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ON THE DERIVATION LIE ALGEBRAS OF FEWNOMIAL SINGULARITIES

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Abstract

Let V be a hypersurface with an isolated singularity at the origin defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The Yau algebra, $L(V)$, is the Lie algebra of derivations of the moduli algebra of V. It is a finite-dimensional solvable algebra and its dimension $\lambda(V)$ is the Yau number. Fewnomial singularities are those which can be defined by an n-nomial in n indeterminates. Yau and Zuo ['A sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularity', *Pure Appl. Math. Q.* **12**(1) (2016), 165–181] conjectured a bound for the Yau number and proved that this conjecture holds for binomial isolated hypersurface singularities. In this paper, we verify this conjecture for weighted homogeneous fewnomial surface singularities.

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1. Introduction

Let $V = V(f)$ be a hypersurface defined by an equation $f = 0$ with a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. For an isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$, we consider the Yau algebra $L(V)$, which is the Lie algebra of derivations of the moduli algebra

$$A(V) := O_n/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n),$$

where O_n is the algebra of convergent power series in n indeterminates (see [6, 16]). The second author proved that $L(V)$ is a finite-dimensional solvable Lie algebra [12]. According to Elashvili and Khimshiashvili [5], the dimension of the Lie algebra $L(V)$ is called the Yau number and it is denoted by $\lambda(V)$. The Yau algebra is an important tool to investigate singularities [10]. It has also been studied in our recent papers [3, 13] in connection with the nonexistence of negative-weight derivations.

The order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity, mult(f), of the singularity $(V, 0)$. The Milnor number μ and

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the Tjurina number τ of $(V, 0)$ are defined respectively by

$$\mu = \dim \mathbb{C}\{x_1, \ldots, x_n\}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n),$$
$$\tau = \dim \mathbb{C}\{x_1, \ldots, x_n\}/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is said to be weighted homogeneous if there exist positive rational numbers $w_1, \ldots, w_n$ (called the weights of $x_1, \ldots, x_n$) and d such that $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with nonzero coefficient. The number d is called the weighted homogeneous degree, w-deg$f$, of f with respect to the weights w_j. The weight type of f is denoted by $(w_1, \ldots, w_n; d)$. Without loss of generality, we can assume that w-deg$f = 1$ and we will often use this in what follows. One can easily compute the Milnor number of weighted homogeneous isolated hypersurface singularities using the weight type. If $(w_1, \ldots, w_n; 1)$ is the weight type of a weighted homogeneous isolated hypersurface singularity, then the Milnor number is $\mu = (w_1^{-1} - 1) \cdots (w_n^{-1} - 1)$ [7]. In 1971, Saito computed the necessary and sufficient condition for V to be defined by a weighted homogeneous polynomial and showed that f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$ [9].

Obviously, the number of monomials in f may depend on the system of coordinates. In order to obtain a rigorous concept, we shall only allow linear changes of coordinates and say that f (or rather its germ at the origin) is a k-nomial if k is the smallest natural number such that f becomes a k-nomial after (possibly) a linear change of coordinates. An isolated hypersurface singularity V is called k-nomial if there exists an isolated hypersurface singularity Y analytically isomorphic to V which can be defined by a k-nomial and k is the smallest such number. A weighted homogeneous polynomial $f(x_1, \ldots, x_n)$ is called a fewnomial if the number of variables coincides with the number of monomials [5], that is, fewnomial singularities are those which can be defined by an n-nomial in n indeterminates.

It is well known that the direct sum of two isolated weighted homogeneous fewnomial singularities is also an isolated weighted homogeneous fewnomial singularity. Ebeling and Takahashi [4] called fewnomial singularities invertible singularities and in the case of three variables divided them into five types (see Proposition 2.9). Khimshiashvili [6] investigated the Yau algebras of binomial singularities and used this to distinguish the analytic isomorphism types of these singularities. The classical theorem of Pursell and Shanks states that the Lie algebra of smooth vector fields on a smooth manifold determines the diffeomorphism type of a manifold. Khimshiashvili [6] proved the analogue of this theorem for Yau algebras of binomial singularities. From results in [11], there exist isolated singularities defined by quadrinomials in three variables which are analytically nonisomorphic but have isomorphic Yau algebras. Thus, there is no analogue of the theorem of Pursell and Shanks outside the class of fewnomial singularities. Fewnomial singularities have been extensively studied since the early stages of mirror symmetry and applied to give topological mirror pairs of Calabi–Yau manifolds.

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It is known that the weight types are topological invariants for one- or two-dimensional weighted homogeneous singularities [8, 15]. It is natural to ask whether we can bound the analytic invariant Yau numbers by only using the topological invariant weight types of the weighted homogeneous isolated hypersurface singularities. In [14], we proposed the following sharp upper estimate conjecture.

CONJECTURE 1.1 [14]. *Let* $(V,0) = \{(x_1,\dots,x_n) \in \mathbb{C}^n : f(x_1,\dots,x_n) = 0\}$ $(n \geq 2)$ *be an isolated singularity defined by the weighted homogeneous polynomial* $f(x_1,\dots,x_n)$ *of weight type* $(w_1,\dots,w_n;1)$. *Then the Yau number satisfies*

$$\lambda(V) \leq n\mu - \sum_i^n \Big(\frac{1}{w_1}-1\Big)\Big(\frac{1}{w_2}-1\Big)\cdots\widehat{\Big(\frac{1}{w_i}-1\Big)}\cdots\Big(\frac{1}{w_n}-1\Big),$$

where μ *is the Milnor number and* $\widehat{(w_i^{-1}-1)}$ *means that* $w_i^{-1}-1$ *is omitted.*

The conjecture holds for binomial isolated hypersurface singularities [14]. In this paper, we verify the conjecture for fewnomial surface singularities. The main difficulty in the proof is computing the Lie algebras of these singularities. The main purpose of this paper is to prove the following theorem.

THEOREM 1.2 (MAIN THEOREM). *Let* $(V,0)$ *be a fewnomial singularity defined by the weighted homogeneous polynomial* $f(x_1,x_2,x_3)$ *(see Proposition 2.9) with weight type* $(w_1,w_2,w_3;1)$. *Then*

$$\begin{aligned}\lambda(V) \leq 3\Big(\frac{1}{w_1}-1\Big)\Big(\frac{1}{w_2}-1\Big)\Big(\frac{1}{w_3}-1\Big) - \Big(\frac{1}{w_1}-1\Big)\Big(\frac{1}{w_2}-1\Big)\\ - \Big(\frac{1}{w_1}-1\Big)\Big(\frac{1}{w_3}-1\Big) - \Big(\frac{1}{w_2}-1\Big)\Big(\frac{1}{w_3}-1\Big).\end{aligned}$$

2. Generalities on fewnomial singularities and Lie algebras

We present here the necessary definitions and results on derivations of Lie algebras which we will use to compute the Yau algebras of a large class of singularities.

A derivation of a commutative associative algebra A is a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}\,(A,A)$ (or $\mathrm{Der}\,A$) with the bracket defined by the commutator of linear endomorphisms.

Let A, B be associative algebras over $\mathbb{C}$. The subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A is called the multiplication algebra $M(A)$ of A. The centroid $C(A)$ is the set of endomorphisms of A which commute with all elements of $M(A)$. Obviously, $C(A)$ is a unital subalgebra of $\mathrm{End}(A)$. We state a particular case of a general result from [2, Proposition 1.2]. Let $S = A \otimes B$ be a tensor product of finite-dimensional associative algebras with units. Then $\mathrm{Der}\,S \cong (\mathrm{Der}\,A) \otimes C(B) + C(A) \otimes (\mathrm{Der}\,B)$. We will only use this result for commutative associative algebras with units, in which case the centroid coincides with the algebra itself and we have the following result.

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THEOREM 2.1 [2]. *For commutative associative algebras A and B,*

$$\operatorname{Der} S \cong (\operatorname{Der} A) \otimes B + A \otimes (\operatorname{Der} B).$$

DEFINITION 2.2. Let J be an ideal in an analytic algebra S. Then $\operatorname{Der}_J S \subseteq \operatorname{Der}_{\mathbb{C}} S$ is the Lie subalgebra of all $\sigma \in \operatorname{Der}_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.

We shall use the following well-known result to compute the derivations.

THEOREM 2.3 [14]. *Let J be an ideal in $R = \mathbb{C}\{x_1, \ldots, x_n\}$. Then there is a natural isomorphism of Lie algebras* $(\operatorname{Der}_J R)/(J \cdot \operatorname{Der}_{\mathbb{C}} R) \cong \operatorname{Der}_{\mathbb{C}} (R/J)$.

We recall the definitions of the Yau algebra $L(V)$ and the Yau number $\lambda(V)$.

DEFINITION 2.4. Let $f(x_1, \ldots, x_n)$ be a complex polynomial and $V = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A(V)$ be the moduli algebra. Then $L(V) := \operatorname{Der}_{\mathbb{C}} (A(V), A(V))$ and $\lambda(V) := \dim L(V)$.

DEFINITION 2.5. A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is called quasi-homogeneous (or weighted homogeneous) if there exist positive rational numbers $w_1, \ldots, w_n$ (the weights of the indeterminates x_j) and d such that $\sum w_j k_j = d$ for each monomial $\prod x_j^{k_j}$ appearing in f with nonzero coefficient. The number d is called the weight-homogeneous degree, w-deg$f$, of f with respect to the weights w_j. The collection $(w; d) = (w_1, \ldots, w_n; d)$ is called the weight-homogeneity type (wt-type) of f.

REMARK 2.6. Suppose that $f(x_1, \ldots, x_n) = 0$ of weight type $(w_1, \ldots, w_n; 1)$ and $g(y_1, \ldots, y_m) = 0$ of weight type $(w_{n+1}, \ldots, w_{n+m}; 1)$ are two weighted homogeneous polynomials which define two isolated hypersurface singularities $(V_f, 0) \subset (\mathbb{C}^n, 0)$ and $(V_g, 0) \subset (\mathbb{C}^m, 0)$. The Thom–Sebastiani sum is $f(x_1, \ldots, x_n) + g(y_1, \ldots, y_m) = 0$ with weight type $(w_1, \ldots, w_{n+m}; 1)$ and defines a weighted homogeneous isolated singularity $(V_{f+g}, 0) \subset (\mathbb{C}^{m+n}, 0)$.

DEFINITION 2.7. A polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is fewnomial if the number of monomials appearing in f does not exceed n. Thus, a direct sum of weighted homogeneous fewnomial isolated singularities is again a weighted homogeneous fewnomial isolated singularity.

REMARK 2.8. It is well known that an isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by an n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. A 3-nomial isolated hypersurface singularity is also called a trinomial singularity.

Ebeling and Takahashi [4] gave the following classification of weighted homogeneous fewnomial singularities in three variables.

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Proposition 2.9 [4]. *Let $f(x_1, x_2, x_3)$ be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then f is analytically equivalent to one of the following five types:*

Type 1.	$x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,	Type 2.	$x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3}$,
Type 3.	$x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_1$,	Type 4.	$x_1^{a_1} + x_2^{a_2} + x_3^{a_3} x_2$,
Type 5.	$x_1^{a_1} x_2 + x_2^{a_2} x_1 + x_3^{a_3}$.		

In the next section we prove that the Types 1–5 satisfy Conjecture 1.1. The following results play an important part in the proof of our main theorem.

Theorem 2.10 [14]. *Let $(V_f, 0) \subset (\mathbb{C}^n, 0)$ and $(V_g, 0) \subset (\mathbb{C}^m, 0)$ be defined by weighted homogeneous polynomials $f(x_1, \ldots, x_n) = 0$ of weight type $(w_1, \ldots, w_n; 1)$ and $g(y_1, \ldots, y_m) = 0$ of weight type $(w_{n+1}, \ldots, w_{n+m}; 1)$, respectively. Let $\mu(V_f)$, $\mu(V_g)$, $A(V_f)$ and $A(V_g)$ be the Milnor numbers and moduli algebras of $(V_f, 0)$ and $(V_g, 0)$, respectively. Then $\lambda(V_{f+g}) = \mu(V_f)\lambda(V_g) + \mu(V_g)\lambda(V_f)$. Further, if both f and g satisfy Conjecture 1.1, then so does $f + g$.*

Proposition 2.11 [14]. *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity of type 1 which is defined by $f = x_1^{a_1} + \cdots + x_n^{a_n}$ $(a_i \geq 3,\ 1 \leq i \leq n)$ with weight type $(a_1^{-1}, \ldots, a_n^{-1}; 1)$. Then the Yau number is*

$$\lambda(V) = n\prod_{i=1}^{n}(a_i - 1) - \sum_{i=1}^{n}(a_1 - 1)(a_2 - 1)\cdots(\widehat{a_i - 1})\cdots(a_n - 1),$$

where $(\widehat{a_i - 1})$ means that $a_i - 1$ is omitted.

3. Proof of the main theorem

In order to prove the main theorem, we need the following propositions.

Proposition 3.1. *Let $(V, 0)$ be a fewnomial isolated singularity of type 2 which is defined by the equation $f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ with weight type $((1 - a_3 + a_2 a_3)/a_1 a_2 a_3, a_3 - 1/a_2 a_3, 1/a_3; 1)$. Then the Yau number of V is*

$$\lambda(V) = \begin{cases} 3a_1a_2a_3 - 2a_1a_3 - 4a_2a_3 + 6a_3 + 2a_1 - 2a_1a_2 + 2a_2 - 7, & a_2 \geq 3, \\ 4a_1a_3 - 3a_3 - 2a_1 - 1, & a_2 = 2. \end{cases}$$

Furthermore,

$$\lambda(V) \leq 3a_1a_2a_3 - 4(a_2a_3 + a_3 - 1) - \frac{(a_1a_2a_3 - 1 + a_3 - a_2a_3)(a_3 - 1)}{1 - a_3 + a_2a_3} - \frac{(a_1a_2a_3 - 1 + a_3 - a_2a_3)}{a_3 - 1}.$$

Proof. It is easy to see that the moduli algebra $A(V) = \mathbb{C}\{x_1, x_2, x_3\}/(f_{x_1}, f_{x_2}, f_{x_3})$ has dimension $(a_1a_2a_3 - 1 + a_3 - a_2a_3)$ and has a monomial basis of the form [1]

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$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0 \le i_1 \le a_1-2; 0 \le i_2 \le a_2-1; 0 \le i_3 \le a_3-1; x_1^{a_1-1}x_3^{i_3}, 0 \le i_3 \le a_3-2\}$, with the following relations:

$$a_1x_1^{a_1-1}x_2 = 0, \quad a_2x_2^{a_2-1}x_3 + x_1^{a_1} = 0, \quad a_3x_3^{a_3-1} + x_2^{a_2} = 0. \tag{3.1}$$

From (3.1),

$$x_1^{a_1-1}x_2 = x_2^{a_2}x_3 = x_3^{a_3} = 0, \tag{3.2}$$

$$x_1^{i_1} = 0,\ i_1 \ge 2a_1-1, \quad x_2^{i_2} = 0,\ i_2 \ge 2a_2, \quad x_3^{i_3} = 0,\ i_3 \ge a_3. \tag{3.3}$$

In order to compute a derivation D of $A(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3, which can be written in terms of the basis. Thus, we can write

$$Dx_j = \sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-1}\sum_{i_3=0}^{a_3-1} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_3=0}^{a_3-2} c^j_{a_1-1,0,i_3}x_1^{a_1-1}x_3^{i_3}, \quad j = 1, 2, 3.$$

Using the relations (3.2)–(3.3), one easily finds the necessary and sufficient conditions defining a derivation of $A(V)$ as follows:

$$a_1c^1_{i_1,0,i_3} = (a_2-1)c^2_{i_1-1,1,i_3} + c^3_{i_1-1,0,i_3+1}, \quad 1 \le i_1 \le a_1-1,\ 1 \le i_3 \le a_3-2; \tag{3.4}$$

$$c^1_{0,i_2,i_3} = 0, \quad 0 \le i_2 \le a_2-3,\ 0 \le i_3 \le a_3-1; \tag{3.5}$$

$$c^1_{0,i_2,0} = 0, \quad a_2-2 \le i_2 \le a_2-1; \tag{3.6}$$

$$c^2_{i_1,0,i_3} = 0, \quad 0 \le i_1 \le a_1-2,\ 0 \le i_3 \le a_3-2; \tag{3.7}$$

$$c^3_{i_1,i_2,0} = 0, \quad 0 \le i_1 \le a_1-2,\ 0 \le i_2 \le a_2-1; \quad c^3_{a_1-1,0,0} = 0; \tag{3.8}$$

$$(a_1-1)c^1_{0,1,i_3} = a_2c^2_{a_1-1,0,i_3-1}, \quad 1 \le i_3 \le a_3-1; \tag{3.9}$$

$$(a_1a_2)c^1_{i_1,0,0} = (a_2-1)((a_3-1)-1)c^3_{i_1-1,0,1}, \quad 1 \le i_1 \le a_1-1; \tag{3.10}$$

$$a_2c^2_{i_1,i_2,0} = (a_3-1)c^3_{i_1,i_2-1,1}, \quad 0 \le i_1 \le a_1-2,\ 1 \le i_2 \le a_2. \tag{3.11}$$

It is easy to see that the number of linearly independent equations is

$$2a_1a_3 + a_2a_3 + 2a_1a_2 - 3a_3 - 2a_1 - 2a_2 + 4.$$

Using (3.4)–(3.11), we obtain the following description of the Lie algebras in question.

The derivations represented by the following vector fields form a basis in $\operatorname{Der} A(V)$:

$$x_2^{i_2}x_3^{i_3}\partial_3, \quad 1 \le i_2 \le a_2-1,\ 2 \le i_3 \le a_3-1; \quad x_1^{a_1-1}x_3^{i_3}\partial_3, \quad 1 \le i_3 \le a_3-2;$$
$$x_2^{i_2}x_3^{i_3}\partial_2, \quad 2 \le i_2 \le a_2-1,\ 1 \le i_3 \le a_3-2; \quad x_2^{i_2}x_3^{a_3-1}\partial_2, \quad 1 \le i_2 \le a_2-1;$$
$$(1-a_3+a_2a_3)x_1^{i_1}\partial_1 + a_1(a_3-1)x_1^{i_1-1}x_2\partial_2 + a_1a_2x_1^{i_1-1}x_3\partial_3, \quad 1 \le i_1 \le a_1-1;$$
$$x_1^{i_1}x_3^{a_3-1}\partial_1, \quad 1 \le i_1 \le a_1-2; \quad x_2^{a_2-1}x_3^{i_3}\partial_1, \quad 1 \le i_3 \le a_3-1;$$
$$(a_3-1)x_1^{i_1}x_2^{i_2}\partial_2 + a_2x_1^{i_1}x_2^{i_2-1}x_3\partial_3, \quad 0 \le i_1 \le a_1-2,\ 2 \le i_2 \le a_2-1;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3, \quad 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-1,\ 2 \le i_3 \le a_3-1;$$

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$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, \quad 1 \le i_1 \le a_1-2, 2 \le i_2 \le a_2-1,\ 1 \le i_3 \le a_3-2;$$
$$(a_2-1)x_1^{i_1}x_3^{i_3}\partial_1 + a_1x_1^{i_1-1}x_2x_3\partial_2, \quad 1 \le i_1 \le a_1-1,\ 1 \le i_3 \le a_3-2;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1, \quad 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-1,\ 1 \le i_3 \le a_3-1;$$
$$x_1^{i_1}x_3^{i_3}\partial_1 + a_1x_1^{i_1-1}x_3^{i_3+1}\partial_3, \quad 1 \le i_1 \le a_1-1,\ 1 \le i_3 \le a_3-2;$$
$$a_3(a_3-1)x_1^{i_1}x_3^{a_3-1}\partial_2 + a_2x_1^{i_1}x_2^{a_2-1}x_3\partial_3, \quad 0 \le i_1 \le a_1-2;$$
$$a_2x_2^{a_2-2}x_3^{i_3}\partial_1 + (a_1-1)x_1^{a_1-1}x_3^{i_3-1}\partial_2, \quad 1 \le i_3 \le a_3-1;$$
$$x_1^{i_1}x_2^{i_2}x_3^{a_3-1}\partial_2, \quad 1 \le i_1 \le a_1-2,\ 1 \le i_2 \le a_2-1;$$
$$x_1^{i_1}x_2^{i_2}\partial_1, \quad 1 \le i_1 \le a_1-2,\ 1 \le i_2 \le a_2-1.$$

Therefore, $\lambda(V) = 3a_1a_2a_3 - 2a_1a_3 - 4a_2a_3 + 6a_3 + 2a_1 - 2a_1a_2 + 2a_2 - 7$. To prove the second part, observe that after simplification the difference between the bound for $\lambda(V)$ stated in the proposition and the value just calculated is

$$a_3(a_1-2) + a_1(a_3-2) + a_2(a_1-1) + a_1a_2 + \frac{a_1a_2(a_3-2)+2}{a_3-1} + \frac{a_1a_3(a_3-1)}{1+a_3(a_2-1)} + 2$$

and this is ≥ 0 because $a_1 \ge 2$, $a_2 \ge 2$ and $a_3 \ge 3$. This gives the required result.

In the case $a_1 \ge 2$, $a_2 = 2$ and $a_3 \ge 3$, the conditions defining a derivation of $A(V)$ are

$$c^1_{0,0,0} = c^1_{0,0,1} = \cdots = c^1_{0,0,a_3-2} = 0, c^1_{0,1,0} = 0; \quad c^3_{0,1,0} = c^3_{1,1,0} = \cdots = c^3_{a_1-2,1,0} = 0;$$
$$c^2_{0,0,0} = c^2_{0,0,1} = \cdots = c^2_{0,0,a_3-2} = 0; \quad c^2_{1,0,0} = c^2_{2,0,0} = \cdots = c^2_{a_1-1,0,0} = 0;$$
$$c^2_{a_1-1,0,1} = c^2_{a_1-1,0,2} = \cdots = c^2_{a_1-1,0,a_3-3} = 0;$$
$$c^3_{0,0,0} = c^3_{1,0,0} = c^3_{2,0,0} = \cdots = c^3_{a_1-1,0,0} = 0;$$
$$a_1c^1_{i_1,0,i_3+1} + a_2c^2_{i_1+a_1-1,0,i_3} = c^3_{i_1-1,0,i_3+2}, \quad 1 \le i_1 \le a_1-1,\ 0 \le i_3 \le a_3-3;$$
$$a_1a_2c^1_{1,0,0} = (a_3+1)c^3_{0,0,1}, a_1a_2c^1_{2,0,0} = (a_3+1)c^3_{1,0,1}, \ldots,$$
$$a_1a_2c^1_{a_1-1,0,0} = (a_3+1)c^3_{a_1-2,0,1};$$
$$(a_1-1)c^1_{0,0,a_3-1} = a_2c^2_{a_1-1,0,a_3-2}; \quad a_2c^2_{i_1,1,0} = a_1c^3_{i_1,0,1}, \quad 0 \le i_1 \le a_1-2;$$
$$c^2_{i_1,0,a_3-1} = a_1a_3c^3_{i_1+a_1,0,0}, \quad 0 \le i_1 \le a_1-2$$

and the following system of equations:

$$c^2_{1,0,1} = c^2_{1,0,2} = \cdots = c^2_{1,0,a_3-2} = 0; \quad c^2_{2,0,1} = c^2_{2,0,2} = \cdots = c^2_{2,0,a_3-2} = 0;$$
$$\vdots$$
$$c^2_{a_1-2,0,1} = c^2_{a_1-2,0,2} = \cdots = c^2_{a_1-2,0,a_3-2} = 0.$$

It is easy to see that the number of linearly independent equations is $2a_1 + 2a_1a_3 - 2$. Using the above conditions, we find that the derivations represented by the following

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vector fields form a basis in $\operatorname{Der} A(V)$:

$$\frac{4(1-a_3+a_2a_3)}{a_1a_2a_3}x_1^{i_1}\partial_1+\frac{4(a_3-1)}{a_2a_3}x_1^{i_1-1}x_2\partial_2+\frac{4}{a_3}x_1^{i_1-1}x_3\partial_3,\quad 1\le i_1\le a_1-1;$$
$$x_1^{i_1}x_2\partial_1,\quad 1\le i_1\le a_1-2;\quad x_1^{i_1}x_3^{a_3-1}\partial_1,\quad 1\le i_1\le a_1-2;$$
$$x_1^{i_1}x_3^{i_3}\partial_1,\quad a_1\le i_1\le 2a_1-2, 0\le i_3\le a_3-2;\quad x_1^{i_1}x_3^{a_3-2}\partial_2,\quad a_1\le i_1\le 2a_1-2;$$
$$x_1^{i_1}x_3^{i_3}\partial_3,\quad a_1-1\le i_1\le 2a_1-2,\ 1\le i_3\le a_3-2;$$
$$\frac{-a_2}{a_1}x_1^{i_1}x_3^{i_3}\partial_1+x_1^{a_3-2+i_1}x_3^{i_3-1}\partial_2,\quad 1\le i_1\le a_1-1,\ 1\le i_3\le a_3-2;$$
$$\frac{1}{a_3-1}x_1^{i_1}x_3^{i_3}\partial_1+x_1^{i_1-1}x_3^{i_3+1}\partial_3,\quad 1\le i_1\le a_1-1,\ 1\le i_3\le a_3-2;$$
$$\frac{a_2}{a_1-1}x_3^{a_3-1}\partial_1+x_1^{a_1-1}x_3^{a_3-2}\partial_2;\quad (a_1a_3)x_1^{i_1}x_3^{a_3-1}\partial_2+x_1^{a_1-1+i_1}\partial_3,\quad 0\le i_1\le a_1-2.$$

Therefore, the Yau number is $\lambda(V)=4a_1a_3-3a_3-2a_1-1$. After simplification, the difference between the bound for $\lambda(V)$ stated in the proposition and the value just calculated is

$$(2a_3-3)+\frac{2a_1(a_3(2a_3-3)-1)}{a_3^2-1}+\frac{2}{a_3-1}\ge 0.$$

This completes the proof. □

Proposition 3.2. *Let $(V,0)$ be a fewnomial isolated singularity of type 3 which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}x_1$ $(a_1\ge 2, a_2\ge 2, a_3\ge 2)$ with weight type*

$$\Big(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3},\frac{1-a_1+a_1a_3}{1+a_1a_2a_3},\frac{1-a_2+a_1a_2}{1+a_1a_2a_3};1\Big).$$

Then the Yau number is

$$\lambda(V)=\begin{cases}12 & \text{if } a_1=2, a_2=2, a_3=2,\\ 3a_1a_2a_3-2(a_1a_2+a_1a_3+a_2a_3)+2(a_1+a_2+a_3)-1 & \text{otherwise.}\end{cases}$$

Furthermore,

$$\lambda(V)\le 3a_1a_2a_3-\frac{a_1a_3(1-a_2+a_1a_2)}{1-a_1+a_1a_3}-\frac{a_1a_2(1-a_3+a_2a_3)}{1-a_2+a_1a_2}-\frac{a_2a_3(1-a_1+a_1a_3)}{1-a_3+a_2a_3}.$$

Proof. It is easy to see that the moduli algebra $A(V)=\mathbb{C}\{x_1,x_2,x_3\}/(f_{x_1},f_{x_2},f_{x_3})$ has dimension $a_1a_2a_3$ and has a monomial basis of the form [1]

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0\le i_1\le a_1-1, 0\le i_2\le a_2-1, 0\le i_3\le a_3-1\}, \tag{3.12}$$

with the following relations:

$$a_2x_2^{a_2-1}x_3+x_1^{a_1}=0,\quad a_3x_3^{a_3-1}x_1+x_2^{a_2}=0,\quad a_1x_1^{a_1-1}x_2+x_3^{a_3}=0. \tag{3.13}$$

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From (3.13),

$$x_1^{a_1}x_2 = x_2^{a_2}x_3 = x_3^{a_3}x_1 = 0, \tag{3.14}$$

$$x_1^{i_1} = 0,\ i_1 \geq 2a_1, \quad x_2^{i_2} = 0,\ i_2 \geq 2a_2, \quad x_3^{i_3} = 0,\ i_3 \geq 2a_3. \tag{3.15}$$

In order to compute a derivation D of $A(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3, which can be written in terms of the basis (3.12). Thus, we can write

$$Dx_j = \sum_{i_1=0}^{a_1-1}\sum_{i_2=0}^{a_2-1}\sum_{i_3=0}^{a_3-1} c^j_{i_1,i_2,i_3} x_1^{i_1}x_2^{i_2}x_3^{i_3}, \quad j = 1,2,3.$$

Using the relations (3.14)–(3.15), one easily finds the necessary and sufficient conditions defining a derivation of $A(V)$ as follows:

$$c^1_{0,i_2,i_3} = 0,\ 1 \leq i_2 \leq a_2-2, 0 \leq i_3 \leq a_3-2; \quad c^1_{0,i_2,a_3-1} = 0,\ 1 \leq i_2 \leq a_2-3; \tag{3.16}$$

$$c^1_{0,a_2-1,0} = 0; \quad c^1_{0,0,i_3} = 0,\ 0 \leq i_3 \leq a_3-1; \quad c^2_{0,0,i_3} = 0,\ 0 \leq i_3 \leq a_3-1; \tag{3.17}$$

$$c^2_{i_1,0,i_3} = 0,\ 1 \leq i_1 \leq a_1-2, 0 \leq i_3 \leq a_3-2; \quad c^2_{a_1-1,0,i_3} = 0,\ 0 \leq i_3 \leq a_3-3; \tag{3.18}$$

$$c^3_{i_1,0,0} = 0,\ 0 \leq i_1 \leq a_1-1; \quad c^3_{i_1,i_2,0} = 0,\ 0 \leq i_1 \leq a_1-3, 1 \leq i_2 \leq a_2-1; \tag{3.19}$$

$$c^3_{a_1-1,i_2,0} = 0,\ 1 \leq i_2 \leq a_2-2; \tag{3.20}$$

$$a_1c^1_{0,a_2-1,i_3} + a_2(a_2-1)c^2_{a_1-1,1,i_3-1} + a_2c^3_{a_1-1,0,i_3} = 0,\ 1 \leq i_3 \leq a_3-1; \tag{3.21}$$

$$a_1a_3c^1_{1,i_2,a_3-1} + a_1c^2_{0,i_2+1,a_3-1} + a_3c^3_{a_1-1,i_2+1,0} = 0,\ 0 \leq i_2 \leq a_2-2; \tag{3.22}$$

$$a_3c^1_{i_1,a_2-1,0} + a_2c^2_{i_1,0,a_3-1} + a_3(a_3-1)c^3_{i_1-1,a_2-1,1} = 0,\ 1 \leq i_1 \leq a_1-1; \tag{3.23}$$

$$a_1(a_1-1)c^1_{1,i_2,i_3} + a_1c^2_{0,i_2+1,i_3} - a_1a_3c^3_{0,i_2,i_3+1} = 0, \quad 1 \leq i_2 \leq a_2-2,\ 1 \leq i_3 \leq a_3-2; \tag{3.24}$$

$$a_3c^1_{i_1,i_2,0} - a_2a_3c^2_{i_1-1,i_2+1,0} + a_3(a_3-1)c^3_{i_1-1,i_2,1} = 0,\ \ 2 \leq i_1 \leq a_1-1,\ 1 \leq i_2 \leq a_2-2; \tag{3.25}$$

$$-a_1a_2c^1_{i_1,0,i_3} + a_2(a_2-1)c^2_{i_1-1,1,i_3} + a_2c^3_{i_1-1,0,i_3+1} = 0,\ \ 2 \leq i_1 \leq a_1-1,\ 1 \leq i_3 \leq a_3-2; \tag{3.26}$$

$$((a_1-1)a_2+1)c^1_{i_1,0,0} - ((a_2-1)a_3+1)c^3_{i_1-1,0,1} = 0, \quad 1 \leq i_1 \leq a_1-1; \tag{3.27}$$

$$((a_1-1)a_2+1)c^1_{1,i_2,0} - ((a_2-1)a_3+1)c^3_{0,i_2,1} = 0, \quad 1 \leq i_2 \leq a_2-2; \tag{3.28}$$

$$((a_1-1)a_2+1)c^1_{1,0,i_3} - ((a_2-1)a_3+1)c^3_{0,0,i_3+1} = 0, \quad 1 \leq i_3 \leq a_3-2; \tag{3.29}$$

$$((a_1-1)a_2+1)c^2_{i_1,1,0} - ((a_3-1)a_1+1)c^3_{i_1,0,1} = 0, \quad 0 \leq i_1 \leq a_1-2; \tag{3.30}$$

$$((a_1-1)a_2+1)c^2_{0,i_2,0} - ((a_3-1)a_1+1)c^3_{0,i_2-1,1} = 0, \quad 2 \leq i_2 \leq a_2-1; \tag{3.31}$$

$$((a_1-1)a_2+1)c^2_{0,1,i_3} - ((a_3-1)a_1+1)c^3_{0,0,i_3+1} = 0, \quad 1 \leq i_3 \leq a_3-2; \tag{3.32}$$

$$a_1c^1_{0,a_2-2,a_3-1} - a_3c^3_{a_1-2,a_2-1,0} = 0; \quad a_2c^2_{a_1-1,0,a_3-2} - a_3c^3_{a_1-2,a_2-1,0} = 0. \tag{3.33}$$

It is easy to see that the number of linearly independent equations is

$$2(a_1a_3 + a_2a_3 + a_1a_2) - 2(a_1 + a_2 + a_3) + 1.$$

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Using (3.16)–(3.33), we see that the derivations represented by the following vector fields form a basis in $\operatorname{Der} A(V)$:

$$
\begin{gathered}
(1-a_3+a_2a_3)x_1x_3^{i_3}\partial_1+(1-a_1+a_1a_3)x_2x_3^{i_3+1}\partial_2+(1-a_2+a_1a_2)x_3^{i_3+1}\partial_3,\\
\hfill 1\le i_3\le a_3-2;\\
(1-a_3+a_2a_3)x_1^{i_1}\partial_1+(1-a_1+a_2a_3)x_1^{i_1-1}\partial_2+(1-a_2+a_1a_2)x_1^{i_1-1}x_3\partial_3,\\
\hfill 1\le i_1\le a_1-1;\\
(1-a_3+a_2a_3)x_1x_2^{i_2}\partial_1+(1-a_1+a_2a_3)x_2^{i_2+1}\partial_2+(1-a_2+a_1a_2)x_2^{i_2}x_3\partial_3,\\
\hfill 1\le i_2\le a_2-2;\\
a_2a_3x_2^{a_2-2}x_3^{a_3-1}\partial_1+a_1a_3x_1^{a_1-1}x_3^{a_3-2}\partial_2+a_1a_2x_1^{a_1-2}x_2^{a_2-1}\partial_3;\\
x_1^{i_1}x_3^{a_3-1}\partial_1,\quad 2\le i_1\le a_1-1;\quad x_1x_2^{a_2-1}x_3^{i_3}\partial_1,\quad 1\le i_3\le a_3-1;\\
x_1x_2^{i_2}x_3^{i_3}\partial_3,\quad 2\le i_2\le a_2-1,\ 2\le i_3\le a_3-1;\\
x_1x_2x_3^{i_3}\partial_3,\quad 2\le i_3\le a_3-1;\quad x_2^{a_2-1}x_3^{i_3}\partial_3,\quad 2\le i_3\le a_3-1;\\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\quad 2\le i_1\le a_1-1,\ 2\le i_2\le a_2-1,\ 1\le i_3\le a_3-1;\\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\quad 2\le i_1\le a_1-1,\ 1\le i_2\le a_2-1,\ 2\le i_3\le a_3-1;\\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\quad 2\le i_1\le a_1-1,\ 1\le i_2\le a_2-1,\ 1\le i_3\le a_3-1;\\
(a_2-1)x_1^{i_1}x_3^{i_3}\partial_1+a_1x_1^{i_1-1}x_2x_3^{i_3}\partial_2,\quad 2\le i_1\le a_1-1,\ 1\le i_3\le a_3-2;\\
(a_1-1)x_1x_2^{i_2}x_3^{i_3}\partial_1+a_3x_2^{i_2}x_3^{i_3+1}\partial_3,\quad 1\le i_2\le a_2-2,\ 1\le i_3\le a_3-2;\\
a_2x_1^{i_1}x_2^{i_2}\partial_1+x_1^{i_1-1}x_2^{i_2+1}\partial_2,\quad 2\le i_1\le a_1-1,\ 1\le i_2\le a_2-2;\\
a_2(a_2-1)x_2^{a_2-1}x_3^{i_3}\partial_1+a_1x_1^{a_1-1}x_2x_3^{i_3-1}\partial_2,\quad 1\le i_3\le a_3-1;\\
(a_3-1)x_1^{i_1}x_2^{i_2}\partial_1+x_1^{i_1-1}x_2^{i_2}\partial_3,\quad 2\le i_1\le a_1-1,\ 1\le i_2\le a_2-1;\\
x_1x_2^{i_2}x_3^{i_3}\partial_1+(a_1-1)x_2^{i_2+1}x_3^{i_3}\partial_2,\quad 1\le i_2\le a_2-2,\ 1\le i_3\le a_3-1;\\
x_1^{i_1}x_3^{i_3}\partial_1+a_1x_1^{i_1-1}x_3^{i_3+1}\partial_3,\quad 2\le i_1\le a_1-1,\ 1\le i_3\le a_3-2;\\
x_1x_2^{i_2}x_3^{a_3-1}\partial_1+(a_1-1)x_1^{a_1-1}x_2^{i_2+1}\partial_3,\quad 1\le i_2\le a_2-2;\\
x_1x_2x_3^{a_3-1}\partial_2, x_1^{a_1-1}x_2^{i_2}x_3\partial_3,\quad 1\le i_2\le a_2-1;\\
a_2x_1^{i_1}x_2^{a_2-1}\partial_1+a_3x_1^{i_1}x_3^{a_3-1}\partial_2,\quad 1\le i_1\le a_1-1;\quad a_1x_2^{a_2-1}x_3^{i_3}\partial_1+a_2x_1^{a_1-1}x_3^{i_3}\partial_3,\\
\hfill 1\le i_3\le a_3-1;\\
x_1x_2^{i_2}x_3^{i_3}\partial_2,\quad 2\le i_2\le a_2-1,\ 1\le i_3\le a_3-1;\quad x_1^{i_1}x_2x_3^{a_3-1}\partial_2,\ 2\le i_1\le a_1-1;\\
x_1x_3^{a_3-1}\partial_1+(a_1-1)x_2x_3^{a_3-1}\partial_2;\quad x_1x_3^{a_3-1}\partial_1+(a_1-1)x_1^{a_1-1}x_2\partial_3;\\
(a_3-1)x_1x_2^{a_2-1}\partial_1+x_2^{a_2-1}x_3\partial_3;\quad x_1^{a_1-1}x_2^{i_2}\partial_2,\quad 2\le i_2\le a_2-1.
\end{gathered}
$$

Therefore, $\lambda(V)=3a_1a_2a_3+2(a_1+a_2+a_3)-2(a_1a_2+a_1a_3+a_2a_3)-1$. To prove the second part, observe that after simplification the difference between the bound for $\lambda(V)$

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stated in the proposition and the value just calculated is

$$1+\frac{a_1a_2(a_1-2)(a_3-2)+\frac{a_1a_3(a_2-2)}{2}+\frac{a_2(a_1a_3-4)}{2}}{1-a_1+a_1a_3}$$
$$+\frac{a_2a_3(a_2-2)(a_1-2)+\frac{a_1a_2(a_3-2)}{2}+\frac{a_3(a_1a_2-4)}{2}}{1-a_2+a_1a_2}$$
$$+\frac{a_1a_3(a_2-2)(a_3-2)+\frac{a_2a_3(a_1-2)}{2}+\frac{a_1(a_2a_3-4)}{2}}{1-a_3+a_2a_3}$$

and this is ≥ 0 since $a_1 \geq 2$, $a_2 \geq 2$ and $a_3 \geq 2$.

In the case $a_1 = 2, a_2 = 2, a_3 = 2$, we have the following basis of $\mathrm{Der}A(V)$:

$$x_1\partial_1 + x_2\partial_2 + x_3\partial_3; \quad x_2\partial_1 + x_3\partial_2 + x_1\partial_3; \quad x_3\partial_1 + x_1\partial_2 + x_2\partial_3;$$
$$\frac{x_2^2\partial_1}{2} + x_2x_3\partial_2; \quad -x_2^2\partial_1 + x_3^2\partial_3; \quad 2x_2x_3\partial_1 + x_3^2\partial_2; \quad 2x_2x_3\partial_1 + x_2^2\partial_3;$$
$$-x_3^2\partial_1 + x_2^2\partial_2; \quad \frac{x_3^2\partial_1}{2} + x_2x_3\partial_3; \quad x_3^3\partial_1; \quad x_3^3\partial_2; \quad x_3^3\partial_3.$$

Therefore, $\lambda(V) = 12$. It follows that in the case of $a_1 = 2, a_2 = 2, a_3 = 2$, equality holds in Conjecture 1.1. □

Proof of the Main Theorem. Let $f \in \mathbb{C}\{x_1, x_2, x_3\}$ be a weighted homogeneous fewnomial isolated surface singularity. Then f is analytically equivalent to one of the five types in Proposition 2.9. It is an immediate corollary of Propositions 2.11, 3.1 and 3.2 that types 1, 2 and 3 satisfy Conjecture 1.1. The last two types are Thom–Sebastiani summations of types 1, 2 and 3. It follows from Theorem 2.10 that types 4 and 5 also satisfy Conjecture 1.1. Therefore, the main theorem is proved. □

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