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ABELIAN SEMIGROUPS OF MATRICES ON $\mathbb{C}^n$ AND HYPERCYCLICITY

ADLENE AYADI[1] AND HABIB MARZOUGUI[2]

[1] *Department of Mathematics, Faculty of Science of Gafsa, University of Gafsa, Gafsa 2112, Tunisia* (adlenesoo@yahoo.fr)
[2] *Department of Mathematics, Faculty of Science of Bizerte, University of Carthage, Zarzouna 7021, Tunisia* (habib.marzougui@fsb.rnu.tn; hmarzoug@ictp.it)

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Abstract We give a complete characterization of a hypercyclic abelian semigroup of matrices on $\mathbb{C}^n$. For finitely generated semigroups, this characterization is explicit and it is used to determine the minimal number of matrices in normal form over $\mathbb{C}$ that form a hypercyclic abelian semigroup on $\mathbb{C}^n$. In particular, we show that no abelian semigroup generated by n matrices on $\mathbb{C}^n$ can be hypercyclic.

Keywords: hypercyclic; matrices; orbit; dense orbit; semigroup; abelian

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1. Introduction

Let $M_n(\mathbb{C})$ be the set of all square matrices over $\mathbb{C}$ of order $n \geqslant 1$ and let $\mathrm{GL}(n,\mathbb{C})$ be the group of invertible matrices of $M_n(\mathbb{C})$. Let G be an abelian subsemigroup of $M_n(\mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of G through v: $G(v) = \{Av \colon A \in G\} \subset \mathbb{C}^n$. A subset $E \subset \mathbb{C}^n$ is called *G-invariant* if $A(E) \subset E$ for any $A \in G$. The orbit $G(v) \subset \mathbb{C}^n$ is *dense* (respectively, *somewhere dense*) in $\mathbb{C}^n$ if $\overline{G(v)} = \mathbb{C}^n$ (respectively, $\mathring{\overline{G(v)}} \neq \emptyset$), where $\bar{E}$ (respectively, $\mathring{E}$) denotes the closure (respectively, the interior) of a subset $E \subset \mathbb{C}^n$. The semigroup G is called *hypercyclic* if there exists a vector $v \in \mathbb{C}^n$ such that $G(v)$ is dense in $\mathbb{C}^n$. We refer the reader to the recent book [**4**] and to [**9**] for a thorough account on hypercyclicity.

Recently, there has been much research around this subject. We mention, in particular, [**1**–**3**, **5**–**8**, **12**] for the abelian case and [**10**] for the non-abelian case. Feldman showed in [**8**] that in $\mathbb{C}^n$ there exists a hypercyclic semigroup generated by an $(n+1)$-tuple of diagonal matrices on $\mathbb{C}^n$, and that no semigroup generated by an n-tuple of diagonalizable matrices on $\mathbb{C}^n$ or $\mathbb{R}^n$ can be hypercyclic. Costakis *et al*. proved in [**7**] that if one removes the diagonalizability condition, there exists an n-tuple of non-diagonalizable matrices on $\mathbb{R}^n$ that is hypercyclic. Recently, Costakis and Parissis proved in [**5**] that the minimal number of matrices in Jordan form on $\mathbb{R}^n$ that form a hypercyclic tuple is $n+1$.

In the non-abelian case, Javaheri shows in [**10**] that there exists a 2-generator hypercylic semigroup in any dimension in both real and complex cases.

The main purpose of this paper is to investigate the following: when can an abelian subsemigroup of $M_n(\mathbb{C})$ be hypercyclic?

Shkarin [**12**] and Abels and Manoussos [**1**] considered the same topic, in particular the minimal number of generators of a finitely abelian hypercyclic semigroup of matrices on $\mathbb{C}^n$ and $\mathbb{R}^n$. They have, independently, proved similar results to Corollaries 1.8 and 1.9. The methods of proof in [**1**, **12**] and in this paper are quite different and have different consequences.

We firstly give a general result answering the above question for any abelian *subsemigroup* of $M_n(\mathbb{C})$, by providing an effective way of checking that a given semigroup is hypercyclic. Note that in [**3**] the authors answer this question for any abelian *subgroup* of $\mathrm{GL}(n, \mathbb{C})$, so this paper can be viewed as a continuation of that work. We point out that, as the results obtained for groups are not used to get those for semigroups, the present paper is almost independent of [**3**].

Secondly, we prove that the minimal number of matrices required to form a hypercyclic abelian semigroup in $\mathcal{K}_\eta(\mathbb{C})$, having a normal form of length r (see the definition below), is exactly $2n - r + 1$ (see Corollary 1.7). In particular, $n + 1$ is the minimal number of matrices on $\mathbb{C}^n$ required to form a hypercyclic abelian semigroup on $\mathbb{C}^n$; this was recently shown in [**2**], answering a question raised by Feldman in [**8**, § 6].

To state our main results, we need to introduce the following notation and definitions.

Let $\mathbb{N}$ be the set of non-negative integers, and set $\mathbb{N}_0 = \mathbb{N}\backslash\{0\}$. Let $n \in \mathbb{N}_0$ be fixed. By a partition of n we mean a finite sequence of positive integers $\eta = (n_1, \dots, n_r)$ such that $\sum_{i=1}^{r} n_i = n$. The number r will be called the *length* of the partition. Given a partition $\eta = (n_1, \dots, n_r)$, we define the following:

$$\mathcal{K}_\eta(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathbb{T}_{n_r}(\mathbb{C}),$$

where $\mathbb{T}_m(\mathbb{C})$ $(m = 1, 2, \dots, n)$ is the set of lower-triangular matrices over $\mathbb{C}$ with only one eigenvalue.

Obviously, $\mathcal{K}_\eta(\mathbb{C})$ is a subsemigroup of $M_n(\mathbb{C})$. We have the following:

- $\mathcal{K}^*_\eta(\mathbb{C}) := \mathcal{K}_\eta(\mathbb{C}) \cap \mathrm{GL}(n, \mathbb{C})$ is a subsemigroup of $\mathrm{GL}(n, \mathbb{C})$;
- $\mathbb{T}^*_m(\mathbb{C}) = \mathbb{T}_m(\mathbb{C}) \cap \mathrm{GL}(m, \mathbb{C})$ is a subgroup of $\mathrm{GL}(m, \mathbb{C})$;
- $\mathcal{B}_0 = (e_1, \dots, e_n)$ is the canonical basis of $\mathbb{C}^n$;
- I_n is the identity matrix on $\mathbb{C}^n$.

For a row vector $v \in \mathbb{C}^n$, we denote by v^{T} the transpose of v. We also have that

- $u_\eta = [e_{\eta,1}, \dots, e_{\eta,r}]^{\mathrm{T}} \in \mathbb{C}^n$, where $e_{\eta,k} = [1, 0, \dots, 0]^{\mathrm{T}} \in \mathbb{C}^{n_k}$, $k = 1, \dots, r$,
- $e^{(k)}_\eta = [(e^{(k)}_\eta)_1, \dots, (e^{(k)}_\eta)_r] \in \mathbb{C}^n$, where, for every $j = 1, \dots, r$,

$$(e^{(k)}_\eta)_j = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } j \neq k, \\ e_{\eta,k} & \text{if } j = k. \end{cases}$$

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The following 'normal form of G' allows us to deduce the results for an arbitrary semigroup.

For every abelian subsemigroup G of $M_n(\mathbb{C})$, there exists a $P \in \mathrm{GL}(n, \mathbb{C})$ such that $P^{-1}GP \subset \mathcal{K}_\eta(\mathbb{C})$ for some partition η of n (see Proposition 2.4).

Given a positive integer $r \in \{1, \dots, r\}$, we say that the semigroup G has 'a normal form of length r' if G has a normal form in $\mathcal{K}_\eta(\mathbb{C})$ for some partition η with length r.

Consider the matrix exponential map $\exp\colon M_n(\mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$. Set $\exp(M) = e^M$.

For such a choice of matrix P, we let

$$\mathfrak{g}_\eta = \exp^{-1}(G) \cap (P(\mathcal{K}_\eta(\mathbb{C}))P^{-1}),$$
$$\mathfrak{g}_\eta(u) = \{Bu\colon B \in \mathfrak{g}_\eta\}, \quad u \in \mathbb{C}^n.$$

Finally, we define

$$G^* = G \cap \mathrm{GL}(n, \mathbb{C}).$$

Our principal results can now be stated as follows.

Theorem 1.1. *Let G be an abelian subsemigroup of $\mathcal{K}_\eta(\mathbb{C})$ for some partition η with length r.*

(1) *The following are equivalent:*

(i) *G is hypercyclic,*

(ii) *the orbit $G(u_\eta)$ is dense in $\mathbb{C}^n$,*

(iii) *$\mathfrak{g}_\eta(u_\eta)$ is an additive subsemigroup, dense in $\mathbb{C}^n$.*

(2) *Assume that G^* is finitely generated by p matrices ($p \geqslant 1$) and let $B_1, \dots, B_p \in \mathcal{K}_\eta(\mathbb{C})$ such that $e^{B_1}, \dots, e^{B_p}$ generate G^*. Then G is hypercyclic if and only if*

$$\sum_{k=1}^{p} \mathbb{N} B_k u_\eta + \sum_{k=1}^{r} 2\mathrm{i}\pi\mathbb{Z} e_\eta^{(k)}$$

is dense in $\mathbb{C}^n$.

An immediate consequence is the following corollary.

Corollary 1.2. *Let G be an abelian subsemigroup of $M_n(\mathbb{C})$ with normal form in $\mathcal{K}_\eta(\mathbb{C})$, where η has length r, and if $P \in \mathrm{GL}(n, \mathbb{C})$ such that $P^{-1}GP \subset \mathcal{K}_\eta(\mathbb{C})$, then we have the conditions below.*

(1) *The following are equivalent:*

(i) *G is hypercyclic,*

(ii) *the orbit $G(Pu_\eta)$ is dense in $\mathbb{C}^n$,*

(iii) *$\mathfrak{g}_\eta(Pu_\eta)$ is dense in $\mathbb{C}^n$.*

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(2) *Assume that G^* is finitely generated by p matrices ($p \geqslant 1$) and let $B_1, \dots, B_p \in M_n(\mathbb{C})$ such that $P^{-1}B_jP \in \mathcal{K}_\eta(\mathbb{C})$ and the e^{B_j}, $1 \leqslant j \leqslant p$, generate G^*. Then G is hypercyclic if and only if*

$$\sum_{k=1}^{p} \mathbb{N}B_kPu_\eta + \sum_{k=1}^{r} 2\mathrm{i}\pi\mathbb{Z}Pe_\eta^{(k)}$$

is dense in $\mathbb{C}^n$.

Remark 1.3. If all matrices of $G \setminus I_n$ are non-invertible (i.e. $G^* = \{I_n\}$), then G is not hypercyclic (see Proposition 4.4).

Corollary 1.4. *If G is an abelian semigroup having a normal form of length $r \in \{1, \dots, n\}$ and generated by $(2n - r)$ matrices of $M_n(\mathbb{C})$, it has no dense orbit.*

Corollary 1.5 (Ayadi [2]). *If G is an abelian semigroup generated by n matrices of $M_n(\mathbb{C})$, it has no dense orbit.*

Theorem 1.6. *For any partition η of length r, there exist $(2n - r + 1)$ matrices in $\mathcal{K}_\eta^*(\mathbb{C})$ that generate a hypercyclic abelian semigroup.*

As a consequence, from Theorem 1.6 and Corollary 1.4, we obtain the following corollary.

Corollary 1.7. *For every $n \in \mathbb{N}_0$, $r \in \{1, \dots, n\}$, the minimum number of matrices of $M_n(\mathbb{C})$ that generate a hypercyclic abelian semigroup having a normal form of length r is exactly $2n - r + 1$.*

In particular, for $r = n$, we obtain Feldman's theorem.

Corollary 1.8 (Feldman [8]). *The minimum number of diagonalizable matrices of $M_n(\mathbb{C})$ that generate a hypercyclic abelian semigroup is $n + 1$.*

For $r < n$, we obtain the following.

Corollary 1.9 (Abels and Manoussos [1]; Shkarin [12]). *The minimum number of non-diagonalizable matrices of $M_n(\mathbb{C})$ that generate a hypercyclic abelian semigroup is $n + 2$.*

For $r = 1$, we obtain the following.

Corollary 1.10 (Abels and Manoussos [1]). *The minimum number of matrices of $\mathbb{T}_n(\mathbb{C})$ that generate a hypercyclic abelian semigroup is $2n$.*

This paper has the following structure. In § 2 we introduce the normal form of an abelian subsemigroup of $M_n(\mathbb{C})$ and we give some related properties. In § 3 we explore the characterization of hypercyclic abelian subsemigroups of $\mathcal{K}_\eta^*(\mathbb{C})$. The first part of Theorem 1.1 is proved in § 4. In § 5, we prove the second part of Theorem 1.1 and Corollaries 1.4 and 1.5. Theorem 1.6 is proved in § 6. In § 7, we give some examples for the cases $n = 1, 2$.

2. The normal form of abelian subsemigroups of $M_n(\mathbb{C})$ and some related properties

First recall the following proposition.

Proposition 2.1. *Let G be an abelian subgroup of $\mathrm{GL}(n,\mathbb{C})$. There then exists a $P \in \mathrm{GL}(n,\mathbb{C})$ such that $P^{-1}GP$ is an abelian subgroup of $\mathcal{K}^*_\eta(\mathbb{C})$, for some partition η of n.*

The proof of Proposition 2.1 results from combining the Lemmas 2.2 and 2.3.

Lemma 2.2. *Let G be an abelian subgroup of $\mathrm{GL}(n,\mathbb{C})$. There then exists a direct sum decomposition*

$$\mathbb{C}^n = \bigoplus_{k=1}^{r} E_k, \quad 1 \leqslant r \leqslant n, \tag{2.1}$$

where E_k is a G-invariant vector subspace of $\mathbb{C}^n$ of dimension n_k, $1 \leqslant k \leqslant r$, such that, for each $A \in G$, the restriction A_k of A to E_k has a unique eigenvalue $\mu_{A,k}$.

Proof. Given $A \in G$, let $\mu_{A,k}$ be an eigenvalue and let $E_{A,k} = \mathrm{Ker}(A - \mu_{A,k} I_n)^n$ be the associated generalized eigenspace. For any $B \in G$ the space $E_{A,k}$ is invariant under B. If B restricted to $E_{A,k}$ has two distinct eigenvalues, then it can be decomposed further. The decomposition (2.1) is the maximal decomposition associated to all $A \in G$. □

The restriction of the group G to each subspace E_k can be put into triangular form. This follows from a standard induction argument (see [**13**, Chapter 1, § 2, Corollary to Theorem 1]), used to prove the following.

Lemma 2.3. *Let G be an abelian subgroup of $\mathrm{GL}(n,\mathbb{C})$. Assume that every element of G has a unique eigenvalue. There then exists a matrix $P \in \mathrm{GL}(n,\mathbb{C})$ such that $P^{-1}GP$ is a subgroup of $\mathbb{T}^*_n(\mathbb{C})$.*

The analogous proposition to Proposition 2.1 for the subsemigroup is the following.

Proposition 2.4. *Let G be an abelian subsemigroup of $M_n(\mathbb{C})$. There then exists a $P \in \mathrm{GL}(n,\mathbb{C})$ such that $P^{-1}GP$ is an abelian subsemigroup of $\mathcal{K}_\eta(\mathbb{C})$ for some partition η of n.*

Proof. For every $A \in G$ there exists $\lambda_A \in \mathbb{C}$ such that $(A - \lambda_A I_n) \in \mathrm{GL}(n,\mathbb{C})$ (it suffices to take λ_A not an eigenvalue of A). Define $\hat{L}$ as the group generated by $L := \{A - \lambda_A I_n \colon A \in G\}$. Then $\hat{L}$ is an abelian subgroup of $\mathrm{GL}(n,\mathbb{C})$ and, by Proposition 2.1, there exists a $P \in \mathrm{GL}(n,\mathbb{C})$ such that $P^{-1}\hat{L}P \subset \mathcal{K}^*_\eta(\mathbb{C})$, for some $\eta \in \mathbb{N}_0^r$ and $r \in \{1,\dots,n\}$. As $P^{-1}LP = \{P^{-1}AP - \lambda_A I_n \colon A \in G\}$, we have $P^{-1}GP \subset \mathcal{K}_\eta(\mathbb{C})$; this proves the proposition. □

The following results follow from basic properties of the matrix exponential map, and their proofs are left to the reader.

Lemma 2.5. $\exp(\mathcal{K}_\eta(\mathbb{C})) = \mathcal{K}^*_\eta(\mathbb{C})$.

Lemma 2.6. *Let $A, B \in \mathcal{K}_\eta(\mathbb{C})$. If $e^A e^B = e^B e^A$, then $AB = BA$.*

Let G be an abelian subsemigroup of $\mathcal{K}^*_\eta(\mathbb{C})$. Define

$$\mathcal{C}(G) := \{A \in \mathcal{K}_\eta(\mathbb{C})\colon AB = BA \ \forall B \in G\}.$$

Since G is abelian, $G \subset \mathcal{C}(G)$.

Lemma 2.7. *Let G be an abelian subsemigroup of $\mathcal{K}^*_\eta(\mathbb{C})$. We have that*

(i) $g_\eta \subset \mathcal{C}(G)$ *and all matrices of* g_η *commute,*

(ii) $\exp(g_\eta) = G$,

(iii) $\exp(\mathcal{C}(G)) = \mathcal{C}(G) \cap \mathrm{GL}(n, \mathbb{C}) = \mathcal{C}(G) \cap \mathcal{K}^*_\eta(\mathbb{C})$.

Proof. (i) By Lemma 2.6, all elements of g_η commute; hence $g_\eta \subset \mathcal{C}(g_\eta)$. Let $B \in g_\eta$ and $A \in G$, so $e^B \in G$. As G is abelian, $Ae^B = e^B A$; hence $e^A e^B = e^B e^A$. Since $A, B \in \mathcal{K}_n(\mathbb{C})$, it follows by Lemma 2.6 that $AB = BA$, and therefore $B \in \mathcal{C}(G)$. We conclude that $g_\eta \subset \mathcal{C}(G)$.

(ii) We have $\exp(g_\eta) \subset G$ by definition. Conversely, let $A \in G$. Since $G \subset \mathcal{K}^*_\eta(\mathbb{C})$, there exists $B \in \mathcal{K}_\eta(\mathbb{C})$ such that $e^B = A$ (see Lemma 2.5). Hence, $B \in \exp^{-1}(G) \cap \mathcal{K}_\eta(\mathbb{C}) = g_\eta$, and then $A \in \exp(g_\eta)$. It follows that $\exp(g_\eta) = G$; this proves (ii).

(iii) Let $A = e^B$, where $B \in \mathcal{C}(G)$, and let $C \in \mathcal{C}(G)$. Then $BC = CB$, and therefore $Ce^B = e^B C$, or $AC = CA$. It follows that $A \in \mathcal{C}(G)$. Since $B \in \mathcal{K}_\eta(\mathbb{C})$, so $A \in \mathcal{K}^*_\eta(\mathbb{C})$, by Lemma 2.5. Conversely, let $A \in \mathcal{C}(G) \cap \mathcal{K}^*_\eta(\mathbb{C})$. By Lemma 2.5 there exists $B \in \mathcal{K}_\eta(\mathbb{C})$ such that $e^B = A$. Let $C \in G$. Then $Ce^B = e^B C$, and hence $e^C e^B = e^B e^C$. Since $B, C \in \mathcal{K}_\eta(\mathbb{C})$, it follows by Lemma 2.6 that $BC = CB$. Therefore, $B \in \mathcal{C}(G)$, and hence $A \in \exp(\mathcal{C}(G))$. □

3. The hypercyclic abelian subsemigroup of $\mathcal{K}^*_\eta(\mathbb{C})$

Let G be an abelian subsemigroup of $\mathcal{K}^*_\eta(\mathbb{C})$. Let $u \in \mathbb{C}^n$ and consider the linear map

$$\begin{aligned} \Phi_u \colon \mathcal{C}(G) &\to \mathbb{C}^n \\ A &\mapsto Au. \end{aligned}$$

Denote by $\mathrm{Vect}(G)$ the vector subspace of $\mathcal{K}_\eta(\mathbb{C})$ generated by G.

Proposition 3.1. *Let G be an abelian subsemigroup of $\mathcal{K}^*_\eta(\mathbb{C})$. If $\mathring{\overline{G(u)}} \neq \emptyset$ (respectively, $\mathring{\overline{g_\eta(u)}} \neq \emptyset$) for some $u \in \mathbb{C}^n$, then Φ_u is a linear isomorphism. Moreover, $\Phi_u(\mathrm{Vect}(G)) = \Phi_u(\mathcal{C}(G)) = \mathbb{C}^n$.*

Proof.

Case 1 ($\overset{\circ}{\overline{G(u)}} \neq \emptyset$)**.** Let us prove that Φ_u is surjective: we have that $\Phi_u(\mathcal{C}(G))$ is a vector subspace of $\mathbb{C}^n$. Since $G \subset \mathcal{C}(G)$, it follows that $G(u) \subset \Phi_u(\mathcal{C}(G))$. As $\mathcal{C}(G)$ is a vector space, $\emptyset \neq \overset{\circ}{\overline{G(u)}} \subset \Phi_u(\mathcal{C}(G))$, and therefore $\Phi_u(\mathcal{C}(G)) = \mathbb{C}^n$. We also have $G(u) \subset \Phi_u(\mathrm{Vect}(G))$, so, as above, $\Phi_u(\mathrm{Vect}(G)) = \mathbb{C}^n$.

Φ_u is injective: let $A \in \mathrm{Ker}(\Phi_u)$, so $Au = 0$. Let $x \in \mathbb{C}^n$; then, by the above working there exists $B \in \mathrm{Vect}(G)$ such that $x = Bu$. As $A \in \mathrm{Ker}(\Phi_u) \subset \mathcal{C}(G)$ then $AB = BA$. Therefore, $Ax = ABu = BAu = B(0) = 0$. It follows that $A = 0$, and hence $\mathrm{Ker}(\Phi_u) = \{0\}$.

Case 2 ($\overset{\circ}{\overline{g_\eta(u)}} \neq \emptyset$)**.** We also have that $\Phi_u(\mathcal{C}(G)) = \mathbb{C}^n$ since $g_\eta \subset \mathcal{C}(G)$ (Lemma 2.7 (i)) and $\emptyset \neq \overset{\circ}{\overline{g_\eta(u)}} \subset \Phi_u(\mathcal{C}(G))$. □

Corollary 3.2. *If G is hypercyclic, then $\mathcal{C}(G) = \mathrm{Vect}(G) = \bar{G}$; in particular, $\bar{G}$ is a vector space of dimension n.*

Proof. If $\overline{G(u)} = \mathbb{C}^n$ for some $u \in \mathbb{C}^n$, then Φ_u is a linear isomorphism (see Proposition 3.1), and hence $\mathcal{C}(G) = \Phi_u^{-1}(\mathbb{C}^n) = \Phi_u^{-1}(\overline{G(u)}) = \overline{\Phi_u^{-1}(G(u))} = \bar{G}$; this proves the corollary. □

We let $U := \prod_{k=1}^{r}(\mathbb{C}^* \times \mathbb{C}^{n_k-1})$. Then, U is open and dense in $\mathbb{C}^n$; moreover, $\mathbb{C}^n \setminus U$ is a union of r G-invariant vector subspaces of $\mathbb{C}^n$ of dimension $n-1$.

Lemma 3.3. *Let $u \in U$.*

(i) *If $B \in \mathcal{K}_\eta(\mathbb{C})$ satisfies $Bu \in U$, then $B \in \mathcal{K}_\eta^*(\mathbb{C})$.*

(ii) *If $\overset{\circ}{\overline{G(u)}} \neq \emptyset$ (respectively, $\overset{\circ}{\overline{g_\eta(u)}} \neq \emptyset$), then $U = \Phi_u(\mathcal{C}(G) \cap \mathcal{K}_\eta^*(\mathbb{C}))$.*

Proof. (i) Write

$$u = [u_1, \dots, u_r]^{\mathrm{T}}, \ Bu = v = [v_1, \dots, v_r]^{\mathrm{T}} \in U,$$

with $u_k = [a_{k,1}, \dots, a_{k,n_k}]^{\mathrm{T}} \in \mathbb{C}^* \times \mathbb{C}^{n_k-1}$, $v_k = [x_{k,1}, \dots, x_{k,n_k}]^{\mathrm{T}} \in \mathbb{C}^* \times \mathbb{C}^{n_k-1}$, and write $B = \mathrm{diag}(B_1, \dots, B_r)$, with $B_k \in \mathbb{T}_{n_k}(\mathbb{C})$, $k = 1, \dots, r$. Let μ_k be the eigenvalue of B_k. From $Bu = v$, we get that $\mu_k a_{k,1} = x_{k,1} \neq 0$ for every $k = 1, \dots, r$. It follows that $\mu_k \neq 0$. Therefore, $B \in \mathrm{GL}(n, \mathbb{C})$, that is, $B \in \mathcal{K}_\eta^*(\mathbb{C})$.

(ii) If $v \in U$, then, by Proposition 3.1, there exists $B \in \mathcal{C}(G)$ such that $Bu = v$; hence, by (i), $B \in \mathcal{K}_\eta^*(\mathbb{C})$ and so $v \in \Phi_u(\mathcal{C}(G) \cap \mathcal{K}_\eta^*(\mathbb{C}))$. Conversely, if $v = Bu$, where $B \in \mathcal{C}(G) \cap \mathcal{K}_\eta^*(\mathbb{C})$, then $x_{k,1} = \mu_k a_{k,1} \neq 0$ for every $k = 1, \dots, r$. It follows that $v \in U$. □

Lemma 3.4. *Let G be an abelian subsemigroup of $\mathcal{K}_\eta^*(\mathbb{C})$. Assume that G has a somewhere dense (respectively, dense) orbit in $\mathbb{C}^n$; then, for every $v \in U$, $G(v)$ is somewhere dense (respectively, dense) in $\mathbb{C}^n$.*

Proof. Let $u \in \mathbb{C}^n$ such that $\overset{\circ}{\overline{G(u)}} \neq \emptyset$. Since $\mathbb{C}^n \setminus U$ is a union of r G-invariant vector subspaces of $\mathbb{C}^n$ with dimension $n-1$, it follows that $u \in U$. Let $v \in U$; then, by Proposition 3.1, $v = Bu$ for some $B \in \mathrm{Vect}(G)$. Moreover, by Lemma 3.3, $B \in \mathcal{K}^*_\eta(\mathbb{C})$. It follows that $G(v) = B(G(u))$ and, since B is invertible, $\overset{\circ}{\overline{G(v)}} \neq \emptyset$.

Now, if $\overline{G(u)} = \mathbb{C}^n$, then $\overline{G(v)} = B(\overline{G(u)}) = B(\mathbb{C}^n) = \mathbb{C}^n$. □

4. Proof of the first part of Theorem 1.1

We require the following result.

Proposition 4.1 (Rossmann [11, Proposition 7′ p. 17]). *The restriction*

$$\exp|_{\mathbb{T}_n(\mathbb{C})} \colon \mathbb{T}_n(\mathbb{C}) \to \mathbb{T}^*_n(\mathbb{C})$$

is a local diffeomorphism; in particular, it is an open map.

Corollary 4.2. *The restriction $\exp|_{\mathcal{K}_\eta(\mathbb{C})} \colon \mathcal{K}_\eta(\mathbb{C}) \to \mathcal{K}^*_\eta(\mathbb{C})$ is a local diffeomorphism; in particular, it is an open map.*

Proof. The proof results from Proposition 4.1 and the fact that

$$\exp|_{\mathcal{K}_\eta(\mathbb{C})} = \exp|_{\mathbb{T}_{n_1}(\mathbb{C})} \oplus \cdots \oplus \exp|_{\mathbb{T}_{n_r}(\mathbb{C})}.$$

□

Recall that $U := \prod_{k=1}^{r}(\mathbb{C}^* \times \mathbb{C}^{n_k-1})$ and, for $u \in \mathbb{C}^n$, the linear map Φ_u is defined as $\Phi_u \colon \mathcal{C}(G) \to \mathbb{C}^n$, $A \mapsto Au$.

Proposition 4.3. *Let G be an abelian subsemigroup of $\mathcal{K}^*_\eta(\mathbb{C})$. Assume that $\overset{\circ}{\overline{G(u_\eta)}} \neq \emptyset$ or $\overset{\circ}{\overline{\mathrm{g}_\eta(u_\eta)}} \neq \emptyset$, where u_η is defined in § 1. Then, $f := \Phi_{u_\eta} \circ \exp|_{\mathcal{K}_\eta(\mathbb{C})} \circ \Phi^{-1}_{u_\eta}$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ is well defined and satisfies the following:*

(i) *f is continuous and open;*

(ii) *$f(Bu_\eta) = e^B u_\eta$ for every $B \in \mathcal{C}(G)$;*

(iii) *$f^{-1}(G(u_\eta)) = \mathrm{g}_\eta(u_\eta)$ and $f(\mathrm{g}_\eta(u_\eta)) = G(u_\eta)$;*

(iv) *$f(\mathbb{C}^n) = U$.*

Proof. (i) By Proposition 3.1, Φ_{u_η} is a linear isomorphism. So $f := \Phi_{u_\eta} \circ \exp|_{\mathcal{K}_\eta(\mathbb{C})} \circ \Phi^{-1}_{u_\eta}$ is well defined and continuous. Moreover, f is a local diffeomorphism by Corollary 4.2, and therefore f is an open map.

(ii) For every $B \in \mathcal{C}(G)$, we have that $\Phi^{-1}_{u_\eta}(Bu_\eta) = B$. Therefore,

$$\begin{aligned} f(Bu_\eta) &= \Phi_{u_\eta} \circ \exp|_{\mathcal{K}_\eta(\mathbb{C})} \circ \Phi^{-1}_{u_\eta}(Bu_\eta) \\ &= \Phi_{u_\eta}(e^B) \\ &= e^B u_\eta. \end{aligned}$$

(iii) We have that

$$\begin{aligned} f^{-1}(G(u_\eta)) &= \Phi_{u_\eta} \circ \exp|_{\mathcal{K}_\eta(\mathbb{C})}^{-1} \circ \Phi_{u_\eta}^{-1}(G(u_\eta)) \\ &= \Phi_{u_\eta}(\exp|_{\mathcal{K}_\eta(\mathbb{C})}^{-1}(G)) \\ &= \Phi_{u_\eta}(\mathrm{g}_\eta) \\ &= \mathrm{g}_\eta(u_\eta). \end{aligned}$$

We also have that

$$\begin{aligned} f(\mathrm{g}_{u_\eta}) &= \Phi_{u_\eta} \circ \exp|_{\mathcal{K}_\eta(\mathbb{C})} \circ \Phi_{u_\eta}^{-1}(\mathrm{g}_\eta(u_\eta)) \\ &= \Phi_{u_\eta}(\exp|_{\mathcal{K}_\eta(\mathbb{C})}(\mathrm{g}_\eta)) \\ &= \Phi_{u_\eta}(G) \\ &= G(u_\eta). \end{aligned}$$

(iv) As $\Phi_{u_\eta}^{-1}\colon \mathbb{C}^n \to \mathcal{C}(G)$ is an isomorphism, we get that

$$\begin{aligned} f(\mathbb{C}^n) &= \Phi_{u_\eta} \circ \exp|_{\mathcal{K}_\eta(\mathbb{C})} \circ \Phi_{u_\eta}^{-1}(\mathbb{C}^n) \\ &= \Phi_{u_\eta}[\exp|_{\mathcal{K}_\eta(\mathbb{C})}(\mathcal{C}(G))] \\ &= \Phi_{u_\eta}(\mathcal{C}(G) \cap \mathcal{K}_\eta^*(\mathbb{C})) && \text{(by Lemma 2.7 (iii))} \\ &= U && \text{(by Lemma 3.3 (ii))}. \end{aligned}$$

□

Proposition 4.4. *Let G be an abelian subsemigroup of $M_n(\mathbb{C})$ and let $u \in \mathbb{C}^n$. Then $G^*(u)$ is somewhere dense (respectively, dense) if and only if $G(u)$ is.*

Proof. The first implication is trivial. Conversely, suppose that $\overset{\circ}{\overline{G(u)}} \neq \emptyset$ (respectively, $\overline{G(u)} = \mathbb{C}^n$). We can assume, using Proposition 2.4, that $G \subset \mathcal{K}_\eta(\mathbb{C})$. We let $G' := G \setminus G^*$.

- If $G' = \emptyset$, then $G = G^*$ and so $\overset{\circ}{\overline{G^*(u)}} \neq \emptyset$ (respectively, $\overline{G^*(u)} = \mathbb{C}^n$).
- If $G' \neq \emptyset$, then
$$G(u) \subset \left(\bigcup_{A \in G'} \mathrm{Im}(A) \right) \cup G^*(u).$$

Since every $A \in G'$ is non-invertible,

$$\mathrm{Im}(A) \subset \bigcup_{k=1}^{r} H_k,$$

where

$$H_k := \{u = [u_1, \dots, u_r]^{\mathrm{T}} \in \mathbb{C}^n \colon u_j \in \mathbb{C}^{n_j},\ u_k \in \{0\} \times \mathbb{C}^{n_k - 1},\ 1 \leqslant j \neq k \leqslant r\}.$$

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It follows that

$$G(u) \subset \left(\bigcup_{k=1}^{r} H_k\right) \cup G^*(u),$$

and so

$$\overline{G(u)} \subset \left(\bigcup_{k=1}^{r} H_k\right) \cup \overline{G^*(u)}.$$

Since H_k has dimension $n-1$, $\mathring{H}_k = \emptyset$, for every $1 \leqslant k \leqslant r$, and therefore $\overset{\circ}{\overline{G^*(u)}} \neq \emptyset$ (respectively, $\overline{G^*(u)} = \mathbb{C}^n$). □

Proof of the first part of Theorem 1.1. Let G be an abelian subsemigroup of $\mathcal{K}_\eta(\mathbb{C})$. From Proposition 4.4 and since $\mathfrak{g}_\eta = \mathfrak{g}^*_\eta := \exp^{-1}(G^*) \cap (\mathcal{K}_\eta(\mathbb{C}))$, we may assume that $G \subset \mathcal{K}^*_\eta(\mathbb{C})$.

(ii) $\Longrightarrow$ (i). This is clear.

(i) $\Longrightarrow$ (ii). This follows directly from Proposition 4.4 and Lemma 3.4 (since $u_\eta \in U$).

(iii) $\Longrightarrow$ (ii). Suppose that $\overline{\mathfrak{g}_\eta(u_\eta)} = \mathbb{C}^n$. Then $\overline{\mathfrak{g}_\eta(u_\eta)} = \overline{\mathfrak{g}^*_\eta(u_\eta)} = \mathbb{C}^n$. By applying Proposition 4.3 to G^*, there exists a continuous map $f\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $f(\mathfrak{g}_\eta(u_\eta)) = G(u_\eta)$ and $f(\mathbb{C}^n) = U$. Hence, one has that $U = f(\overline{\mathfrak{g}_\eta(u_\eta)}) \subset \overline{G(u_\eta)}$. Therefore, $\overline{G(u_\eta)} = \mathbb{C}^n$.

(ii) $\Longrightarrow$ (iii). Suppose that $\overline{G(u_\eta)} = \mathbb{C}^n$. Since f is an open map, we have that $f^{-1}(G(u_\eta)) = \mathfrak{g}_\eta(u_\eta)$, and thus

$$\mathbb{C}^n = f^{-1}(\overline{G(u_\eta)}) \subset \overline{f^{-1}(G(u_\eta))} = \overline{\mathfrak{g}_\eta(u_\eta)}.$$

Hence, $\overline{\mathfrak{g}_\eta(u_\eta)} = \mathbb{C}^n$. □

5. Proof of the second part of Theorem 1.1, Corollaries 1.4 and 1.5

Lemma 5.1 (Ayadi and Marzougui [3, Proposition 3.5]). *Let $A, B \in \mathbb{T}_n(\mathbb{C})$. If $e^A = e^B$, then $A = B + 2ik\pi I_n$ for some $k \in \mathbb{Z}$.*

Proposition 5.2. *Let G be an abelian subsemigroup of $\mathcal{K}^*_\eta(\mathbb{C})$ and let $B_1, \dots, B_p \in \mathcal{K}_\eta(\mathbb{C})$ $(p \geqslant 1)$ such that $e^{B_1}, \dots, e^{B_p}$ generate G. We have that*

$$\mathfrak{g}_\eta(u_\eta) = \sum_{k=1}^{p} \mathbb{N} B_k u_\eta + \sum_{k=1}^{r} 2i\pi \mathbb{Z} e^{(k)}_\eta.$$

Proof. First we determine $\mathfrak{g}_\eta$. Let $C \in \mathfrak{g}_\eta$. Then $C = \operatorname{diag}(C_1, \dots, C_r) \in \mathcal{K}_\eta(\mathbb{C})$ and $e^C \in G$. So

$$e^C = \operatorname{diag}(e^{C_1}, \dots, e^{C_r}) = e^{m_1 B_1} \cdots e^{m_p B_p}$$

for some $m_1, \ldots, m_p \in \mathbb{N}$. Since $B_1, \ldots, B_p \in \mathfrak{g}_\eta$, they pairwise commute (see Lemma 2.7 (i)). Therefore, $e^C = e^{m_1 B_1 + \cdots + m_p B_p}$. Set $B_j = \operatorname{diag}(B_{j,1}, \ldots, B_{j,r})$; then

$$e^{C_k} = e^{m_1 B_{1,k} + \cdots + m_p B_{p,k}}, \quad k = 1, \ldots, r.$$

As $C \in \mathfrak{g}_\eta$, we also have that $CB_j = B_j C$, so $C_k B_{j,k} = B_{j,k} C_k$, $j = 1, \ldots, p$. It follows that

$$C_k = m_1 B_{1,k} + \cdots + m_p B_{p,k} + 2\mathrm{i}\pi s_k I_{n_k}$$

for some $s_k \in \mathbb{Z}$ (see Lemma 5.1). Therefore,

$$\begin{aligned} C &= \operatorname{diag}\left(\sum_{j=1}^{p} m_j B_{j,1} + 2\mathrm{i}\pi s_1 I_{n_1}; \ldots, \ldots; \sum_{j=1}^{p} m_j B_{j,r} + 2\mathrm{i}\pi s_r I_{n_r}\right) \\ &= \sum_{j=1}^{p} m_j B_j + \operatorname{diag}(2\mathrm{i}\pi s_1 I_{n_1}, \ldots, 2\mathrm{i}\pi s_r I_{n_r}). \end{aligned}$$

Set $J_k := \operatorname{diag}(J_{k,1}, \ldots, J_{k,r})$, where

$$J_{k,i} = \begin{cases} 0 \in \mathbb{T}_{n_i}(\mathbb{C}) & \text{if } i \neq k, \\ I_{n_k} & \text{if } i = k. \end{cases}$$

We have that

$$\operatorname{diag}(2\mathrm{i}\pi s_1 I_{n_1}, \ldots, 2\mathrm{i}\pi s_r I_{n_r}) = \sum_{k=1}^{r} 2\mathrm{i}\pi s_k J_k,$$

and therefore

$$C = \sum_{j=1}^{p} m_j B_j + \sum_{k=1}^{r} 2\mathrm{i}\pi s_k J_k.$$

We conclude that

$$\mathfrak{g}_\eta = \sum_{j=1}^{p} \mathbb{N} B_j + \sum_{k=1}^{r} 2\mathrm{i}\pi \mathbb{Z} J_k.$$

Second, we determine $\mathfrak{g}_\eta(u_\eta)$. Let $B \in \mathfrak{g}_\eta$. We have that

$$B = \sum_{j=1}^{p} m_j B_j + \sum_{k=1}^{r} 2\mathrm{i}\pi s_k J_k$$

for some $m_1, \ldots, m_p \in \mathbb{N}$ and $s_1, \ldots, s_r \in \mathbb{Z}$. We also have that

$$\begin{aligned} J_k u_\eta &= \operatorname{diag}(J_{k,1}, \ldots, J_{k,r})[e_{\eta,1}, \ldots, e_{\eta,r}]^{\mathrm{T}} \\ &= [(e_\eta^{(k)})_1, \ldots, (e_\eta^{(k)})_r]^{\mathrm{T}} \\ &= e_\eta^{(k)}. \end{aligned}$$

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Hence,

$$Bu_\eta = \sum_{j=1}^{p} m_j B_j u_\eta + \sum_{k=1}^{r} 2\mathrm{i}\pi s_k e_\eta^{(k)},$$

and therefore

$$g_\eta(u_\eta) = \sum_{j=1}^{p} \mathbb{N} B_j u_\eta + \sum_{k=1}^{r} 2\mathrm{i}\pi \mathbb{Z} e_\eta^{(k)}.$$

This proves the proposition. □

Proof of the second part of Theorem 1.2. This results directly from Proposition 5.2 and the first part of Theorem 1.1. □

Lemma 5.3. *Let $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_m$, with $u_k \in \mathbb{C}^n$, $k = 1, \ldots, m$. If $m \leqslant 2n$, then H is nowhere dense in $\mathbb{C}^n$.*

Proof. By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, the proof comes from [**12**, Lemma 2.1]. □

Proof of Corollary 1.4. First, it is clear by Lemma 5.3 that if $H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_m$, $u_k \in \mathbb{C}^n$ with $m \leqslant 2n$, then H cannot be dense. Now, by applying Corollary 1.2 for $p = 2n - r$, one has that $m = p + r = 2n$, and Corollary 1.4 follows. □

Proof of Corollary 1.5. This follows from the fact that $n \leqslant 2n - r$, since $r \leqslant n$, and by applying Corollary 1.2. □

6. Proof of Theorem 1.6

We construct, for every $r \in \{1, \ldots, n\}$ and for every partition η of n of length r, $(2n-r+1)$ matrices $A_1, \ldots, A_{2n-r+1} \in \mathcal{K}_\eta^*(\mathbb{C})$ generating a hypercyclic abelian semigroup.

We repeatedly use the following multidimensional version of Kronecker's theorem.

Kronecker's theorem

Let $\alpha_1, \ldots, \alpha_n$ be negative real numbers such that the numbers $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$. Then the set

$$\mathbb{N}^n + \mathbb{N}[\alpha_1, \ldots, \alpha_n]^{\mathrm{T}} := \{[s_1, \ldots, s_n]^{\mathrm{T}} + k[\alpha_1, \ldots, \alpha_n]^{\mathrm{T}} \colon k, s_1, \ldots, s_n \in \mathbb{N}\}$$

is dense in $\mathbb{R}^n$.

We deduce the complex version as follows.

Corollary 6.1. *Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be negative real numbers such that the numbers $1, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are linearly independent over $\mathbb{Q}$. Then $\mathbb{N}^n + \mathrm{i}\mathbb{N}^n + \mathbb{N}[\alpha_1 + \mathrm{i}\beta_1, \ldots, \alpha_n + \mathrm{i}\beta_n]^{\mathrm{T}}$ is dense in $\mathbb{C}^n$.*

Proof. This is clear by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in the obvious way. □

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Recall that $e_\eta^{(k)} = [(e_\eta^{(k)})_1, \dots, (e_\eta^{(k)})_r] \in \mathbb{C}^n$, where, for every $j = 1, \dots, r$,

$$(e_\eta^{(k)})_j = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } j \neq k, \\ e_{\eta,k} & \text{if } j = k. \end{cases}$$

An equivalent formulation is

$$e_\eta^{(1)} = e_1, \dots, e_\eta^{(k)} = e_{\ell_k}, \quad \text{where } \ell_1 = 1,\ \ell_k := \sum_{j=1}^{k-1} n_j + 1,\ k = 2, \dots, r.$$

Proposition 6.2. *Let $n \in \mathbb{N}_0$ and $r \in \{1, \dots, n\}$. There then exist $(2n-r+1)$ vectors $u_1, \dots, u_{2n-r+1}$ of $\mathbb{C}^n$ such that*

$$\sum_{k=1}^{2n-r+1} \mathbb{N}u_k + \sum_{k=1}^{r} 2\mathrm{i}\pi\mathbb{Z}e_\eta^{(k)}$$

is dense in $\mathbb{C}^n$.

Proof. Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ be negative real numbers such that the numbers $1, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are linearly independent over $\mathbb{Q}$. Define $(e_{i_{r+1}}, \dots, e_{i_n}) := \mathcal{B}_0 \setminus (e_{\ell_1}, \dots, e_{\ell_r})$, and define the matrix S by

$$Se_k = \begin{cases} 2\mathrm{i}\pi e_\eta^{(k)} & \text{if } 1 \leqslant k \leqslant r, \\ e_{i_k} & \text{if } r+1 \leqslant k \leqslant n. \end{cases}$$

We see that $S \in \mathrm{GL}(n, \mathbb{C})$. Set $u = [\alpha_1 + \mathrm{i}\beta_1, \dots, \alpha_n + \mathrm{i}\beta_n]^{\mathrm{T}}$ and define

$$u_k := \begin{cases} Se_{r+k} & \text{if } 1 \leqslant k \leqslant n-r, \\ \mathrm{i}Se_{r-n+k} & \text{if } n-r+1 \leqslant k \leqslant 2n-r, \\ Su & \text{if } k = 2n-r+1. \end{cases}$$

Set

$$H := \sum_{k=1}^{2n-r+1} \mathbb{N}u_k + \sum_{k=1}^{r} 2\mathrm{i}\pi\mathbb{Z}e_\eta^{(k)}$$

and

$$H' := \sum_{k=1}^{n-r} \mathbb{N}e_{r+k} + \sum_{k=1}^{n} \mathrm{i}\mathbb{N}e_k + \mathbb{N}u + \sum_{k=1}^{r} \mathbb{Z}e_k.$$

We then have that

$$\begin{aligned} S(H') &= \sum_{k=1}^{n-r} \mathbb{N}Se_{r+k} + \sum_{k=1}^{n} \mathrm{i}\mathbb{N}Se_k + \mathbb{N}Su + \sum_{k=1}^{r} \mathbb{Z}Se_k \\ &= \sum_{k=1}^{n-r} \mathbb{N}u_k + \sum_{k=1}^{n} \mathbb{N}\mathrm{i}Se_k + \mathbb{N}u_{2n-r+1} + \sum_{k=1}^{r} \mathbb{Z}2\mathrm{i}\pi e_\eta^{(k)} \\ &= \sum_{k=1}^{n-r} \mathbb{N}u_k + \sum_{k=n-r+1}^{2n-r} \mathbb{N}u_k + \mathbb{N}u_{2n-r+1} + \sum_{k=1}^{r} 2\mathrm{i}\pi\mathbb{Z}e_\eta^{(k)} \\ &= H. \end{aligned}$$

Since $\mathbb{N}^n + \mathrm{i}\mathbb{N}^n + \mathbb{N}u \subset H'$, we see that H' is dense in $\mathbb{C}^n$ by Corollary 6.1, and thus so is H. This proves the proposition. $\square$

Proof of Theorem 1.6. By Proposition 6.2, there exist $u_1, \dots, u_{2n-r+1} \in \mathbb{C}^n$ such that

$$H := \sum_{k=1}^{2n-r+1} \mathbb{N}u_k + \sum_{k=1}^{r} 2\mathrm{i}\pi\mathbb{Z}e_\eta^{(k)}$$

is dense in $\mathbb{C}^n$. Set

$$u_k = [u_{k,1}, \dots, u_{k,r}]^{\mathrm{T}}$$

with $u_{k,j} = [x_{j,1}^{(k)}, \dots, x_{j,n_j}^{(k)}]^{\mathrm{T}}$. Let $B_1, \dots, B_{2n-r+1}$ be defined by

$$B_k = \mathrm{diag}(B_{k,1}, \dots, B_{k,r}),$$

where

$$B_{k,j} = \begin{bmatrix} x_{j,1}^{(k)} & & & & 0 \\ \vdots & \ddots & & & \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{j,n_j}^{(k)} & 0 & \dots & 0 & x_{j,1}^{(k)} \end{bmatrix}, \quad 1 \leqslant j \leqslant r,\ 1 \leqslant k \leqslant 2n-r+1.$$

Then $B_k u_\eta = u_k$.

Let G be the subsemigroup of $\mathcal{K}_\eta^*(\mathbb{C})$ generated by $e^{B_1}, \dots, e^{B_{2n-r+1}}$.

Firstly, we check that G is abelian. For this, it suffices to show that $B_k B_{k'} = B_{k'} B_k$ for every $k, k' = 1, \dots, 2n-r+1$.

Set $B_{k,j} := N_{k,j} + x_{j,1}^{(k)} I_{n_j}$, where

$$N_{k,j} = \begin{bmatrix} 0 & 0 \\ T_{k,j} & 0 \end{bmatrix} \in \mathbb{T}_{n_j}(\mathbb{C}) \quad \text{with } T_{k,j} = [x_{j,2}^{(k)}, \dots, x_{j,n_j}^{(k)}]^{\mathrm{T}},\ j = 1, \dots, r.$$

We see that $N_{k,j} N_{k',j} = N_{k',j} N_{k,j} = 0$ for every $j = 1, \dots, r$. Hence, $B_{k,j} B_{k',j} = B_{k',j} B_{k,j}$ and so $B_k B_{k'} = B_{k'} B_k$.

Secondly, by Proposition 5.2, we have that

$$\begin{aligned} g_\eta(u_\eta) &= \sum_{k=1}^{2n-r+1} \mathbb{N}B_k u_\eta + \sum_{k=1}^{r} 2\mathrm{i}\pi\mathbb{Z}e_\eta^{(k)} \\ &= \sum_{k=1}^{2n-r+1} \mathbb{N}u_k + \sum_{k=1}^{r} 2\mathrm{i}\pi\mathbb{Z}e_\eta^{(k)} \\ &= H. \end{aligned}$$

Therefore, $\overline{g_\eta(u_\eta)} = \mathbb{C}^n$ and, by Theorem 1.1, $\overline{G(u_\eta)} = \mathbb{C}^n$. $\square$

7. Examples

Example 7.1. Let G be the subsemigroup of $\mathbb{C}^*$ generated by $a_1 = e^{2\pi}$, $a_2 = e^{-2(\sqrt{2}+\mathrm{i}\sqrt{3})\pi}$. Then G is hypercyclic.

Proof. In this case, we have that $\eta = (1)$, $u_\eta = 1$ and $g_\eta = \exp^{-1}(G)$. By Proposition 5.2,

$$g_\eta(1) = 2\pi\mathbb{N} - 2(\sqrt{2} + \mathrm{i}\sqrt{3})\pi\mathbb{N} + 2\mathrm{i}\pi\mathbb{Z} = 2\pi L,$$

where

$$L := \mathbb{N} - (\sqrt{2} + \mathrm{i}\sqrt{3})\mathbb{N} + \mathrm{i}\mathbb{Z}.$$

As 1, $\sqrt{2}$ and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$, by Corollary 6.1, $\overline{\mathbb{N}-(\sqrt{2}+\mathrm{i}\sqrt{3})\mathbb{N}+\mathrm{i}\mathbb{N}} \subset L$ is dense in $\mathbb{C}$, and so is L. Therefore, $\overline{g_\eta(1)} = \mathbb{C}$ and, by Theorem 1.1, $\overline{\mathrm{G}(1)} = \mathbb{C}$. $\square$

Example 7.2. Let G be the semigroup generated by

$$A_1 = \operatorname{diag}(e^{2\pi}, e^{2\pi}), \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 2\pi & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 0 \\ 2\mathrm{i}\pi & 1 \end{bmatrix}$$

and

$$A_4 = e^{-2\pi(\sqrt{2}+\mathrm{i}\sqrt{3})} \begin{bmatrix} 1 & 0 \\ 2\pi(1-\mathrm{i}\sqrt{5}) & 1 \end{bmatrix}.$$

Then G is abelian and hypercyclic.

Proof. By construction, G is an abelian subsemigroup of $\mathbb{T}_2^*(\mathbb{C})$ and we have that $u_\eta = e_1$ and $A_k = e^{B_k}$, $k = 1, \dots, 4$, where

$$B_1 = \operatorname{diag}(2\pi, 2\pi), \qquad B_2 = \begin{bmatrix} 0 & 0 \\ 2\pi & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 & 0 \\ 2\mathrm{i}\pi & 0 \end{bmatrix}$$

and

$$B_4 = \begin{bmatrix} -2\pi(\sqrt{2}+\mathrm{i}\sqrt{3}) & 0 \\ 2\pi(1-\mathrm{i}\sqrt{5}) & -2\pi(\sqrt{2}+\mathrm{i}\sqrt{3}) \end{bmatrix}.$$

By Proposition 5.2,

$$\begin{aligned} \boldsymbol{g}_\eta(e_1) &= \sum_{k=1}^{4} \mathbb{N}B_k e_1 + 2\mathrm{i}\pi\mathbb{Z}e_1 \\ &= 2\pi L, \end{aligned}$$

where

$$L := \mathbb{N}e_1 + \mathbb{N}e_2 + \mathrm{i}\mathbb{N}e_2 + \mathbb{N}[-\sqrt{2} - \mathrm{i}\sqrt{3}, 1 - \mathrm{i}\sqrt{5}]^{\mathrm{T}} + \mathrm{i}\mathbb{Z}e_1.$$

We let

$$K := \mathbb{N}e_1 + \mathbb{N}e_2 + \mathrm{i}\mathbb{N}e_2 + \mathbb{N}[-\sqrt{2} - \mathrm{i}\sqrt{3}, 1 - \mathrm{i}\sqrt{5}]^{\mathrm{T}} + \mathrm{i}\mathbb{N}e_1.$$

Then

$$K = \mathbb{N}^2 + \mathrm{i}\mathbb{N}^2 + \mathbb{N}[-\sqrt{2} - \mathrm{i}\sqrt{3}, 1 - \mathrm{i}\sqrt{5}]^{\mathrm{T}} \subset L.$$

By Corollary 6.1, K is dense in $\mathbb{C}^2$ since $1, -\sqrt{2}, -\sqrt{3}$ and $-\sqrt{5}$ are linearly independent over $\mathbb{Q}$, and so is L. We conclude by Theorem 1.1 that $\overline{G(e_1)} = \mathbb{C}^2$. □

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