximal linear subspaces. These arise from:*

(I) *the* 4*-dimensional vector subspace* $v\wedge V\subset\bigwedge^2 V$ *for a fixed* $v\in V$*;*

(II) *the* 3*-dimensional subspace* $\bigwedge^2 U\subset\bigwedge^2 V$ *for a fixed three-dimensional vector subspace* $U\subset V$.

Thus, there are two different formats to set up $\mathbb{P}^{n-1}\subset Y$. Case (I) gives $\mathbb{P}^3_v\subset\mathrm{Grass}(2,5)$. A section of $\mathrm{Grass}(2,5)$ by a general $\mathbb{P}^7$ containing $\mathbb{P}^3_v$ is a 4-fold Y^4 whose unprojection is $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. Case (II) gives $\mathrm{Grass}(2, U) = \mathbb{P}^2_U \subset \mathrm{Grass}(2, 5)$. A section of $\mathrm{Grass}(2, 5)$ by a general $\mathbb{P}^6$ containing $\mathbb{P}^2_U$ is a 3-fold Y^3 whose unprojection is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$.

The proof is a lovely exercise. Hint: use local and Plücker coordinates

$$\begin{pmatrix} 1 & 0 & a_1 & a_2 & a_3 \\ 0 & 1 & b_1 & b_2 & b_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ & b_1 & b_2 & b_3 \\ & & m_{12} & m_{13} \\ & & & m_{23} \end{pmatrix} \tag{2.1}$$

with Plücker equations $m_{12} = a_1 b_2 - a_2 b_1$, etc.; permute the indices and choose signs pragmatically to make this true. Prove that in Plücker $\mathbb{P}^9$, the tangent plane $m_{12} = m_{13} = m_{23} = 0$ intersects $\mathrm{Grass}(2, 5)$ in the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.

2.2 Tom_1 and Jer_{12} in equations

Tom_1 is

$$\begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{pmatrix} \tag{2.2}$$

with $y_{1\ldots4}$ arbitrary elements, and the six entries m_{ij} linear combinations of a regular sequence $x_{1\ldots4}$ of length four. Expressed vaguely, there are 'two constraints on these six entries'; these two coincidences take the simplest form when $m_{23} = m_{45} = 0$. In this case, the Pfaffian equations all reduce to binomials, and can be seen as the 2×2 minors of an array: as a slogan,

$$4 \times 4 \text{ Pfaffians of } \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ & 0 & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & 0 \end{pmatrix} = 2 \times 2 \text{ minors of } \begin{pmatrix} * & y_3 & y_4 \\ y_1 & m_{24} & m_{25} \\ y_2 & m_{34} & m_{35} \end{pmatrix}. \tag{2.3}$$

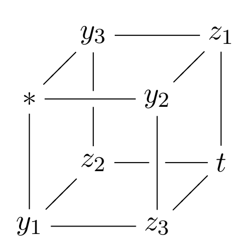
That is, the 4×4 Pfaffians on the left equal the five 2×2 minors of the array on the right. To see the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and its linear projection from a point, replace the star entry by the unprojection variable s.

In a similar style, Jer_{12} is

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & y_{34} & y_{35} \\ & & & y_{45} \end{pmatrix} \tag{2.4}$$

with y_{34}, y_{35}, y_{45} arbitrary, and the seven entries m_{ij} linear combinations of $x_{1\ldots4}$. Vaguely, there are 'three constraints on these seven entries'; most simply, these take the form $m_{15} = m_{23} = 0$, $m_{24} = m_{14}$. We leave the reader to see this as the linear projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, starting from the hint:

$$4 \times 4 \text{ Pfaffians of } \begin{pmatrix} t & z_1 & z_2 & 0 \\ & 0 & z_2 & z_3 \\ & & y_3 & y_2 \\ & & & y_1 \end{pmatrix} = 2 \times 2 \text{ minors of} \tag{2.5}$$



https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

that is, on the right, take 2×2 minors of the three square faces out of t, together with the 'diagonal' minors $y_1 z_1 = y_2 z_2 = y_3 z_3$, then replace the star by an unprojection variable. Compare (9.6).

2.3 General conclusions

Definition 2.2. Tom_i and Jer_{ij} are matrix formats that specify unprojection data, namely a codimension 3 scheme Y defined by a 5×5 Pfaffian ideal, containing a codimension 4 complete intersection D. Given a regular sequence $x_{1\dots 4}$ in a regular ambient ring R generating the ideal I_D, the ideal of Y is generated by the Pfaffians of a 5×5 skew matrix M with entries in R, subject to the following conditions.

Tom_i: the six entries $m_{jk} \in I_D$ for all $j, k \neq i$; in other words, the four entries m_{ij} of the ith row and column are free choices, but the other entries of M are required to be in I_D. See (4.6) for an example.

Jer_{ij}: the seven entries $m_{kl} \in I_D$ if either k or l equals i or j. See (4.7) for an example. The bound entries are the *pivot* m_{ij} and the two rows and columns through it. The three free entries are the Pfaffian partners m_{kl}, m_{km}, m_{lm} of the pivot, where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$. In Y, the pivot vanishes twice on D.

Case (I) in Lemma 2.1 is the ancestor of our Tom constructions and (II) that of Jerry. Our main aim in what follows is to work out several hundred applications of the same formalism to biregular models of Fano 3-folds, when our 'constraints'

$$m_{ij} = \text{linear combination of } x_{1\dots 4} \tag{2.6}$$

are not linear, do not necessarily reduce to a simple normal form and display a rich variety of colourful and occasionally complicated behaviour.

Nevertheless, the same general tendencies recur again and again. Tom tends to be fatter than Jerry. Jerry tends to have a singular locus of bigger degree than Tom, and the unprojected varieties X have different topologies, in fact different Euler numbers. For example, Y^4 in Case (I) has two lines of transversal nodes; Y^3 in Case (II) has three nodes. If we only look at 3-folds in 2.1 (cutting Y^4 by a hyperplane), the unprojected varieties X are then the familiar del Pezzo 3-folds of index 2, namely the flag manifold of $\mathbb{P}^2$ versus $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; see Section 7 (especially Remark 7.2) for the number of nodes (2 and 3 in the two cases) via enumerative geometry. Tom equations often relate to extensions of $\mathbb{P}^2 \times \mathbb{P}^2$ such as the 'extrasymmetric 6×6 format'; Jerry equations often relate to extensions of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such as the 'rolling factors format' (an anticanonical divisor in a scroll) or the 'double Jerry format'; Section 9 gives a brief discussion.

3. The main result

3.1 Numerical data of Fano 3-folds

Let X be a Fano 3-fold. As explained in [ABR02], the numerical data of X consists of an integer genus $g \geqslant -2$ plus a basket $\mathcal{B} = \{\frac{1}{r}(1, a, r-a)\}$ of terminal cyclic orbifold points; this data determines the Hilbert series $P_X(t) = \sum_{a\geqslant 0} h^0(X, nA)t^n$ of $R(X, A)$, and is equivalent to it. At present we only treat cases when the ring is generated as simply as possible, and not (say) cases that fall in a monogonal or hyperelliptic special case. The database [BK] lists cases of small codimension, including 145 candidate cases in codimension 4 from Altınok's thesis [Alt98]. We sometimes say Fano 3-fold to mean numerical candidate; the abuse of terminology is fairly

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

harmless, because practically all the candidates in codimension $\leqslant 5$ (possibly all of them) give rise to quasismooth Fano 3-folds; in fact usually more than one family, as we now relate.

3.2 Type I centre and Type I projection

An orbifold point $P \in X$ of type $\frac{1}{r}(1, a, r-a)$ with $r \geqslant 2$ is a *Type I centre* if its orbinates are restrictions of global forms $x \in H^0(A)$, $y \in H^0(aA)$, $z \in H^0((r-a)A)$ of the same weight. The condition means that after projecting, the exceptional locus of the projection is a weighted projective plane $\mathbb{P}(1, a, r-a)$ that is embedded projectively normally.

One may view a projection $P \in X \dashrightarrow Y \supset D$ in simple terms: in geometry, as the map $(x_1, \dots, x_n) \mapsto (x_1, \dots, \widehat{x_i}, \dots, x_n)$ analogous to linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ from centre $P_i = (0, \dots, 1, \dots, 0)$; or, in algebra, as eliminating a variable, corresponding to passing to a graded subring $k[x_1, \dots, \widehat{x_i}, \dots, x_n]$; to be clear, the distinguishing characteristic is not the eliminated variable x_i, rather the point P_i and the complementary system of variables x_j that vanish there.

We take the more sophisticated view of [CPR00, Section 2.6.3], of a projection as an intrinsic biregular construction of Mori theory; namely a diagram

$$\begin{array}{ccc} & & P \in X \subset \mathbb{P}(a_0, \dots, a_n) \\ & \nearrow & \\ E \subset X_1 & & \\ & \searrow & \\ & & D \subset Y \subset \mathbb{P}(a_0, \dots, \widehat{a_k}, \dots, a_n) \end{array} \tag{3.1}$$

consisting of an extremal extraction $\sigma\colon X_1 \to X$ centred at P followed by the anticanonical morphism $\varphi\colon X_1 \to Y$. In more detail, we have the following result.

Lemma 3.1. *The Type I assumption implies that $-K_{X_1}$ is semiample. The anticanonical morphism $\varphi\colon X_1 \to Y$ contracts only curves C with $-K_{X_1}C = 0$ meeting the exceptional divisor $E = \mathbb{P}(1, a, r-a) \subset X_1$ transversely in one point.*

Proof. A theorem of Kawamata [Kaw96] (discussed also in [CPR00, 3.4.2]) says that the $(1, a, r-a)$ weighted blowup $\sigma\colon X_1 \to X$ is the unique Mori extremal extraction whose centre meets the $\frac{1}{r}(1, a, r-a)$ orbifold point $P \in X$. It has exceptional divisor the weighted plane $E = \mathbb{P}(1, a, r-a)$ with discrepancy $1/r$. Thus, $-K_{X_1} = -K_X - \frac{1}{r}E$, and the anticanonical ring of X_1 consists of forms of weight d in $R(X, -K_X)$ vanishing to order $\geqslant d/r$ on E. The homogenising variable x_k of degree r with $x_k(P) = 1$ does not vanish at all, so is eliminated. By assumption, the orbinates x, y, z at P are global forms of weights $1, a, r-a$ vanishing to order exactly $1/r, a/r, (r-a)/r$, so these extend to regular elements of $R(X_1, -K_{X_1})$. Locally at P, appropriate monomials in x, y, z base the sheaves $\mathcal{O}_X(d)$ modulo any power of the maximal ideal m_P, so we can adjust the remaining generators x_l of $R(X, -K_X)$ to vanish to order $\geqslant \operatorname{wt} x_l/r$, and so they lift to $R(X_1, -K_{X_1})$. It follows that $-K_{X_1}$ is semiample and the anticanonical morphism $\varphi\colon X_1 \to Y$ takes E isomorphically to $D \subset Y$. □

In our cases, φ contracts a nonempty finite set of flopping curves to singular points of Y on D, and Y is a codimension 3 Fano 3-fold. The anticanonical model Y is not $\mathbb{Q}$-factorial because the divisor $D \subset Y$ is not $\mathbb{Q}$-Cartier. It is the *midpoint of a Sarkisov link* (compare [CPR00, 4.1(3)]); we develop this idea in Part II. The ideal case is when each $\Gamma_i \subset X_1$ is a copy of $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1,-1)$ or, equivalently, Y has only ordinary nodes on D. We prove that this happens generically in all our families.

In other situations, Type I allows φ to be an isomorphism, typically for X of large index. At the other extreme, the Type I condition on its own does not imply that $-K_{X_1}$ is big, and φ could be an elliptic Weierstrass fibration over $D=\mathbb{P}(1,a,r-a)$, although this never happens for codimension 4 Fano 3-folds. Also, φ might contract a surface to a curve of canonical singularities of Y; then $X \dashrightarrow Y$ is a 'bad link' in the sense of [CPR00, 5.5]. We know examples of this if X is not required to be $\mathbb{Q}$-factorial and prime, but none with these conditions.

Example. Consider the general codimension 2 complete intersection

$$X_{12,14} \subset \mathbb{P}(1,1,4,6,7,8)_{\langle x,a,b,c,d,e\rangle}. \tag{3.2}$$

The coordinate point $P_e=(0,\dots,0,1)$ is necessarily contained in X: near it, the two equations $f_{12}: be=F_{12}$ and $g_{14}: ce=G_{14}$ express b and c as implicit functions of the other variables, so that X is locally the orbifold point $\frac{1}{8}(1,1,7)$ with orbinates x,a,d.

Eliminating e from f_{12}, g_{14} projects $X_{12,14}$ birationally to the hypersurface $Y_{18}:(bG-cF=0) \subset \mathbb{P}(1,1,4,6,7)_{\langle x,a,b,c,d\rangle}$. Note that Y contains the plane $D=\mathbb{P}(1,1,7)_{\langle x,a,d\rangle}=V(b,c)$, and has in general $24=\frac{1}{7}\times 12\times 14$ ordinary nodes at the points $F=G=0$ of D.

In this case, the Kustin–Miller unprojection of the 'opposite' divisor $(b=F=0)\subset Y$ completes the 2-ray game on X_1 to a Sarkisov link, in the style of Corti and Mella [CM04]: the flop $X_1 \to Y \leftarrow Y^+$ blows this up to a $\mathbb{Q}$-Cartier divisor, and the unprojection variable $z_2=c/b=G/F$ then contracts it to a nonorbifold terminal point $P_z \in Z_{14} \subset \mathbb{P}(1,1,4,7,2)_{\langle x,a,b,d,z\rangle}$.

3.3 Main theorem

Write $P\in X$ for the numerical type of a codimension 4 Fano 3-fold of index 1 marked with a Type I centre. There are 115 or 116 candidates for X (depending on how you count the initial case); some have two or three centres, and treating them separately makes 162 cases for $P\in X$.

Theorem 3.2. *Let $P\in X$ be as above; then the projected variety is realised as a codimension 3 Fano $Y\subset w\mathbb{P}^6$, and Y can be made to contain a coordinate stratum $D=\mathbb{P}(1,a,r-a)$ of $w\mathbb{P}^6$ in several ways.*

For every numerical case $P\in X$, there are several formats, at least one Tom and one Jerry (see Definition 2.2) for which the general $D\subset Y$ only has nodes on D, and unprojects to a quasismooth Fano 3-fold $X\subset w\mathbb{P}^7$. In different formats, the resulting Y have different numbers of nodes on D, so that the unprojected quasismooth varieties X have different Betti numbers. Therefore, in each of the 115 numerical cases for X, the Hilbert scheme has at least two components containing quasismooth Fano 3-folds.

3.4 Discussion of the result

The theorem constructs around 320 different families of quasismooth Fano 3-folds. We do not burden the journal pages with the detailed lists, the *Big Table* in the Graded ring database [BK]; the case worked out in Section 4 may be adequate for most readers. Our data and the software tools for manipulating them are available from [BK].

Our 162 cases for $P\in X$ project to $D\subset Y\subset w\mathbb{P}^6$; of the 69 codimension 3 families of Fano 3-folds Y that are 5×5 Pfaffians, 67 are the images of projections, each having up to four candidate planes $D\subset Y$. For each of the 162 candidate pairs $D\subset Y$, we study five Tom and ten

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

Jerry formats, of which at least one Tom and one Jerry succeeds (often one more, occasionally two), so that Theorem 3.2 describes around 450 constructions of pairs $P \in X$ of quasismooth Fano 3-folds with marked centre of projection, giving around 320 different families of X.

Theorem 3.2 covers codimension 4 Fano 3-folds of index 1 for which there exists a Type I centre. If one believes the possible conjecture raised in [ABR02, 4.8.3] that every Fano 3-fold in the Mori category (that is, with terminal singularities) admits a $\mathbb{Q}$-smoothing, this also establishes the components of the Hilbert scheme of codimension 4 Fano 3-folds in these numerical cases. The main novelty of this paper (and this was a big surprise to us) is that in every case, the moduli space has two, three or four different components.

An important remaining question is which X are prime. In some cases, our Tom or Jerry matrices have a zero entry, possibly after massaging. Then three of the Pfaffian equations are binomial, which implies that X has class group of rank $\rho \geqslant 2$. This happens in the ancestral examples of Section 2 and the easier cases 4.2–4.3 of Section 4. Our Big Table confirms that if we set aside all these cases with a zero, each of our numerical possibilities for Type I centres $P \in X$ admits exactly one Tom and one Jerry construction that is potentially prime. Compare Takagi's cases discussed in Section 1. We return to this question in Part II.

3.5 Flowchart

Our proof in Sections 5–8 applies computer algebra calculations and verifications to a couple of thousand cases; any of these could in principle be done by hand. We go to the database for candidates for $P \in X$, figure out the weights of the coordinates of $D \subset Y \subset w\mathbb{P}^6$ and the matrix of weights and list all inequivalent Tom and Jerry formats. Section 5 gives criteria for a format to fail. In the cases that pass these tests, Section 6 contains an algorithm to produce $D \subset Y$ in the given format, and to prove that it has only allowed singularities (that is, only nodes on D). Section 7 contains the Chern class calculation for the number of nodes, so proving that the different constructions build topologically distinct varieties. Section 8 gives 'quick start-up' instructions; do not under any circumstances read the `README` file.

3.6 Further outlook

The reducibility phenomenon appearing in this paper is characteristic of Gorenstein in codimension 4; we have several current preprints and work in progress addressing different aspects of this. See for example [Rei].

This paper concentrates on 115 numerical cases of codimension 4 Fano 3-folds of index 1. Most of the remaining numerical cases from Altınok's list of 145 [Alt98] can be studied in terms of more complicated Type II or Type IV unprojections, when the unprojection divisor is not projectively normal; see [Rei] for an introduction. We believe that codimension 5 is basically similar: most cases have two or more Type I centres that one can project to smaller codimension, leading to parallel unprojection constructions.

The methods of this paper apply also to other categories of varieties, most obviously K3 surfaces and Calabi–Yau 3-folds. K3 surfaces are included as general elephants $S \in |-K_X|$ in our Fano 3-folds, although the K3 is unobstructed, so that passing to the elephant hides the distinction between Tom and Jerry. We can also treat some of the Fano 3-folds of index > 1 of Suzuki's thesis [BS07, Suz04]; we have partial results on the existence of some of these families, and hope eventually to cover the cases not excluded by Prokhorov's birational methods [Pro10].

This paper uses Type I projections $X \dashrightarrow Y$ to study the biregular question of the existence and moduli of X; however, in each case, the Kawamata blowup $X_1 \to X$ initiates a 2-ray game on X_1, with the anticanonical model $X_1 \to Y$ and its flop $Y \leftarrow Y^+$ as first step. In many cases, we know how to complete this to a Sarkisov link using Cox rings, in the spirit of [BCZ05, BZ10, CM04, CPR00]; we return to this in Part II.

4. Extended example

The case $g=0$ plus basket $\left\{\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{5}(1,1,4)\right\}$ gives the codimension 4 candidate $X \subset \mathbb{P}^7(1,1,2,3,3,4,4,5)$ with Hilbert numerator

$$1 - 2t^6 - 3t^7 - 3t^8 - t^9 + t^9 + 4t^{10} + 6t^{11} + \cdots + t^{22}. \tag{4.1}$$

It has three different possible Type I centres, namely the $\frac{1}{3}$, $\frac{1}{4}$ or $\frac{1}{5}$ points. We project away from each of these, obtaining consistent results; each case leads to four unprojection constructions for X, two Toms and two Jerries:

from $\frac{1}{3}$: gives $\mathbb{P}(1,1,2) \subset Y \subset \mathbb{P}(1,1,2,3,4,4,5)$ with matrix of weights

$$\begin{pmatrix} 2 & 2 & 3 & 4 \\ & 3 & 4 & 5 \\ & & 4 & 5 \\ & & & 6 \end{pmatrix} \quad \text{and} \quad \begin{array}{l} \text{Tom}_2 \text{ has 13 nodes} \\ \text{Tom}_1 \text{ has 14 nodes} \\ \text{Jer}_{45} \text{ has 16 nodes} \\ \text{Jer}_{25} \text{ has 17 nodes} \end{array} \tag{4.2}$$

from $\frac{1}{4}$: gives $\mathbb{P}(1,1,3) \subset Y \subset \mathbb{P}(1,1,2,3,3,4,5)$ with matrix of weights

$$\begin{pmatrix} 2 & 3 & 3 & 4 \\ & 3 & 3 & 4 \\ & & 4 & 5 \\ & & & 5 \end{pmatrix} \quad \text{and} \quad \begin{array}{l} \text{Tom}_3 \text{ has 9 nodes} \\ \text{Tom}_1 \text{ has 10 nodes} \\ \text{Jer}_{35} \text{ has 12 nodes} \\ \text{Jer}_{15} \text{ has 13 nodes} \end{array} \tag{4.3}$$

from $\frac{1}{5}$: gives $\mathbb{P}(1,1,4) \subset Y \subset \mathbb{P}(1,1,2,3,3,4,4)$ with matrix of weights

$$\begin{pmatrix} 2 & 2 & 3 & 3 \\ & 3 & 4 & 4 \\ & & 4 & 4 \\ & & & 5 \end{pmatrix} \quad \text{and} \quad \begin{array}{l} \text{Tom}_4 \text{ has 8 nodes} \\ \text{Tom}_2 \text{ has 9 nodes} \\ \text{Jer}_{24} \text{ has 11 nodes} \\ \text{Jer}_{14} \text{ has 12 nodes.} \end{array} \tag{4.4}$$

Specifically, we assert that *in each of these twelve cases, if we pour general elements of the ideal I_D and general elements of the ambient ring into the Tom or Jerry matrix M as specified in Definition 2.2, the Pfaffians of M define a Fano 3-fold Y having only the stated number of nodes on D, and the resulting X is quasismooth.* Section 6 verifies this claim by cheap computer algebra, although we work out particular cases here without such assistance. Section 7 computes the number of nodes in each case from the numerical data. Imposing the unprojection plane D on the general quasismooth Y_t introduces singularities on $Y = Y_0$, nodes in general, which are then resolved on the quasismooth X_1. Each node thus gives a conifold transition, replacing a vanishing cycle S^3 by a flopping line $\mathbb{P}^1$, and therefore adds 2 to the Euler number of X; so the four different X have different topology.

The unprojection formats and nonsingularity algorithms establish the existence of four different families of quasismooth Fano 3-folds X. The rest of this section analyses these in reasonably natural formats; an ideal would be to free ourselves from unprojection and computer algebra, although we do not succeed completely.

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

For illustration, work from $\frac{1}{3}$; take $X \subset \mathbb{P}^7(1,1,2,3,3,4,4,5)_{\langle x,a,b,c,d,e,f,g\rangle}$ and assume that $P_d = (0,0,0,0,1,0,0,0)$ is a Type 1 centre on X of type $\frac{1}{3}(1,1,2)$. The assumption means that $P \in X$ is quasismooth with orbinates x, a, b. The cone over X is thus a manifold along the d-axis and, therefore, by the implicit function theorem, four of the generators of I_X form a regular sequence locally at P_d, with independent derivatives, say $cd = \cdots$, $de = \cdots$, $df = \cdots$, $dg = \cdots$ of degrees 6, 7, 7, 8. Eliminating d gives the Type I projection $X \dashrightarrow Y$, where $Y \subset \mathbb{P}^6(1,1,2,3,4,4,5)$ has Hilbert numerator

$$1 - t^6 - t^7 - 2t^8 - t^9 + t^{10} + 2t^{11} + t^{12} + t^{13} - t^{19}. \tag{4.5}$$

Let Y be a 5×5 Pfaffian matrix with weights as in (4.2). Since rows 2 and 3 have the same weights, we can interchange the indices 2 and 3 throughout; thus, Tom_2 is equivalent to Tom_3, Jer_{25} to Jer_{35}, and so on.

4.1 Failure

Some Tom and Jerry cases fail, either for coarse or for more subtle reasons; for example, it sometimes happens that for reasons of weight, one of the variables x_i cannot appear in the matrix, so the variety is a cone, which we reject. Section 5 discusses failure systematically.

In the present case $D = \mathbb{P}(1,1,2)_{\langle x,a,b\rangle}$, the generators of $I_D = (c,e,f,g)$ all have weight $\geqslant 3$, but wt $m_{12}, m_{13} = 2$. Thus, requiring $m_{12}, m_{13} \in I_D$ forces them to be zero, making the Pfaffians $\mathrm{Pf}_{12.34}$ and $\mathrm{Pf}_{12.35}$ reducible. This kills Tom_4, Tom_5, Jer_{1i} for any i and Jer_{23}. The same argument says that Tom_2 has $m_{13}=0$ and Jer_{25} has $m_{12}=0$, a key simplification in treating them: a zero in M makes three of the Pfaffians binomial.

We see below that Jer_{24} fails for an interesting new reason. The other cases all work, as we could see from the nonsingularity algorithm of Section 6. Tom_2 and Jer_{25} are simpler, and we start with them, whereas Tom_1 and Jer_{45} involve heavier calculations; they are more representative of constructions that possibly lead to prime X.

4.2 $\mathbf{Tom_2}$

The analysis of the matrix proceeds as

$$\begin{pmatrix} K_2 & 0 & c & e \\ & L_3 & M_4 & N_5 \\ & & f & g \\ & & & \langle c,e,f,g\rangle_6 \end{pmatrix} \mapsto \begin{pmatrix} b & 0 & c & e \\ & L_3 & M_4 & N_5 \\ & & f & g \\ & & & 0 \end{pmatrix} \mapsto \begin{pmatrix} b & c & e \\ d & M & N \\ L & f & g \end{pmatrix}. \tag{4.6}$$

Here $m_{13}=0$ is forced by low degree, K_2, L_3, M_4, N_5 are general forms of the given degrees, that we can treat as tokens (independent indeterminates), and the four entries $m_{14}, m_{15}, m_{34}, m_{35}$ are general elements of I_D that we write c, e, f, g by choice of coordinates. Next, m_{45} can be whittled away to 0 by successive row–column operations that do not harm the remaining format; seeing this is a 'crossword puzzle' exercise that uses the fact that $m_{13}=0$ and all the entries in Row 2 are general forms. For example, subtracting a suitable multiple of Row 1 from Row 5 (and then the same for the columns) kills the c in m_{45}, while leaving m_{15} and m_{35} unchanged (because $m_{11}=m_{13}=0$) and modifying N_5 by a multiple of K_2, which is harmless because N_5 is just a general ring element of weight five.

The two zeros imply that all the Pfaffians are binomial and, as in 2.2, putting in the unprojection variable d of weight four gives the 2×2 minors of the matrix on the right. The equations describe X inside the projective cone over $w(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}(2,3^3,4^3,5^2)$ with

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

vertex $\mathbb{P}^1_{\langle x,a\rangle}$ as the complete intersection of three general forms of degrees $3, 4, 5$ expressing L, M, N in terms of the other variables. (It is still considerably easier to do the nonsingularity computation after projecting to smaller codimension.)

4.3 Jer$_{25}$

We start from

$$\begin{pmatrix} 0 & b & L_3 & f \\ & c & e & g \\ & & M_4 & \lambda_1 e \\ & & & \mu_3 c + \nu_2 e \end{pmatrix}, \tag{4.7}$$

where $m_{12} = 0$ is forced by low degree, and we put tokens b, L, M in place of the free entries m_{13}, m_{14}, m_{34}. We have cleaned out m_{35} and m_{45} as much as we can; the quantities $b, L, M, \lambda, \mu, \nu$ are general ring elements of the given weights.

We have to adjoin d together with unprojection equations for dc, de, df, dg. There are various ways of doing this, including the systematic method of writing out the Kustin–Miller homomorphism between resolution complexes, that we use only as a last resort. An *ad hoc* parallel unprojection method is to note that g appears only as the entry m_{25}, so we can project it out to a codimension 2 complete intersection (c.i.) containing the plane $c = e = f = 0$:

$$\begin{pmatrix} \mu b & \nu b - \lambda L & M \\ L & -b & 0 \end{pmatrix} \begin{pmatrix} c \\ e \\ f \end{pmatrix} = 0. \tag{4.8}$$

The equations for dc, de, df come from Cramer's rule, and we can write the unprojection in rolling factors format:

$$\bigwedge^2 \begin{pmatrix} b & L & f & d \\ c & e & g & M \end{pmatrix} \quad \text{and} \quad \begin{matrix} \mu b^2 + \nu bL - \lambda L^2 + df, \\ \mu bc + \nu cL - \lambda eL + Mf, \\ \mu c^2 + \nu ce - \lambda e^2 + Mg. \end{matrix} \tag{4.9}$$

The first set of equations of (4.9), with the entries viewed as indeterminates, defines $w(\mathbb{P}^1 \times \mathbb{P}^3) \subset \mathbb{P}(2, 3, 3, 3, 4, 4, 4, 5)_{\langle b,c,d,L,e,f,M,g\rangle}$; the second set is a single quadratic form evaluated on the rows, so defines a divisor in the cone over this with vertex $\mathbb{P}^1_{\langle x,a\rangle}$. Finally, setting L, M general forms gives X as a complete intersection in this.

4.4 Jer$_{24}$ fails

The matrix has the form

$$\begin{pmatrix} 0 & b & c & L_4 \\ & c & f & g \\ & & e & M_5 \\ & & & \langle c, e, f, g\rangle_6 \end{pmatrix} \mapsto \begin{pmatrix} 0 & b & c & L_4 \\ & c & f & g \\ & & e & M_5 \\ & & & 0 \end{pmatrix}. \tag{4.10}$$

The entries in the rows and columns through the pivot $m_{24} = f$ are general elements of the ideal $I_D = (c, e, f, g)$. As before, $m_{12} = 0$ is forced by degrees. Although (5) of 5.2 fails this for a mechanical reason, we discuss it in more detail as an instructive case, giving a perfectly nice construction of the unprojected variety X, that happens to be slightly too singular. First, please check that the entry m_{45} can be completely taken out by row and column operations. For example, to get rid of the e term in m_{45}, add α_3 times Row 3–Row 5; in m_{25} this changes g to $g + \alpha c$, that we rename g.

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

One sees that the equations of the unprojected variety X take the form

$$\bigwedge^2 \begin{pmatrix} b & c & e & f \\ d & L & M & g \end{pmatrix} = 0 \quad \text{and} \quad \begin{cases} bf = c^2, \\ bg = cL, \\ dg = L^2, \end{cases} \tag{4.11}$$

(exercise, hint: project out f or g). In straight projective space, these equations define $\mathbb{P}^1 \times Q \subset \mathbb{P}^1 \times \mathbb{P}^3$, where $Q \subset \mathbb{P}^3$ is the quadric cone. This is singular in codimension 2, so the 3-fold X cannot have isolated singularities.

4.5 $\mathbf{Tom_1}$

The matrix and its clean form are

$$\begin{pmatrix} b & K_2 & L_3 & M_4 \\ & c & e & g \\ & & f & \langle c,e,f,g\rangle_5 \\ & & & \langle c,e,f,g\rangle_6 \end{pmatrix} \mapsto \begin{pmatrix} b & K & L & M \\ & c & e & g \\ & & f & \lambda_1 e \\ & & & \mu_3 c + \nu_2 e \end{pmatrix}, \tag{4.12}$$

where K, L, M and λ, μ, ν are general forms, that we treat as tokens. We add a multiple of Column 2–Column 5 to clear c from m_{35}, so we cannot use the same operation to clear e from m_{45}. The nonsingularity algorithm of Section 6 ensures that for general choices this has only nodes on D.

We show how to exhibit X as a triple parallel unprojection from a hypersurface in the product of three codimension 2 c.i. ideals (compare 9.1). Since g only appears as m_{25}, it is eliminated by writing the two Pfaffians $\mathrm{Pf}_{12.34}$ and $\mathrm{Pf}_{13.45}$ as

$$\begin{pmatrix} L & -K & b \\ \mu K & \nu K - \lambda L & M \end{pmatrix} \begin{pmatrix} c \\ e \\ f \end{pmatrix} = 0; \tag{4.13}$$

in the same way, $\mathrm{Pf}_{12.45}$ and $\mathrm{Pf}_{12.35}$ eliminate f:

$$\begin{pmatrix} M & \lambda b & -K \\ \mu b & M + \nu b & -L \end{pmatrix} \begin{pmatrix} c \\ e \\ g \end{pmatrix} = 0. \tag{4.14}$$

Cramer's rule applied to these gives the unprojection equations for d:

$$\begin{aligned} dc &= KM + \nu bK - \lambda bL, & df &= -\mu K^2 + \nu KL - \lambda L^2, \\ de &= LM - \mu bK, & dg &= M^2 + \nu bM - \lambda\mu b^2. \end{aligned} \tag{4.15}$$

The combination eliminating d, f and g is

$$eKM - cLM - \lambda beL + \mu bcK + \nu beK = 0. \tag{4.16}$$

This is a hypersurface $Z_{10} \subset \mathbb{P}^4(1,1,2,3,4)_{\langle x,a,b,c,e\rangle}$ contained in the product ideal of $I_d = (c, e)$, $I_f = (b, M_4)$, $I_g = (K_2, L_3)$. The unprojection planes Π_d, Π_f, Π_g are projectively equivalent to $\mathbb{P}(1,1,2)$, $\mathbb{P}(1,1,3)$, $\mathbb{P}(1,1,4)$, but we cannot normalise all three of them to coordinate planes at the same time. Their pairwise intersection is

$$\begin{aligned} \Pi_d \cap \Pi_f &= \text{the 4 zeros of } M_4 \text{ on the line } b = c = e = 0, \\ \Pi_d \cap \Pi_g &= \text{the 3 zeros of } L_3 \text{ on the line } c = e = K = 0, \\ \Pi_f \cap \Pi_g &= \text{the 2 zeros of } K_2 \text{ on the line } b = L = M = 0. \end{aligned}$$

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

Nonsingularity based on (4.16). All the assertions we need for Y and X are most simply derived from (4.16). The linear system $|I_d \cdot I_f \cdot I_g \cdot \mathcal{O}_{\mathbb{P}}(10)|$ of hypersurfaces through the three unprojection planes has base locus the planes themselves, together with the curve $(b = c = K_2 = 0)$, which is in the base locus because the term $eLM \in I_d \cdot I_f \cdot I_g$ has degree 11 and so does not appear in the equation of Z. This curve is a pair of generating lines $(K = 0) \subset \mathbb{P}(1, 1, 4)_{\langle x,a,e\rangle}$. One sees that for general choices, one of the terms cLM or λbeL in Z provides a nonzero derivative LM or λeL at every point along this curve away from the three planes.

The singular locus of Z on $\Pi_d = \mathbb{P}(1, 1, 2)$ is given by

$$\frac{\partial Z}{\partial c} = -LM + \mu bK = 0, \quad \frac{\partial Z}{\partial e} = KM - \lambda bL + \nu bK = 0. \tag{4.17}$$

For general choices, these are $21 = \frac{7\times 6}{2}$ reduced points of $\mathbb{P}(1, 1, 2)$, including the four points of $\Pi_d \cap \Pi_f$ and the three points of $\Pi_d \cap \Pi_g$; after unprojecting Π_f and Π_g, this leaves 14 nodes of Tom_1, as we asserted in (4.2). The calculations for the other planes are similar.

We believe that $Z_{10} \subset \mathbb{P}^4(1, 1, 2, 3, 4)$ has class group $\mathbb{Z}^4$ generated by the hyperplane section $A = -K_Z$ and the three planes Π_d, Π_f, Π_g, so that X is prime.

4.6 Jer_{45}

The tidied up matrix is

$$\begin{pmatrix} b & -L_2 & c & e \\ & M_3 & e & g \\ & & f & \lambda_2 c \\ & & & m_{45} \end{pmatrix}, \tag{4.18}$$

with pivot $m_{45} = \delta_3 c + \gamma_2 e + \beta_2 f + \alpha_1 g$; we use row and column operations and changes of coordinates in $I_D = (c, e, f, g)$ to clean c and f out of m_{24}, but we cannot modify the pivot m_{45} without introducing multiples of b, L, M into Row 4 or Row 5, spoiling the Jer_{45} format.

We get parallel unprojection constructions for X by eliminating f or g or both. First, subtract α times Row 2 from Row 4, and ditto with the columns, to take g out of m_{45}. This spoils the format by $c \mapsto c - \alpha b \notin I_D$ in m_{14}, but does not change the Pfaffian ideal. The new matrix only contains g in m_{25}; the two Pfaffians not involving it are $\mathrm{Pf}_{12.34}$ and the modified $\mathrm{Pf}_{13.45}$, giving

$$\begin{pmatrix} M & L & b \\ \delta L + \lambda c - \alpha\lambda b & \gamma L - \alpha M & \beta L - e \end{pmatrix} \begin{pmatrix} c \\ e \\ f \end{pmatrix} = 0. \tag{4.19}$$

Eliminating $f = m_{34}$ is similar, with $\mathrm{Pf}_{12.35}$ and modified $\mathrm{Pf}_{12.45}$ giving

$$\begin{pmatrix} \lambda b & M & L \\ \delta b - \beta M & \gamma b + e - \beta L & \alpha b - c \end{pmatrix} \begin{pmatrix} c \\ e \\ g \end{pmatrix} = 0. \tag{4.20}$$

We derive the unprojection equations for d using Cramer's rule:

$$\begin{aligned} dc &= -L(e - \beta L) - \gamma Lb + \alpha Mb, \\ de &= M(e - \beta L) + \lambda b(c - \alpha b) + \delta Lb, \\ df &= -\lambda L(c - \alpha b) - \delta L^2 + \gamma LM - \alpha M^2, \\ dg &= \lambda b(e - \beta L) + M(g - \delta b) + \gamma\lambda b^2 + \beta M^2. \end{aligned} \tag{4.21}$$

This is also a triple parallel unprojection, but with a difference: the hypersurface $Z_{10} \subset \mathbb{P}(1, 1, 2, 3, 4)$ obtained by eliminating f from (4.19) or g from (4.20) or d from the first two rows

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

of (4.21) is now

$$e(e-\beta L)L+\delta cbL+\gamma ebL+\lambda bc(c-\alpha b)+M(ce-\beta cL-\alpha be)=0. \tag{4.22}$$

It is in the intersection of the three codimension 2 c.i. unprojection ideals $I_d=(c,e)$, $I_f=(b,e-\beta L)$, $I_g=(c-\alpha b,L)$, but not in their product: the first four terms are clearly in the product ideal. The interesting part is the bracket in the last term, which cannot be in the product since it has terms of degree 2, but is in $I_d\cap I_f\cap I_g$, because

$$c(e-\beta L)-\alpha be=e(c-\alpha b)-\beta Lc. \tag{4.23}$$

The slogan is *like lines on a quadric*; the three ideals have linear combinations of b,c as first generator, and of e,L as second generator, like three disjoint lines $x=z=0$, $y=t=0$ and $x=t, y=z$ on $Q:(xy=zt)$. One analyses the singularities of Z_{10} from this much as before; we believe that $\mathrm{Cl}\,Z=\langle A,D_1,D_2,D_3\rangle$, so that the triple unprojection X is prime.

5. Failure

We give reasons for failure following the introductory discussion in Section 4; we do not need to treat all the possible tests in rigorous detail, or the logical relations between them. For the structure of our proof, the point of this section is merely to give cheap preliminary tests to exclude all the candidates $D\subset Y$ that will not pass the nonsingularity algorithm in Section 6.

5.1 Easy fail at a coordinate point

Consider a coordinate point $P_i=P_{x_i}\in Y$. In either of the following cases, P_i cannot be a hyperquotient point, let alone terminal, and we can safely fail the candidate $D\subset Y$:

(1) x_i does not appear in the matrix M;

(2) x_i does not appear as a pure power in any entry of M, which thus has rank zero at P_i.

5.2 Fishy zero in M and excess singularity

Suppose that we can arrange that $m_{12}=0$, if necessary after row and column operations; then the subscheme $Z=V(\{m_{1i},m_{2i}\mid i=3,4,5\})$ is in the singular locus of Y. Indeed, the three Pfaffians $\mathrm{Pf}_{12,ij}$ are in I_Z^2, so do not contribute to the Jacobian at points of Z. The case that $\dim Z=0$ and $Z\subset D$ is perfectly acceptable and happens in a fraction of our successful constructions (see Tom_2 and Jer_{25} in Section 4). Notice that $\dim Z=0$ if and only if the six forms m_{1i},m_{2i} make up a regular sequence for $\mathbb{P}^6$; in the contrary case, the zero is *fishy*. Thus, any little coincidence between the six m_{1i},m_{2i} fails $D\subset Y$. The tests we implement are:

(3) two collinear zeros in M; see 4.1 for an example;

(4) two of the m_{1i},m_{2i} coincide; see Section 4, Jer_{24};

(5) an entry m_{1i} or m_{2i} is in the ideal generated by the other five.

In fact, the tricky point here is how to read our opening 'Suppose that we can arrange that $m_{12}=0$'. The row and column operations clearly need a modicum of care to preserve the format (that is, the entries we require to be in I_D). The harder point is that we may need a particular change of basis in I_D for the zero to appear. For example, in the Tom_5 format for $\mathbb{P}^2\subset Y\subset\mathbb{P}(1^6,2)$, with matrix of weights $\begin{smallmatrix}1&1&1&2\\&1&1&2\\&&1&2\\&&&2\end{smallmatrix}$, the lowest degree Pfaffian is quadratic in three variables of weight one, so we can write it $xy-z^2$. Mounting this as a Pfaffian in these coordinates, we can force a fishy zero, with two equal entries z arising from the term z^2. (The same applies to several candidates, but this is the only one that fails solely for this reason.)

5.3 More sophisticated and *ad hoc* reasons for failure

For the unprojected X to have terminal singularities, Y itself must also: it is the anticanonical model of the weak Fano 3-fold X_1. We can test for this at a coordinate point P of index $r > 1$: by Mori's classification, Y is either quasismooth at P or a hyperquotient singularity with local weights $\frac{1}{r}(1, a, r-a, 0)$ or $\frac{1}{4}(1, 1, 3, 2)$. Thus, we can fail the candidate $D \subset Y$ if:

(6) a coordinate point off D is a nonterminal hyperquotient singularity;

(7) a coordinate point on D is a nonterminal hyperquotient singularity.

These tests dispatch most of the remaining failing candidates.

(8) *Ad hoc* fail. Just two cases have nonisolated singularities not revealed by the elementary tests so far:

(a) Tom_4 for $\mathbb{P}(1, 2, 3) \subset Y \subset \mathbb{P}(1^2, 2, 3^2, 4^2)$ with weights $\begin{array}{cccc} 2 & 2 & 3 & 3 \\ & 3 & 4 & 4 \\ & & 4 & 4 \\ & & & 5 \end{array}$;

(b) Jer_{12} for $\mathbb{P}(1, 2, 3) \subset Y \subset \mathbb{P}(1^2, 2^2, 3^2, 4)$ with weights $\begin{array}{cccc} 2 & 2 & 2 & 3 \\ & 3 & 3 & 4 \\ & & 3 & 4 \\ & & & 4 \end{array}$.

Each of these has a $\frac{1}{2}(1, 1, 1, 0; 0)$ hyperquotient singularity at the $\frac{1}{2}$ point of D. Such a point may be terminal if it is an isolated double point, but the format of the matrix prevents this. The second case also fails at the index 4 point P_7 lying off D: it is a hyperquotient singularity of the exceptional type $\frac{1}{4}(1, 1, 3, 2; 2)$ with the right quadratic part to be terminal. However, it lies on a curve of double points along the line $\mathbb{P}(2, 4)$ joining P_7 to the $\frac{1}{2}$ point on D: in local coordinates x, a, e, b at P_7, the equation is $xa = e^2 + b \times$ terms in $(x, a, e)^2$.

6. Nonsingularity and proof of Theorem 3.2

To prove Theorem 3.2, we need to run through a long list of candidate 3-folds $D \subset Y \subset w\mathbb{P}^6$ with choice of format Tom_i or Jer_{ij}. We exclude many of these by the automatic methods of Section 5. In every remaining case, we run a nonsingularity algorithm to confirm that the candidate can be unprojected to a codimension 4 Fano 3-fold X with terminal singularities (in fact, we conclude also quasismooth). For the proof of Theorem 3.2, we check that at least one Tom and one Jerry works for each case $D \subset Y$.

We outline the proof as a pseudocode algorithm; our implementation is discussed in Section 8. The justification of the algorithm is that it works in practice. *A priori*, it could fail, for example the singular locus of Y on D could be more complicated than a finite set of nodes, or all three coordinate lines of D could contain a node, but by good luck such accidents never happen.

6.1 Nonsingularity analysis

We work with any $D \subset Y$ not failed in Section 5. The homogeneous ideal I_Y is generated by the 4×4 Pfaffians of M. Differentiating the five equations Pf with respect to the seven variables gives the 5×7 Jacobian matrix $J(\mathrm{Pf})$. Its ideal $I_{\mathrm{Sing}\,Y} = \bigwedge^3 J(\mathrm{Pf})$ of 3×3 minors defines the singular locus of Y; more precisely, it generates the ideal sheaf $\mathcal{I}_{\mathrm{Sing}\,Y} \subset \mathcal{O}_{\mathbb{P}^6}$. Our claim is that the only singularities of Y lie on D, and are nodes. For this, we check that:

(a) $\mathrm{Sing}\,Y \subset D$ or equivalently $I_D \subset \mathrm{Rad}(I_{\mathrm{Sing}\,Y})$;

(b) the restriction $\mathcal{I}_{\mathrm{Sing}\,Y} \cdot \mathcal{O}_D$ defines a reduced subscheme of D.

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

In fact, (b) together with Lemma 7.1 imply that Y has only nodes. In practice, we may work on a standard affine piece of D containing all the singular points: it turns out in every case that some 1-stratum of D is disjoint from the singular locus.

6.2 Proof of Theorem 3.2

We start with the data for a candidate $P \in X \subset w\mathbb{P}^7$: a genus $g \geqslant -2$ and a basket $\mathcal{B}$ of terminal quotient singularities or, equivalently, the resulting Hilbert series (see [ABR02]). We give a choice of eight ambient weights W_X of $w\mathbb{P}^7$ and a choice of Type I centre $P = \frac{1}{r}(1, a, r-a)$ from the basket. The Type I definition predicts that the ambient weights of $Y \subset w\mathbb{P}^6$ are $W_X \backslash \{r\}$ and that $D = \mathbb{P}(1, a, r-a)$ can be chosen to be a coordinate stratum of $w\mathbb{P}^6$. We analyse all possible Tom and Jerry formats for $D \subset Y \subset w\mathbb{P}^6$.

Step 1. Set up coordinates $x_1, x_2, x_3, x_4,\ y_1, y_2, y_3$ on $w\mathbb{P}^6$; here $x_{1\ldots4}$ is a regular sequence generating I_D and y_1, y_2, y_3 are coordinates on D.

Step 2. The numerics of [CR02] determine the weights d_{ij} of the 5×5 skew matrix M from the Hilbert numerator of $Y \subset w\mathbb{P}^6$.

Step 3. Set each entry m_{ij} of M equal to a general form, respectively a general element of the ideal I_D of the given degree d_{ij}, according to the chosen Tom or Jerry format (see Definition 2.2).

Tidy up the matrix M as much as possible while preserving its Tom or Jerry format. Some entries of M may already be zero. Use coordinate changes on $w\mathbb{P}^6$ to set some entries of M equal to single variables. If possible, use row and column operations to simplify M further. Check every zero of M for failure for the mechanical reasons discussed in 5.2, followed by the other failing conditions of 5.1. Now any candidate that passes these tests actually works.

Step 4. Carry out the singularity analysis of 6.1.

Step 5. Calculate the number of nodes as in Section 7; check that no two sets of unprojection data give the same number of nodes.

Step 6 (Optional). Apply the Kustin–Miller algorithm [KM83] to construct the equations of X. This is not essential to prove that X exists, but knowing the full set of equations is useful if we want to put the equations in a codimension 4 format, for example by projecting from another Type I centre.

7. Number of nodes

The unprojection divisor $D = V(x_{1\ldots4}) \subset \mathbb{P}^6$ is a codimension 4 c.i., with conormal bundle $\mathcal{I}_D/\mathcal{I}_D^2$ the direct sum of four orbifold line bundles $\mathcal{O}_D(-x_i)$ on D. The ideal sheaf $\mathcal{I}_Y$ is generated by five Pfaffians that vanish on D, so each is $\mathrm{Pf}_i = \sum a_{ij} x_j$. Thus, the Jacobian matrix Jac restricted to D is the 5×4 matrix $(\overline{a}_{ij})$, where bar is restriction mod $I_D = (x_{1\ldots4})$; the induced homomorphism to the conormal bundle

$$\mathcal{J} \colon \bigoplus_5 \mathcal{O}_{\mathbb{P}}(-\,\mathrm{Pf}_i) \twoheadrightarrow \mathcal{I}_Y/(\mathcal{I}_D \cdot \mathcal{I}_Y) \to \mathcal{I}_D/\mathcal{I}_D^2 \tag{7.1}$$

has generic rank three. Its cokernel $\mathcal{N}$ is the conormal sheaf to D in Y. It is a rank one torsion-free sheaf on D whose second Chern class $c_2(\mathcal{N})$ counts the nodes of Y on D. The more precise result is as follows.

Lemma 7.1. (I) *The cokernel* $\mathcal{N}$ *is an orbifold line bundle at points of* D *where* rank $\mathcal{J} = 3$, *that is, at quasismooth points of* Y.

(II) *Assume that* $P \in D$ *is a nonsingular point (not orbifold), and that* rank $\mathcal{J} = 2$ *at* P *and* $= 3$ *in a punctured neighbourhood of* P *in* D*; then* $\mathcal{N}$ *is isomorphic to a codimension* 2 c.i. *ideal* (f, g) *locally at* P. *This coincides locally with the ideal* $\bigwedge^3 \mathrm{Jac} \cdot \mathcal{O}_D$ *generated by the* 3×3 *minors of the Jacobian matrix.*

(III) *Assume that* $\bigwedge^3 \mathrm{Jac} \cdot \mathcal{O}_D$ *is reduced (locally the maximal ideal* m_P *at each point). Then* Y *has an ordinary node at* P.

(IV) *If this holds everywhere, then* $c_2(\mathcal{N})$ *is the number of nodes of* Y *on* D.

Proof. The statement is the hard part; the proof is just commutative algebra over a regular local ring. The rank one sheaf $\mathcal{N}$ is the quotient of a rank four locally free sheaf by the image of the 5×4 matrix $\mathrm{Jac} = (\overline{a}_{ij})$, of generic rank three. It is a line bundle where the rank is three, and where it drops to two, we can use a 2×2 nonsingular block to take out a rank two locally free summand. The cokernel is therefore locally generated by two elements, so is locally isomorphic to an ideal sheaf (f, g), a c.i. because the rank drops only at P.

The minimal free resolution of $\mathcal{N}$ is the Koszul complex of f, g; now (7.1) is also part of a free resolution of $\mathcal{N}$, so covers the Koszul complex. This means that the matrix $\mathrm{Jac} = (\overline{a}_{ij})$ can be written as its 2×2 nonsingular block and a complementary 2×3 block of rank one, whose two rows are $g \cdot v$ and $-f \cdot v$ for v a 3-vector with entries generating the unit ideal. Therefore, $\bigwedge^3 \mathrm{Jac}$ generates the same ideal (f, g).

If $(f, g) = (y_1, y_2)$ is the maximal ideal at $P \in D$, then the shape of $\bigwedge^3 \mathrm{Jac}$ says that two of the Pfaffians $\mathrm{Pf}_1, \mathrm{Pf}_2$ express two of the variables x_1, x_2 as implicit functions; then a linear combination p of the remaining three has $\partial p / \partial x_3 = y_1$ and $\partial p / \partial x_4 = y_2$, so that Y is a hypersurface with an ordinary node at P. $\square$

We now show how to resolve $\mathcal{N}$ by an exact sequence involving direct sums of orbifold line bundles on D, and deduce a formula for $c_2(\mathcal{N})$.

$\mathbf{Tom_1}$

The matrix is

$$M = \begin{pmatrix} K & L & M & N \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{pmatrix}, \tag{7.2}$$

where m_{ij} are linear forms in $x_{1\dots 4} \in \mathcal{I}_D$ with coefficients in the ambient ring. When we write out $\mathrm{Jac} = (\overline{a}_{ij})$, the only terms that contribute are the derivatives $\partial / \partial x_{1\dots 4}$, with the x_i set to zero; thus, only the terms that are exactly linear in the x_i contribute. Since Pf_1 is of order $\geqslant 2$ in the x_i, the corresponding row of the matrix J is zero and we omit it in (7.3). Moreover, the first row K, L, M, N of M provides a syzygy $\Sigma_1 = K \,\mathrm{Pf}_2 + L \,\mathrm{Pf}_3 + M \,\mathrm{Pf}_4 + N \,\mathrm{Pf}_5 \equiv 0$ between the four remaining Pfaffians. Hence, we can replace J by the resolution

$$\mathcal{N} \leftarrow \sum_{1\dots 4} \mathcal{O}(-d_i) \leftarrow \sum_{j \neq 1} \mathcal{O}(-a_j) \leftarrow \mathcal{O}(-\sigma_1) \leftarrow 0, \tag{7.3}$$

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

where $d_i = \mathrm{wt}\, x_i$, $a_j = \mathrm{wt}\,\mathrm{Pf}_j$ and $\sigma_1 = \mathrm{wt}\,\Sigma_1$, and leave the reader to think of names for the maps. Therefore, $\mathcal{N}$ has total Chern class

$$\prod_{i=1}^{4}(1-d_ih)\times(1-\sigma_1 h)\Big/\prod_{j\neq 1}(1-a_jh). \tag{7.4}$$

The number of nodes $c_2(\mathcal{N})$ is then the h^2 term in the expansion of (7.4); recall that we view $h = c_1(\mathcal{O}_D(1))$ as an orbifold class, so that $h^2 = 1/ab$ for $D = \mathbb{P}(1, a, b)$.

Jer_{12}

The pivot m_{12} appears in three Pfaffians $\mathrm{Pf}_i = \mathrm{Pf}_{12,jk}$ for $\{i, j, k\} = \{3, 4, 5\}$ as the term $m_{12}m_{jk}$, together with two other terms $m_{1j}m_{2k}$ of order $\geqslant 2$ in $x_{1\dots4}$. The Jacobian matrix restricted to D thus has three corresponding rows that are m_{jk} times the same vector $\partial m_{12}/\partial x_{1\dots4}$. This proportionality gives three syzygies Σ_l between these three rows, yoked by a second syzygy T in degree $t = \text{adjunction number} - \mathrm{wt}\, m_{12}$. In other words, the conormal bundle has the resolution

$$\mathcal{N} \leftarrow \bigoplus_4 \mathcal{O}(-d_i) \leftarrow \bigoplus_5 \mathcal{O}(-a_j) \leftarrow \bigoplus_3 \mathcal{O}(-\sigma_l) \leftarrow \mathcal{O}(-t) \leftarrow 0, \tag{7.5}$$

so that the total Chern class of $\mathcal{N}$ is the alternate product

$$\frac{\prod_4(1-d_ih)\prod_3(1-\sigma_lh)}{\prod_5(1-a_jh)(1-th)}, \tag{7.6}$$

with $c_2(\mathcal{N})$ equal to the h^2 term in this expansion.

Example 7.2. We read the number of nodes mechanically from the Hilbert numerator, the matrix of weights and the choice of format. As a baby example, the 'interior' projections of the two del Pezzo 3-folds of degree 6 discussed in 2.2 have 2 and 3 respective nodes. These numbers are the coefficients of h^2 in the formal power series

$$\frac{(1-h)^4(1-3h)}{(1-2h)^4} = 1 + h + 2h^2 \quad\text{and}\quad \frac{(1-h^4)(1-3h)^3}{(1-2h)^5(1-4h)} = 1 + h + 3h^2. \tag{7.7}$$

As a somewhat more strenuous example, in (4.2),

Tom_1 has $\mathrm{wt}\, x_{1\dots4} = 3, 4, 4, 5$, $\mathrm{wt}\,\mathrm{Pf}_{2\dots5} = 8, 8, 7, 6$, $\Sigma_1 = 10$, so that

$$c(\mathcal{N}) = \frac{\prod_{a\in[3,4,4,5,10]}(1-ah)}{\prod_{b\in[6,7,8,8]}(1-bh)} = 1 + 3h + 28h^2, \text{giving } \frac{28}{1\cdot1\cdot2} = 14 \text{ nodes.}$$

Jer_{25} has the same x_i, $\mathrm{Pf}_{1\dots5} = 9, 8, 8, 7, 6$, $\Sigma_l = 10, 11, 12$, adjunction number $= 19$, $\mathrm{wt}\, m_{25} = 5$, so that $c(\mathcal{N}) = \frac{\prod_{a\in[3,4,4,5,10,11,12]}(1-ah)}{\prod_{b\in[6,7,8,8,9,14]}(1-bh)} = 1 + 3h + 34h^2$, giving $\frac{34}{1\cdot1\cdot2} = 17$ nodes.

Try the other cases in (4.2)–(4.4) as homework.

8. Computer code and the GRDB database

A Big Table with the detailed results of the calculations proving Theorem 3.2 is online at the Graded ring database webpage

http://grdb.lboro.ac.uk + Downloads.

This website makes available computer code implementing our calculations systematically, together with the Big Table they generate. The code is for the Magma system [Mag97], and installation instructions are provided; at heart, it only uses primary elements of any computer algebra system, such as polynomial ideal calculations and matrix manipulations. The code runs online in the Magma calculator

http://magma.maths.usyd.edu.au/calc.

All the data on the codimension 4 Fano 3-folds we construct is available on *webloc. cit.*: follow the link to Fano 3-folds, select Fano index $f = 1$ (the default value), codimension $= 4$ and Yes for Projections of Type I, then submit. The result is data on the 116 Fano 3-folds with a Type I projection (the 116th is an initial case with 7×12 resolution, that projects to the complete intersection $Y_{2,2,2} \subset \mathbb{P}^6$ containing a plane, so is not part of our story here). The + link reveals additional data on each Fano 3-fold.

The computer code follows closely the algorithm outlined as the proof of Theorem 3.2. For each Tom and Jerry format, we build a matrix with random entries; some of these can be chosen to be single variables, since we assume that Y is general for its format. We use row and column operations to simplify the matrix further without changing the format. The first failure tests (fishy zeroes, cone points and points of embedding dimension 6) are now easy, and inspection of the equations on affine patches at coordinate points on Y is enough to determine whether their local quotient weights are those of terminal singularities. An ideal inclusion test checks that the singularities lie on D. By good fortune, in every case that passes the tests so far, the singular locus lies on one standard affine patch of D. We pass to this affine patch and check that $\mathcal{I}_{\mathrm{Sing}\,Y} \cdot \mathcal{O}_D$ defines a reduced scheme there. We calculate the length of the quotient $\mathcal{O}_D/(\mathcal{I}_{\mathrm{Sing}\,Y} \cdot \mathcal{O}_D)$ on this patch, providing an alternative to the computation of Section 7 (and a comforting sanity check).

The random entries in the matrix are not an issue: our nonsingularity requirements are open, so if one choice leads to a successful $D \subset Y$, any general choice also works. The only concern is false negative reports, for example an alleged nonreduced singular locus on D. To tackle such hiccups, if a candidate fails at this stage (in practice, a rare occurrence), we simply rerun the code with a new random matrix; the fact that the code happens to terminate justifies the proof.

The conclusion is that every possible Tom and Jerry format for every numerical Type I projection either fails one of the human-readable tests of Section 5 (and we have made any number of such hand calculations) or is shown to work by constructing a specific example.

To complete the proof of Theorem 3.2, we check that the final output satisfies the following two properties.

(a) Every numerical candidate admits at least one Tom and one Jerry unprojection.

(b) Whenever a candidate has more than one Type I centre, the successful Tom and Jerry unprojections of any two correspond one-to-one, with compatible numbers of nodes: the difference in Euler number computed by the nodes is the same whichever centre we calculate from; compare (4.2)–(4.4).

The polynomial ideal calculations of nonsingularity analysis of 6.1 (that is, the inclusion $I_D \subset \mathrm{Rad}(I_{\mathrm{Sing}\,Y})$ and the statement that $\mathcal{I}_{\mathrm{Sing}\,Y} \cdot \mathcal{O}_D$ is reduced) are the only points where we use computer power seriously (other than to handle hundreds of repetitive calculations accurately). In cases with two or three centres, even this could be eliminated by projecting to a complete intersection and applying Bertini's theorem, as in Section 4.

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

9. Codimension 4 Gorenstein formats

The Segre embeddings $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ are well-known codimension 4 projectively Gorenstein varieties with 9×16 resolution. Singularity theorists consider the affine cones over them to be rigid, because they have no nontrivial infinitesimal deformations or small analytic deformation. Nevertheless, both are sections of higher dimensional graded varieties in many different nontrivial ways. Each of these constructions appears at many points in the study of algebraic surfaces by graded rings methods.

9.1 Parallel unprojection and extrasymmetric format

The extrasymmetric 6×6 format occurs frequently, possibly first in Dicks' thesis [Dic88]. It is a particular case of triple unprojection from a hypersurface in the product of three codimension 2 c.i. ideals. Start from the 'undeformed' 6×6 skew matrix

$$M_0 = \begin{pmatrix} b_3 & -b_2 & x_1 & a_3 & a_2 \\ & b_1 & a_3 & x_2 & a_1 \\ & & a_2 & a_1 & x_3 \\ & & & -b_3 & b_2 \\ & & & & -b_1 \end{pmatrix}, \tag{9.1}$$

with the 'extrasymmetric' property that the top right 3×3 block is symmetric, and the bottom right 3×3 block equals minus the top left block. So, instead of fifteen independent entries, it has only nine independent entries and six repeats.

Direct computation reveals that the 4×4 Pfaffians of M_0 fall under the same numerics: of its fifteen Pfaffians, nine are independent and six repeats. One sees that they generate the same ideal as the 2×2 minors of the 3×3 matrix

$$N_0 = \begin{pmatrix} x_1 & a_3 + b_3 & a_2 - b_2 \\ a_3 - b_3 & x_2 & a_1 + b_1 \\ a_2 + b_2 & a_1 - b_1 & x_3 \end{pmatrix}. \tag{9.2}$$

Here N_0 is the generic 3×3 matrix (written as symmetric plus skew), with minors defining Segre $\mathbb{P}^2 \times \mathbb{P}^2$, and thus far we have not gained anything, beyond representing $\mathbb{P}^2 \times \mathbb{P}^2$ as a nongeneric section of $\operatorname{Grass}(2, 6)$.

However, M_0 can be modified to preserve the codimension 4 Gorenstein property while destroying the sporadic coincidence with $\mathbb{P}^2 \times \mathbb{P}^2$. The primitive one-parameter way of doing this is to choose the triangle $(1, 2, 6)$ and multiply the entries m_{12}, m_{16}, m_{26} by a constant r_3. This gives

$$M_1 = \begin{pmatrix} r_3 b_3 & -b_2 & x_1 & a_3 & r_3 a_2 \\ & b_1 & a_3 & x_2 & r_3 a_1 \\ & & a_2 & a_1 & x_3 \\ & & & -b_3 & b_2 \\ & & & & -b_1 \end{pmatrix}. \tag{9.3}$$

One checks that the three Pfaffians $\operatorname{Pf}_{12.i6}$ for $i = 3, 4, 5$ are r_3 times others, whereas three other repetitions remain unchanged. So, the 4×4 Pfaffians of M_1 still define a Gorenstein codimension 4 subvariety with 9×16 resolution. We can view it as the Tom_3 unprojection of the codimension 3 Pfaffian ideal obtained by deleting the final column, with x_3 as unprojection variable.

If $r_3 = \rho^2$ is a perfect square, then floating the square root ρ to the complementary entries m_{34}, m_{35}, m_{45} restores the original extrasymmetry. In general, this is a 'twisted form' of $\mathbb{P}^2 \times \mathbb{P}^2$: changing the sign of ρ swaps the two factors.

A more elaborate version of this depends on eight parameters:

$$M_2 = \begin{pmatrix} r_3s_0b_3 & -r_2s_0b_2 & x_1 & r_2s_1a_3 & r_3s_1a_2 \\ & r_1s_0b_1 & r_1s_2a_3 & x_2 & r_3s_2a_1 \\ & & r_1s_3a_2 & r_2s_3a_1 & x_3 \\ & & & -r_0s_3b_3 & r_0s_2b_2 \\ & & & & -r_0s_1b_1 \end{pmatrix}. \tag{9.4}$$

Now the same three Pfaffians $\mathrm{Pf}_{12.i6}$ are divisible by r_3, and the complementary three are divisible by s_3 with the same quotient, so one has to do a little cancellation to see the irreducible component. The necessity of cancelling these terms (although cheap in computer algebra as the colon ideal) has been a headache in the theory for decades, since it introduces apparent uncertainty as to the generators of the ideal.

The right way to view this is as the triple parallel unprojection of the hypersurface

$$V(a_1a_2b_3r_3s_3 + a_1a_3b_2r_2s_2 + a_2a_3b_1r_1s_1 + b_1b_2b_3r_0s_0) \tag{9.5}$$

in the product ideal $\prod_{i=1}^3 (a_i, b_i)$. Then

$$x_1 = \frac{a_2a_3r_1s_1 + b_2b_3r_0s_0}{a_1} = -\frac{a_2b_3r_2s_2 + a_2b_3r_3s_3}{b_1},$$

etc., and the ideal is generated by the Pfaffians of the three matrices

$$\begin{pmatrix} x_2 & b_1r_0s_0 & a_1r_3s_3 & a_3 \\ & -a_1r_2s_2 & -b_1r_1s_1 & b_3 \\ & & x_3 & a_2 \\ & & & b_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 & b_3r_0s_0 & a_3r_2s_2 & a_2 \\ & -a_3r_1s_1 & -b_3r_3s_3 & b_2 \\ & & x_2 & a_1 \\ & & & b_1 \end{pmatrix}, \quad \begin{pmatrix} x_3 & b_2r_0s_0 & a_2r_1s_1 & a_1 \\ & -a_2r_3s_3 & -b_2r_2s_2 & b_1 \\ & & x_1 & a_3 \\ & & & b_3 \end{pmatrix}.$$

If the r_i and s_i are nonzero constants, one still needs the square root of the discriminant $\prod_{i=0}^3 (r_is_i)$ to get back to $\mathbb{P}^2 \times \mathbb{P}^2$.

9.2 Double Jerry

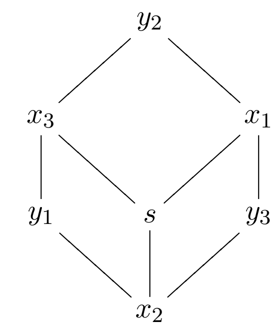
The equations of Segre $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ are the minors of a $2 \times 2 \times 2$ array; they admit several extensions, and it seems most likely that there is no irreducible family containing them all. One family consists of various 'rolling factors' formats discussed below; here we treat 'double Jerry'.

Start from the equations written as

$$\begin{aligned} sy_i &= x_jx_k \quad \text{for } \{i, j, k\} = \{1, 2, 3\}, \\ tx_i &= y_jy_k \quad \text{for } \{i, j, k\} = \{1, 2, 3\}, \\ st &= x_iy_i \quad \text{for } i = 1, 2, 3 \end{aligned} \tag{9.6}$$

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

corresponding to a hexagonal view of the cube centred at vertex s (with three square faces $\Box s x_i y_k x_j$, and t behind the page, cf. (2.5)):



(9.7)

Eliminating both s and t gives the codimension 2 c.i.

$$(x_1y_1 = x_2y_2 = x_3y_3) \subset \mathbb{P}^5, \tag{9.8}$$

containing the two codimension 3 complete intersections $\mathbf{x}=0$ and $\mathbf{y}=0$ as divisors. We can view $\mathbf{x}$ as a row vector and $\mathbf{y}$ a column vector, and the two equations (9.8) as the matrix products

$$\mathbf{x}A\mathbf{y} = \mathbf{x}B\mathbf{y} = 0, \quad \text{where } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{9.9}$$

The unprojection equations for s and t separately take the form

$$t\mathbf{x} = (A\mathbf{y}) \times (B\mathbf{y}) \quad \text{and} \quad s\mathbf{y} = (\mathbf{x}A) \times (\mathbf{x}B), \tag{9.10}$$

where $\times$ is cross product of vectors in $\mathbb{C}^3$, with the convention that the cross product of two row vectors is a column vector and vice versa. For example, $\mathbf{x}A = (x_1, -x_2, 0)$, $\mathbf{x}B = (0, x_2, -x_3)$ and the equations $s\mathbf{y} = (\mathbf{x}A) \times (\mathbf{x}B)$ giving the first line of (9.6) are deduced via Cramer's rule from (9.8).

We can generalise this at a stroke to A, B general 3×3 matrices. That is, for $\mathbf{x}$ a row vector and $\mathbf{y}$ a column vector, $\mathbf{x}A\mathbf{y} = \mathbf{x}B\mathbf{y} = 0$ is a codimension 2 c.i.; since these are general bilinear forms in $\mathbf{x}$ and $\mathbf{y}$, it represents a universal solution to two elements of the product ideal $(x_1, x_2, x_3) \cdot (y_1, y_2, y_3)$. It has two single unprojections:

$$\mathbf{x}A\mathbf{y} = \mathbf{x}B\mathbf{y} = 0 \quad \text{and} \quad s\mathbf{y} = (\mathbf{x}A) \times (\mathbf{x}B), \tag{9.11}$$

$$\mathbf{x}A\mathbf{y} = \mathbf{x}B\mathbf{y} = 0 \quad \text{and} \quad t\mathbf{x} = (A\mathbf{y}) \times (B\mathbf{y}), \tag{9.12}$$

either of which is a conventional 5×5 Pfaffian, and a parallel unprojection putting those equations together with a ninth *long equation*

$$st = \text{something complicated}. \tag{9.13}$$

The equation certainly exists by the Kustin–Miller theorem. It can be obtained easily in computer algebra by coloning out any of $x_1, x_2, x_3, y_1, y_2, y_3$ from the ideal generated by the eight equations (9.11) and (9.12). Its somewhat amazing right-hand side has 144 terms, each bilinear in x, y and biquadratic in A, B. Taking a hint from $144 = 12 \times 12$, we suspect that it may have a product structure of the form

$$\mathbf{x}(A \wedge B) \times (A \wedge B)\mathbf{y}, \tag{9.14}$$

with '$\times$' and '$\wedge$' still requiring elucidation.

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

If the entries of A and B are constants, one gets back to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ after coordinate changes based on the three roots $(\lambda_i : \mu_i)$ of the relative characteristic equation $\det(\lambda A - \mu B) = 0$ and the three eigenvectors $v_i = \ker(\lambda_i A - \mu_i B)$. Swapping the roots permutes the three factors.

The significance of the double Jerry parallel unprojection format is that it covers any Jerry case where the pivot is one of the generators of I_D. Indeed, if the regular sequence generating I_D is s, x_1, x_2, x_3, a Jerry matrix for D is

$$\begin{pmatrix} s & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & y_3 & -y_2 \\ & & & y_1 \end{pmatrix}, \quad \text{where} \quad \begin{matrix} (m_{13}, m_{14}, m_{15}) = \mathbf{x}A, \\ (m_{23}, m_{14}, m_{15}) = \mathbf{x}B \end{matrix} \tag{9.15}$$

for some 3×3 matrices A, B. Unprojecting D gives a double Jerry.

9.3 Rolling factors format

Rolling factors view a divisor $X \subset V$ on a normal projective variety $V \subset \mathbb{P}^n$ as residual to a nice linear system. This phenomenon occurs throughout the literature, with typical cases a divisor on the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$, or on a rational normal scroll $\mathbb{F}$, or on a cone over a Veronese embedding. A divisor $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ in the linear system $|ah_1 + (a+2)h_2| = |-K_V + bH|$ is of course defined by a single bihomogeneous equation in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^3$, but to get equations in the homogeneous coordinate ring of Segre $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ we have to add $|2h_1|$. This is a type of hyperquotient, given by one equation in a nontrivial eigenspace.

Dicks' thesis [Dic88] discussed the generic pseudoformat

$$\begin{gathered} \bigwedge^2 \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} = 0 \quad \text{and} \\ \begin{aligned} m_1a_1 + m_2a_2 + m_3a_3 + m_4a_4 &= 0 \\ m_1b_1 + m_2b_2 + m_3b_3 + m_4b_4 &\equiv n_1a_1 + n_2a_2 + n_3a_3 + n_4a_4 = 0 \\ & \qquad n_1b_1 + n_2b_2 + n_3b_3 + n_4b_4 = 0. \end{aligned} \end{gathered} \tag{9.16}$$

One sees that under fairly general assumptions the 'scroll' V defined by the first set of equations of (9.16) is codimension 3 and Cohen–Macaulay, with resolution

$$\mathcal{O}_V \leftarrow R \leftarrow 6R \leftarrow 8R \leftarrow 3R \leftarrow 0.$$

On the right, the identity is a preliminary condition on quantities in the ambient ring. If we assume (say) that R is a regular local ring and $a_i, b_i, m_i, n_i \in R$ satisfy it (and are 'fairly general'), the second set defines an elephant $X \in |-K_V|$ (anticanonical divisor) which is a codimension 4 Gorenstein variety with 9×16 resolution.

The identity in (9.16) is a quadric of rank sixteen. It is a little close-up view of the 'variety of complexes' discussed in [Rei, Section 10]. To use this method to build genuine examples, we have to decide how to map a regular ambient scheme into this quadric; there are several different solutions. If we take the a_i, b_i to be independent indeterminates, the first set of equations gives the cone on Segre $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, and the second set consists of a single quadratic form q in four variables evaluated on the two rows, so that $X \subset V$ is given by $q(\mathbf{a}) = \varphi(\mathbf{a}, \mathbf{b}) = q(\mathbf{b}) = 0$, with φ the associated symmetric bilinear form (cf. (4.9)). This format seems to be the only commonly occurring codimension 4 Gorenstein format that tends not to have any Type I projection.

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press

On the other hand, if there are coincidences between the a_i, b_i, there may be other ways of choosing the m_i, n_i to satisfy the identity in (9.16) without the need to take m_i, n_i quadratic in the a_i, b_i: for example, if $a_2 = b_1$, we can roll $a_1 \to a_2$ and $b_1 \to b_2$.

ACKNOWLEDGEMENT

This research is supported by the Korean Government WCU Grant R33-2008-000-10101-0.

REFERENCES

Alt98 S. Altınok, *Graded rings corresponding to polarised K3 surfaces and* $\mathbb{Q}$*-Fano 3-folds*, PhD thesis, University of Warwick (1998) vii+93 pp., available at http://www.warwick.ac.uk/staff/Miles.Reid/doctors/Selma/.

Alt05 S. Altınok, *Constructing new K3 surfaces*, Turkish J. Math. **29** (2005), 175–192.

ABR02 S. Altınok, G. Brown and M. Reid, *Fano 3-folds, K3 surfaces and graded rings*, in *Topology and geometry: commemorating SISTAG (National University of Singapore, 2001)*, Contemporary Mathematics, vol. 314 eds A. J. Berrick *et al.* (American Mathematical Society, Providence, RI, 2002), 25–53.

BK G. Brown, A. M. Kasprzyk *et al.*, *Graded ring database*, http://grdb.lboro.ac.uk.

BS07 G. Brown and K. Suzuki, *Fano 3-folds with divisible anticanonical class*, Manuscripta Math. **123** (2007), 37–51.

BZ10 G. Brown and F. Zucconi, *The graded ring of a rank 2 Sarkisov link*, Nagoya Math. J. **197** (2010), 1–44.

BCZ05 G. Brown, A. Corti and F. Zucconi, *Birational geometry of 3-fold Mori fibre spaces*, in *The Fano Conference, Proceedings*, eds A. Collino, A. Conte and M. Marchisio (Università di Torino, Torino, 2005), 235–275.

CM04 A. Corti and M. Mella, *Birational geometry of terminal quartic 3-folds, I*, Amer. J. Math. **126** (2004), 739–761.

CR02 A. Corti and M. Reid, *Weighted Grassmannians*, in *Algebraic geometry* (de Gruyter, Berlin, 2002), 141–163.

CPR00 A. Corti, A. Pukhlikov and M. Reid, *Birationally rigid Fano hypersurfaces*, in *Explicit birational geometry of 3-folds*, eds A. Corti and M. Reid (Cambridge University Press, Cambridge, 2000), 175–258.

Dic88 D. Dicks, *Surfaces with* $p_g = 3$, $K^2 = 4$ *and extension–deformation theory*, PhD thesis, University of Warwick (1988) vi+125 pp.

Ian00 A. Iano-Fletcher, *Working with weighted complete intersections*, in *Explicit birational geometry of 3-folds*, eds A. Corti and M. Reid (Cambridge University Press, Cambridge, 2000), 101–173.

Kaw96 Y. Kawamata, *Divisorial contractions to 3-dimensional terminal quotient singularities*, in *Higher-dimensional complex varieties (Trento, 1994)* (de Gruyter, Berlin, 1996), 241–246.

KM83 A. Kustin and M. Miller, *Constructing big Gorenstein ideals from small ones*, J. Algebra **85** (1983), 303–322.

Mag97 Magma (John Cannon's computer algebra system), *The Magma algebra system I: the user language*, J. Symbolic Comput. **24** (1997), 235–265, see also http://magma.maths.usyd.edu.au/magma.

PR04 S. A. Papadakis and M. Reid, *Kustin–Miller unprojection without complexes*, J. Algebraic Geom. **13** (2004), 563–577.

Pro10 Y. Prokhorov, $\mathbb{Q}$*-Fano threefolds of large Fano index, I*, Doc. Math. **15** (2010), 843–872.

Rei80 M. Reid, *Canonical 3-folds*, in *Journées de géométrie algébrique d'Angers, 1979* (Sijthoff & Noordhoff, Alphen aan den Rijn, 1980), 273–310.

Rei M. Reid, Graded rings and birational geometry, *Proceedings of Algebraic Geometry Symposium (Kinosaki, October 2000)*, ed. K. Ohno, 1–72, available at www.warwick.ac.uk/staff/Miles.Reid/3folds.

Suz04 K. Suzuki, *On Fano indices of $\mathbb{Q}$-Fano 3-folds*, Manuscripta Math. **114** (2004), 229–246.

Tak02 H. Takagi, *On classification of $\mathbb{Q}$-Fano 3-folds of Gorenstein index 2. I, II*, Nagoya Math. J. **167** (2002), 117–155, 157–216.

Gavin Brown g.d.brown@lboro.ac.uk
School of Mathematics, Loughborough University, LE11 3TU, UK

Michael Kerber michael.kerber@ist.ac.at
Institute of Science and Technology (IST) Austria, 3400 Klosterneuburg, Austria

Miles Reid miles.reid@warwick.ac.uk
Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

https://doi.org/10.1112/S0010437X11007226 Published online by Cambridge University Press