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THE LINDELÖF DEGREE OF SCATTERED SPACES AND THEIR PRODUCTS

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Abstract

Different methods are used to show that a finite or countable product of Lindelöf scattered spaces is Lindelöf. Also, a technique of Kunen is modified to yield results concerning the Lindelöf degree of the G_δ- and G_α-topologies on the countable product of compact scattered spaces.

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Introduction

Since 1947, when R. H. Sorgenfrey [7] gave an example of a Lindelöf space whose cartesian product with itself was not normal, numerous questions have arisen concerning the products of Lindelöf spaces. We examine here the products of Lindelöf scattered spaces. Telgársky [9] has shown that the product of a Lindelöf C-scattered space with a Lindelöf space is Lindelöf. We show by a different method that the finite product of Lindelöf scattered spaces is Lindelöf. By looking at $\mathcal{P}$-spaces and also by examining the totally Lindelöf property, we are able to show that a countable product of Lindelöf scattered spaces is Lindelöf.

In Section 3 we look at the Lindelöf degree of the G_δ- and G_α-topologies on countable products of compact scattered or Lindelöf scattered spaces. A technique of Kunen [3] is modified to yield some results here.

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All spaces are assumed to be Hausdorff regular. A space X is said to be scattered if every non-empty subspace of X contains an isolated point. All ordinal spaces are scattered. Given a space X, X_δ will represent the set X with the topology generated by the G_δ-sets of X. Similarly, X_α will denote the set X with the topology generated by the G_α-sets (sets which are the intersection of no more than α open sets) of X. $L(X)$ denotes the Lindelöf degree of X (see Juhász [2]). $|X|$ denotes the cardinality of X. The notation $p_n(Y)$ is used for the nth projection of Y, a subset of a product space.

2. Finite and countable products

As indicated by the following lemma, the G_δ-topology on a space X may be useful in determining the Lindelöf degree of the product of X with another space.

LEMMA 2.1. *If $L(X_\delta) \leqslant \beta$, then $L(X \times Y) \leqslant \beta$ for every Lindelöf space Y.*

PROOF. Let $\mathcal{C}$ be an open cover of $X \times Y$. Without loss of generality, we may assume that every member of $\mathcal{C}$ is of the form $G \times H$ where G and H are open in X and Y, respectively.

For each $x \in X$, $\{x\} \times Y$ is Lindelöf and hence it can be covered by a countable subfamily $\mathcal{C}_x \subseteq \mathcal{C}$. For each $x \in X$, let $G_x = \bigcap\{G\colon G \times H \in \mathcal{C}_x\}$. Since each G_x is a G_δ-set and $L(X_\delta) \leqslant \beta$, there is a subfamily $\{G_{x(\gamma)}\colon \gamma \leqslant \beta\}$ of $\{G_x\colon x \in X\}$ which covers X. Then $\bigcup\{\mathcal{C}_{x(\gamma)}\colon \gamma \leqslant \beta\}$ is a subfamily of $\mathcal{C}$ of cardinality no greater than β which covers $X \times Y$.

Of course this result can easily be generalized to higher cardinalities:

$$\text{If } L(X_\alpha) \leqslant \beta \text{ and } L(Y) \leqslant \alpha, \quad \text{then } L(X \times Y) \leqslant \alpha \cdot \beta.$$

But of interest to us here is the countable case as stated in Lemma 2.1.

P. Meyer [4] showed that a compact space X is scattered if and only if X_δ is Lindelöf. We give a simple proof of a strengthening in one direction of this result without the Cantor-Bendixon decomposition type argument of Meyer.

THEOREM 2.2. *If X is scattered and $L(X) = \omega$, then $L(X_\delta) = \omega$.*

PROOF. Let $\mathcal{C}$ be a cover of X by G_δ-sets. Let $U = \{x \in X\colon x \in H$ and H open in X implies H cannot be covered by a countable subfamily of $\mathcal{C}\}$. U is closed.

Suppose $U \neq \varnothing$. Then U has an isolated point x and there is an open set $G \subseteq X$ such that $G \cap U = \{x\}$. Choose $C(x) \in \mathcal{C}$ such that $x \in C(x)$. We may assume, without loss of generality, that $C(x) = \bigcap\{G(n): n < \omega\}$ where, for each $n < \omega$, $G(n)$ is open and $G(n+1) \subseteq \overline{G(n+1)} \subseteq G(n) \subseteq G$. We consider $\overline{G(n)} - G(n+1)$ for each $n < \omega$. Each $y \in (\overline{G(n)} - G(n+1)) \subseteq X - U$ has a neighborhood $H(y)$ which can be covered by a countable subfamily of $\mathcal{C}$. Furthermore, the family $\{H(y): y \in \overline{G(n)} - G(n+1)\}$ has a countable subcover since $L(\overline{G(n)} - G(n+1)) = \omega$. Hence each $\overline{G(n)} - G(n+1)$ can be covered by a countable subfamily $\mathcal{C}(n) \subseteq \mathcal{C}$. Then $\{C: C \in \mathcal{C}(n), n < \omega\} \cup \{C(x)\}$ is a countable subfamily of $\mathcal{C}$ covering G, which contradicts $x \in U$. Thus $U = \varnothing$.

Since $U = \varnothing$, there is a neighborhood $H(x)$ of x, for each $x \in X$, such that $H(x)$ can be covered by a countable subfamily of $\mathcal{C}$. X is Lindelöf, so $\{H(x): x \in X\}$ can be reduced to a countable subcover which in turn yields a countable subcover of $\mathcal{C}$.

An extensive study of covering properties of C-scattered spaces was made by Telgársky [8], [9]. A space X is said to be C-scattered if every non-empty closed subspace has a point with a compact neighborhood in that subspace. It was shown by Telgársky [9] that the product of a Lindelöf C-scattered space with a Lindelöf space is Lindelöf. By Lemma 2.1 and Theorem 2.2 we have the following corollaries.

Corollary 2.3. *If $L(X) = L(Y) = \omega$ and X is scattered, then $L(X \times Y) = \omega$.*

Corollary 2.4. *A finite product of Lindelöf scattered spaces is Lindelöf.*

Corollary 2.5. *If X is Lindelöf and scattered and if each point of X is a G_δ, then $|X| < \omega$.*

A space X is a $\mathcal{P}$-space if every G_δ-set in X is open. Combining N. Noble's [5] results on $\mathcal{P}$-spaces with Theorem 2.2, we can give a simple proof that the countable product of Lindelöf scattered spaces is Lindelöf.

Theorem 2.6. [5] *A countable product of Lindelöf $\mathcal{P}$-spaces is Lindelöf.*

Corollary 2.7. *A countable product of Lindelöf scattered spaces is Lindelöf.*

Proof. Let $\{X(n): n < \omega\}$ be a family of Lindelöf scattered spaces. Then by Theorem 2.2, each $(X(n))_\delta$ is a Lindelöf $\mathcal{P}$-space. $\prod_{n<\omega}(X(n))_\delta$ is Lindelöf by Theorem 2.6 and since $\prod_{n<\omega}(X(n))_\delta$ maps continuously onto $\prod_{n<\omega}X(n)$, $\prod_{n<\omega}X(n)$ is also Lindelöf.

Another method of determining the Lindelöf degree of a countable product of Lindelöf scattered spaces is by means of totally Lindelöf spaces. J. E. Vaughan has examined this property and its related properties in several papers; [10] and [11] are primary sources. There are spaces which are Lindelöf but not totally Lindelöf [10]. We begin with some definitions.

A filter base $\mathscr{G}$ is said to be finer than the filter base $\mathscr{F}$ if every member of $\mathscr{F}$ contains a member of $\mathscr{G}$. A filter base is said to be total [10] if each finer filter base has an adherent point (that is, each finer filter base clusters).

A space X is totally Lindelöf [11] if given a filter base $\mathscr{F}$ on X which is stable under countable intersections (that is, if $F(n) \in \mathscr{F}$ for all $n < \omega$, then there exists $F \in \mathscr{F}$ such that $F \subseteq \bigcap\{F(n); n < \omega\}$), there is a filter base $\mathscr{G}$ on X such that

(i) $\mathscr{G}$ is stable under countable intersections,

(ii) $\mathscr{G}$ is finer than $\mathscr{F}$, and

(iii) $\mathscr{G}$ is total.

With the following lemma we will be able to establish a relationship between Lindelöf scattered spaces and totally Lindelöf spaces.

Lemma 2.8. *If X is the union of a countable number of subsets each of which is totally Lindelöf, then X is totally Lindelöf.*

Proof. Let $X = \bigcap\{A(n): n < \omega\}$ where each $A(n)$ is totally Lindelöf. Let $\mathscr{F}$ be a filter base on X which is stable under countable intersections.

For each $n < \omega$, we define a family $\mathscr{F}(n)$ as follows: if there exists $F \in \mathscr{F}$ for which $F \cap A(n) = \varnothing$, let $\mathscr{F}(n) = \varnothing$; otherwise let $\mathscr{F}(n) = \{F \cap A(n): F \in \mathscr{F}\}$.

We observe that there exists $n^* < \omega$ for which $\mathscr{F}(n^*) \neq \varnothing$. If this were not the case, then for each $n < \omega$, we could choose $F(n) \in \mathscr{F}$ for which $F(n) \cap A(n) = \varnothing$. Since $\mathscr{F}$ is stable under countable intersections, there exists $G \subseteq \bigcap\{F(n); n < \omega\}$ and since $X = \bigcup\{A(n): n < \omega\}$, there exists $m < \omega$ such that $G \cap A(m) \neq \varnothing$. But $(G \cap A(m)) \subseteq (F(m) \cap A(m)) = \varnothing$ yields a contradiction.

To see that $\mathscr{F}(n^*)$ is stable under countable intersections, let $\{F(n): n < \omega\}$ be a countable subfamily of $\mathscr{F}(n^*)$. For each $n < \omega$, there exists $G(n) \in \mathscr{F}$ such that $F(n) = G(n) \cap A(n^*)$. Since $\mathscr{F}$ is stable there exists $G \in \mathscr{F}$ such that $G \subseteq \bigcap\{G(n): n < \omega\}$. Now $G \cap A(n^*) \neq \varnothing$ and $(G \cap A(n^*)) \subseteq \bigcap\{G(n) \cap A(n^*): n < \omega\} = \bigcap\{F(n): n < \omega\}$.

Since $\mathscr{F}(n^*)$ is stable and $A(n^*)$ is totally Lindelöf, $\mathscr{F}(n^*)$ has a finer filter base $\mathscr{G}(n^*)$ which is total and stable under countable intersections. We note that $\mathscr{G}(n^*)$ is finer than $\mathscr{F}$ and thus X is totally Lindelöf.

Theorem 2.9. *If X is Lindelöf and scattered, then X is totally Lindelöf.*

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PROOF. Suppose X is Lindelöf and scattered. Let $A = \{x \in X$: every neighborhood of x fails to be totally Lindelöf$\}$.

If $X = \varnothing$, we are finished because for each $x \in X$, there is a neighborhood $N(x)$ which is totally Lindelöf. The family $\{N(x): x \in X\}$ can be reduced to a countable subcover of X and Lemma 2.8 can be applied.

If $A \neq \varnothing$, then A has an isolated point a and there exists an open set $G \subseteq X$ such that $G \cap A = \{a\}$. We may assume, without loss of generality, that every point of X except a has a neighborhood which is totally Lindelöf. Suppose $\mathcal{F}$ is a filter base on X which is stable under countable intersections and suppose some finer filter base $\mathcal{G}$, which is stable under countable intersections, does not cluster at a. Then there exists $G \in \mathcal{G}$ such that $a \notin \overline{G}$. If we can show that $\overline{G}$ is totally Lindelöf, then $\mathcal{G}$ will cluster in $\overline{G}$ and we will be finished. For each $x \in \overline{G}$, there is a neighborhood $N(x)$ which is totally Lindelöf. Now $\{N(x): x \in X\}$ is an open cover of $\overline{G}$ which is Lindelöf. Hence $\overline{G}$ is a countable union of subsets each of which is totally Lindelöf and by Lemma 2.8, $\overline{G}$ is totally Lindelöf.

With Theorem 2.9 and the following theorem of Vaughan, we are able to reach our conclusion about countable products of Lindelöf scattered spaces in Theorem 2.11.

THEOREM 2.10. [11] *A countable product of totally Lindelöf spaces is Lindelöf.*

THEOREM 2.11. *A countable product of Lindelöf scattered spaces is Lindelöf.*

3. The G_δ- and G_α-topologies

K. Kunen [3] has shown with a most beautiful technique that the Lindelöf degree of the box product of a countable number of compact scattered spaces is no greater than c, the cardinality of the continuum. This technique is modified to reach conclusions about the cartesian product.

The Cantor-Bendixon decomposition of a space X is a non-increasing sequence of closed sets of X defined inductively as follows: Let

$X^{(0)} = X$,

$X^{(\alpha+1)} = \{x \in X^{(\alpha)}: x$ is not isolated in $X^{(\alpha)}\}$, and

$X^{(\lambda)} = \bigcap\{X^{(\alpha)}: \alpha < \lambda\}$ for λ a limit ordinal.

X is scattered if and only if there exists an α such that $X^{(\alpha)} = \varnothing$. If X is scattered and compact, then the first α for which $X^{(\alpha)} = \varnothing$ is a successor ordinal $\alpha = \beta + 1$ and $X^{(\beta)}$ is finite. In this case, we say the rank of X is β.

THEOREM 3.1. *If $X(n)$ is compact and scattered for each $n < \omega$, then $L((\prod_{n<\omega} X(n))_\delta) \leq c$.*

PROOF. Let $\mathcal{C}$ be a cover of $\prod_{n<\omega} X(n)$ by G_δ-sets. Without loss of generality, we may assume $\mathcal{C}$ to be a closed cover and if $C \in \mathcal{C}$, then $C = \bigcap \{G(C)_i : i < \omega\}$ with $G(C)_i$ open in $\prod_{n<\omega} X(n)$ for each $i < \omega$.

Consider the tree $T = \bigcup \{c^\gamma : \gamma < \omega_1\}$. For $t \in T$, denote the domain of t by $\operatorname{dom}(t)$ and for $\xi < c$, let $t\xi$ be the extension of t where $t\xi(\operatorname{dom}(t) + 1) = \xi$. $t \restriction \gamma$ will denote the restriction of t to γ and 0 is the empty function ($\operatorname{dom}(0) = \varnothing$).

We will define, by induction on $\operatorname{dom}(t)$, closed Kunen sets $K(t)$ in such a way that a subfamily of $\{K(t): t \in T\}$ refines $\mathcal{C}$ and since $|T| = c$, we will be finished. Our sets $K(t)$ will be required to satisfy conditions similar to those in Kunen's theorem, namely:

(i) $K(0) = \prod_{n<\omega} X(n)$,

and for each $t \in T$,

(ii) $K(t) \subseteq \bigcup \{K(t\xi) : \xi < c\}$,

(iii) $K(t) = \bigcap \{K(t | \gamma) : \gamma < \operatorname{dom}(t)\}$ if $\operatorname{dom}(t)$ is a limit ordinal, and

(iv) for each $\xi < c$, either there exists $C \in \mathcal{C}$ such that $K(t\xi) \subseteq C$ or there exists $n < \omega$ for which rank $p_n(K(t\xi)) <$ rank $p_n(K(t))$.

If these conditions are met, then we will have our refinement. The argument is like Kunen's. If $x \in \prod_{n<\omega} X(n)$, then by (i), (ii), and (iii), there is a function $t \in T$ such that $x \in K(t \restriction \gamma)$ for every $\gamma < \omega_1$. The ranks of $p_n(K(t \restriction \gamma))$, for each n, are non-increasing and thus eventually constant. So by (iv), we must eventually get inside a covering set (that is, inside a member of the cover $\mathcal{C}$).

Our modification of the Kunen technique comes in the way we define our Kunen sets, $K(t)$. We define $K(0) = \prod_{n<\omega} X(n)$ and we take intersections at the limit stages. Now suppose $K(t)$ has already been defined; we will define $K(t\xi)$ for each $\xi < c$. We let $\beta_n =$ rank $p_n(K(t))$ and $Z(n) = (p_n(K(t)))^{(\beta_n)}$. Since each $Z(n)$ is finite, there exists a subfamily $\mathcal{C}' \subseteq \mathcal{C}$ of cardinality c such that $\mathcal{C}'$ covers $\prod_{n<\omega} Z(n)$. Let $\mathcal{G} = \{G : G$ is of the form $G = G(C)_i$ for some $C \in \mathcal{C}'$, $i < \omega\}$. The sets $K(t\xi)$, $\xi < c$, will list the c sets K such that either (a) $K = C \cap K(t)$ for some $C \in \mathcal{C}'$ or (b) K is a box, where for some n,

(1) $p_n(K) = p_n(K(t)) - \bigcup \{p_n(G) : G \in \mathcal{G}'\}$ where $\mathcal{G}' \subseteq \mathcal{G}$ is finite and $Z(n) \subseteq \bigcup \{p_n(G) : G \in \mathcal{G}'\}$, and

(2) $p_m(K) = p_m(K(t))$ for each $m \neq n$.

Conditions (i) and (iii) are obviously met; condition (iv) will be satisfied because of (b) (1) of the definition. We show that condition (ii) is met by assuming $x \in K(t)$ and $x \notin K(t\xi)$ of type (b). Then for each $n < \omega$ and for each finite subfamily $\mathcal{G}' \subseteq \mathcal{G}$, $Z(n) \subseteq \bigcup \{p_n(G) : G \in \mathcal{G}'\}$ implies $x(n) = p_n(x) \in \bigcup \{p_n(G) : G \in \mathcal{G}'\}$. Furthermore, there exists $z(n) \in Z(n)$ such that for every

$G \in \mathcal{G}$, $z(n) \in p_n(G)$ implies $x(n) \in p_n(G)$. So if $C \in \mathcal{C}'$ and if $z(n) \in p_n(C) = p_n(\bigcap \{G(C)_i: i < \omega\}) = \bigcap \{p_n(G(C)_i): i < \omega\}$, then $x(n) \in p_n(C)$. Defining $z \in \prod_{n<\omega} Z(n)$ so that $p_n(z) = z(n)$ for each $n < \omega$, we have for each $C \in \mathcal{C}'$, $z \in C$ implies $x \in C$. Thus choosing $C \in \mathcal{C}'$ such that $z \in C$, we have $x \in C \cap K(t)$, a Kunen set of type (a).

The following corollary easily follows from Theorem 3.1.

COROLLARY 3.2. *If $X(n)$ is a σ-compact, scattered space for each $n < \omega$, then $L((\prod_{n<\omega} X(n))_\delta) \leqslant c$.*

With appropriate changes in the proof of Theorem 3.1, we may further extend the result.

THEOREM 3.3. *Under GCH, if $X(n)$ is compact and scattered for each $n < \omega$ and if α is a limit cardinal with* $\mathrm{cf}(\alpha) > \omega$, *then $L((\prod_{n<\omega} X(n))_\alpha) \leqslant \alpha$.*

Given the closed cover $\mathcal{C}$ of $\prod_{n<\omega} X(n)$ by G_α-sets, if $C \in \mathcal{C}$, then $C = \bigcap \{G(C)_\beta: \beta < \alpha\}$ with $G(C)_\beta$ open in $\prod_{n<\omega} X(n)$ for each $\beta < \alpha$. The proof requires using the tree $T = \bigcup \{\alpha^\gamma: \gamma < \omega_1\}$ and the Kunen sets $K(t)$ are defined, by induction on $\mathrm{dom}(t)$ to meet the conditions (i)–(iv) of the proof of Theorem 3.1. To define the sets $K(t\xi)$, for $\xi < \alpha$, we follow the route of that proof, but use the family $\mathcal{C} = \{G: G$ is of the form $G = G(C)_\beta$ for some $C \in \mathcal{C}', \beta < \alpha\}$.

We turn our attention now to Lindelöf scattered spaces. It is known that if X is a scattered Lindelöf space and α is the first ordinal such that $X^{(\alpha)} = \varnothing$, then either

(a) $\mathrm{cf}(\alpha) = \omega$, or

(b) α is a successor ordinal $\beta + 1$ and $|X^{(\beta)}| \leqslant \omega$.

If condition (b) is met, we may still call β the rank of X.

There are spaces which satisfy condition (a) but which are not Lindelöf. For example, if $X = \omega_1 \dot{\cup} \omega_\omega$ (the disjoint union), then $\alpha = \omega_\omega$ and X is not Lindelöf.

Question: If $X(n)$ is Lindelöf and scattered for each $n < \omega$, then what can be said about $L((\prod_{n<\omega} X(n))_\delta)$?

To answer this question, it may be necessary to consider several cases depending upon the type of Cantor-Bendixon decomposition each space $X(n)$ possesses.

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