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RESEARCH ARTICLE

Open-closed Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric DM stacks

Bohan Fang[1], Chiu-Chu Melissa Liu[2] and Hsian-Hua Tseng[3]

[1]Beijing International Center for Mathematical Research, Peking University, 5 Yiheyuan Road, Beijing 100871, China;
E-mail: bohanfang@gmail.com.
[2]Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027, USA;
E-mail: ccliu@math.columbia.edu.
[3]Department of Mathematics, Ohio State University, 100 Math Tower, 231 West 18th Avenue, Columbus, OH 43210;
E-mail: hhtseng@math.ohio-state.edu.

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Abstract
We study open-closed orbifold Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric Deligne-Mumford stacks (with possibly nontrivial generic stabilisers K and semi-projective coarse moduli spaces) relative to Lagrangian branes of Aganagic-Vafa type. An Aganagic-Vafa brane in this paper is a possibly ineffective C^∞ orbifold that admits a presentation $[(S^1 \times \mathbb{R}^2)/G_\tau]$, where G_τ is a finite abelian group containing K and $G_\tau/K \cong \boldsymbol{\mu}_{\mathfrak{m}}$ is cyclic of some order $\mathfrak{m} \in \mathbb{Z}_{>0}$.

1. We present foundational materials of enumerative geometry of stable holomorphic maps from bordered orbifold Riemann surfaces to a 3-dimensional Calabi-Yau smooth toric DM stack $\mathcal{X}$ with boundaries mapped into an Aganagic-Vafa brane $\mathcal{L}$. All genus open-closed Gromov-Witten invariants of $\mathcal{X}$ relative to $\mathcal{L}$ are defined by torus localisation and depend on the choice of a framing $f \in \mathbb{Z}$ of $\mathcal{L}$.
2. We provide another definition of all genus open-closed Gromov-Witten invariants in (1) based on algebraic relative orbifold Gromov-Witten theory, which agrees with the definition in (1) up to a sign depending on the choice of orientation on moduli of maps in (1). This generalises the definition in [57] for smooth toric Calabi-Yau 3-folds and specifies an orientation on moduli of maps in (1) compatible with the canonical orientation on moduli of relative stable maps determined by the complex structure.
3. When $\mathcal{X}$ is a toric Calabi-Yau 3-orbifold (i.e., when the generic stabiliser K is trivial), so that $G_\tau = \boldsymbol{\mu}_{\mathfrak{m}}$, we define generating functions $F^{\mathcal{X},(\mathcal{L},f)}_{g,h}$ of open-closed Gromov-Witten invariants of arbitrary genus g and number h of boundary circles; it takes values in $H^*_{\mathrm{CR}}(\mathcal{B}\boldsymbol{\mu}_{\mathfrak{m}};\mathbb{C})^{\otimes h}$, where $H^*_{\mathrm{CR}}(\mathcal{B}\boldsymbol{\mu}_{\mathfrak{m}};\mathbb{C}) \cong \mathbb{C}^{\mathfrak{m}}$ is the Chen-Ruan orbifold cohomology of the classifying space $\mathcal{B}\boldsymbol{\mu}_{\mathfrak{m}}$ of $\boldsymbol{\mu}_{\mathfrak{m}}$.
4. We prove an open mirror theorem that relates the generating function $F^{\mathcal{X},(\mathcal{L},f)}_{0,1}$ of orbifold disk invariants to Abel-Jacobi maps of the mirror curve of $\mathcal{X}$. This generalises a conjecture by Aganagic-Vafa [6] and Aganagic-Klemm-Vafa [5] (proved in full generality by the first and the second authors in [33]) on the disk potential of a smooth semi-projective toric Calabi-Yau 3-fold.

Contents

1. Introduction

Open Gromov-Witten (GW) invariants of toric Calabi-Yau 3-folds have been studied extensively by both mathematicians and physicists. They correspond to 'A-model topological open string amplitudes' in the physics literature and can be interpreted as intersection numbers of certain moduli spaces of holomorphic maps from bordered Riemann surfaces to the 3-fold with boundaries in a Lagrangian submanifold. The physics prediction of these open GW invariants comes from string dualities: *mirror symmetry* relates the A-model topological string theory of a Calabi-Yau 3-fold X to the B-model topological string theory of the mirror Calabi-Yau 3-fold $\check{X}$; the *large- N duality* relates the A-model topological string theory on Calabi-Yau 3-folds (of complex dimension three) to the Chern-Simons theory on 3-manifolds (of real dimension three).

1.1. Open GW invariants of smooth toric Calabi-Yau 3-folds

Aganagic-Vafa [6] introduce a class of Lagrangian submanifolds in smooth semi-projective toric Calabi-Yau 3-folds, which are diffeomorphic to $S^1 \times \mathbb{R}^2$. By mirror symmetry, Aganagic-Vafa and Aganagic-Klemm-Vafa [6, 5] relate genus-zero open GW invariants (disk invariants) of a smooth toric Calabi-Yau 3-fold X relative to such a Lagrangian submanifold L to the classical Abel-Jacobi map of the mirror Calabi-Yau 3-fold $\check{X}$, which can be further related to the Abel-Jacobi map to the mirror curve of X. This conjecture is proved in full generality in [33].

By the large-N duality, Aganagic-Klemm-Mariño-Vafa propose the topological vertex [4], an algorithm of computing all genera generating functions $F_{\beta',\mu_1,\dots,\mu_h}$ of open GW invariants of (X, L) obtained by fixing a topological type of the map (determined by the degree $\beta' \in H_2(X, L; \mathbb{Z})$ and winding numbers $\mu_1, \dots, \mu_h \in H_1(L; \mathbb{Z}) = \mathbb{Z}$) and summing over the genus of the domain. The algorithm of the topological vertex is proved in full generality in [60].

Bouchard-Klemm-Mariño-Pasquetti propose the Remodeling Conjecture [8], an algorithm for constructing the B-model topological open string amplitudes in all genera of $\check{X}$ following [58], using Eynard-Orantin's topological recursion from the theory of matrix models [30]. Combined with the mirror symmetry prediction, this gives an algorithm for computing generating functions $F_{g,h}$ of open GW invariants of (X, L) obtained by fixing a topological type of the domain (determined by the genus g and number h of boundary circles) and summing over the topological types of the map. Eynard-Orantin studied the Remodeling Conjecture for any smooth symplectic toric Calabi-Yau threefolds [31].

1.2. Open GW invariants for 3-dimensional Calabi-Yau smooth toric DM stacks

There have been attempts to generalise some of the above results to 3-dimensional Calabi-Yau smooth toric Deligne-Mumford (DM) stacks. The closed GW theory of orbifolds has been studied for a long time. The physics literature dates back to the early 1990s, such as [17, 74], which studies the quantum cohomology ring of orbifolds. The mathematical definition is given by Chen-Ruan [22] for symplectic orbifolds and by Abramovich-Graber-Vistoli [2, 3] for smooth DM stacks. Toric varieties are defined by a fan, while smooth toric DM stacks are defined by a stacky fan [7]. The coarse moduli of a smooth toric DM stack $\mathcal{X}$ is a toric variety X_Σ defined by a simplicial fan Σ. A toric orbifold is a smooth toric DM stack with a trivial generic stabiliser. Any smooth toric DM stack $\mathcal{X}$ is a K-gerbe over its rigidification $\mathcal{X}^{\mathrm{rig}}$, where K is the generic stabiliser (which is a finite abelian group) and $\mathcal{X}^{\mathrm{rig}}$ is a toric orbifold.

The definition of Aganagic-Vafa branes can be extended to the setting of 3-dimensional Calabi-Yau smooth toric DM stacks with semi-projective coarse moduli spaces. These branes are diffeomorphic to $[(S^1 \times \mathbb{R}^2)/G_\tau]$, where G_τ is a finite abelian group containing the generic stabiliser K. The open GW invariants of 3-dimensional Calabi-Yau smooth toric DM stacks are defined via localisation [66],

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generalising the methods in [49]. By localisation, open and closed GW invariants of a smooth toric Calabi-Yau 3-fold can be obtained by glueing the GW vertex, a generating function of open GW invariants of $\mathbb{C}^3$, which can be reduced to a generating function of certain cubic Hodge integrals [29]. Similarly, open and closed orbifold GW invariants of a 3-dimensional Calabi-Yau smooth toric DM stack can be obtained by glueing the orbifold GW vertex, a generating function of open GW invariants of $[\mathbb{C}^3/G]$ (where G is a finite abelian subgroup of $SL(3,\mathbb{C})$), which can be reduced to a generating function of certain cubic abelian Hurwitz-Hodge integrals [66]. The GW vertex has been evaluated in the general case [57, 60]. The orbifold GW vertex has been evaluated for $[\mathbb{C}^2/\mathbb{Z}_n]\times\mathbb{C}$, where $\mathbb{C}^2/\mathbb{Z}_n$ is the A_{n-1} surface singularity [77, 68, 69, 67], but not in the general case.

As for mirror symmetry, a mirror theorem for disk invariants of $[\mathbb{C}^3/\mathbb{Z}_4]$ is proved in [11]. The Remodeling Conjecture is also expected to predict higher genus open GW invariants of toric Calabi-Yau 3-orbifolds via mirror symmetry [8, 9].

1.3. Summary of results

In this paper, we study open-closed orbifold GW invariants of a 3-dimensional Calabi-Yau smooth toric DM stack $\mathcal{X}$ relative to an Aganagic-Vafa A-brane $\mathcal{L}$ at all genera.

Open GW invariants of the pair $(\mathcal{X},\mathcal{L})$ count holomorphic maps from orbicurves to $\mathcal{X}$ with boundaries mapped to $\mathcal{L}$. Open-closed orbifold GW invariants of the pair $(\mathcal{X},\mathcal{L})$ depend on the following discrete data:

(i) topological type (g,h) of the domain orbicurve $(\Sigma,\partial\Sigma)$, where g is the genus and h is the number of boundary holes;
(ii) number of interior marked points n;
(iii) topological type of the map $u:(\Sigma,\partial\Sigma=\coprod_{i=1}^h R_i)\to(\mathcal{X},\mathcal{L})$ given by the degree $\beta'=u_*[\Sigma]\in H_2(\mathcal{X},\mathcal{L};\mathbb{Z})$ and each $[u_*(R_i)]\in H_1(\mathcal{L};\mathbb{Z})\cong\mathbb{Z}\times G_\tau$, collectively denoted by $\vec{\mu}=([u_*(R_1)],\dots,[u_*(R_h)])$;
(iv) framing $f\in\mathbb{Z}$ of the Aganagic-Vafa A-brane $\mathcal{L}$.

Let $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X},\mathcal{L}\mid\beta',\vec{\mu})$ be the moduli space parametrising holomorphic maps with discrete data (i)-(iii). We use the evaluation maps ev_i, $i=1,\dots,n$ at interior points to pull back classes in the orbifold Chen-Ruan cohomology $H^*_{\mathrm{CR}}(\mathcal{X})$ of $\mathcal{X}$ to obtain open-closed GW invariants. More precisely, $\mathcal{L}$ intersects a unique 1-dimensional orbit $\mathfrak{o}_\tau\cong\mathbb{C}^*\times\mathcal{B}G_\tau$. Given $\gamma_1,\dots,\gamma_n\in H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$, we define open-closed orbifold GW invariant $\langle\gamma_1,\dots,\gamma_n\rangle^{\mathcal{X},(\mathcal{L},f)}_{g,\beta',\vec{\mu}}$ via localisation using a circle action determined by the framing $f\in\mathbb{Z}$; this is a rational number depending on f and can be viewed as an equivariant invariant. We also provide another definition based on algebraic relative Gromov-Witten theory, which agrees with the above definition up to a sign depending on the choice of orientation on $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X},\mathcal{L}\mid\beta',\vec{\mu})$. This generalises the definition in [57] for smooth toric Calabi-Yau 3-folds and specifies an orientation on $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X},\mathcal{L}\mid\beta',\vec{\mu})$ compatible with the canonical orientation on moduli of relative stable maps determined by the complex structure.

When $\mathcal{X}$ is a symplectic toric Calabi-Yau 3-orbifolds (i.e., when the generic stabiliser K is trivial), $G_\tau\cong\boldsymbol{\mu}_{\mathfrak{m}}$ is cyclic. In this case, for each topological type (g,h) of the domain bordered Riemann surface, we define a generating function $F^{\mathcal{X},(\mathcal{L},f)}_{g,h}$ of open-closed GW invariants that takes value in $H^*_{\mathrm{CR}}(\mathcal{B}G;\mathbb{C})^{\otimes h}$, where $H^*_{\mathrm{CR}}(\mathcal{B}G;\mathbb{C})=\oplus_{\lambda\in G}\mathbb{C}\mathbf{1}_\lambda$.

In particular, the disk potential $F^{\mathcal{X},(\mathcal{L},f)}_{0,1}$ takes values in $H^*_{\mathrm{CR}}(\mathcal{B}G;\mathbb{C})$. When $\mathcal{L}$ is an outer brane,[1] the A-model disk potential is

$$F^{\mathcal{X},(\mathcal{L},f)}_{0,1}(\boldsymbol{\tau}_2,X)=\sum_{\beta',n\geq 0}\ \sum_{(\mu,\lambda)\in H_1(\mathcal{L};\mathbb{Z})\cong\mathbb{Z}\times G_\tau}\frac{\langle(\boldsymbol{\tau}_2)^n\rangle^{\mathcal{X},(\mathcal{L},f)}_{0,\beta',(\mu,\lambda)}}{n!}\cdot X^\mu(\xi_0)^{\bar{\lambda}}\mathbf{1}_{\lambda^{-1}}$$

[1]We work with both inner and outer branes. See Section 3.3 for the definition.

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where $\boldsymbol{\tau}_2$ is certain equivariant second Chen-Ruan cohomology class of $\mathcal{X}$, ξ_0 is an $\mathfrak{m}$th root of -1, and $\bar{\lambda} \in \{0, 1, \ldots, \mathfrak{m}-1\}$ corresponds to $\lambda \in G$ under a group isomorphism $G_\tau \cong \mathbb{Z}/\mathfrak{m}\mathbb{Z}$. The precise definition of $\boldsymbol{\tau}_2$ and $F_{(0,1)}^{\mathcal{X},(\mathcal{L},f)}$ will be given in Section 3.13. (Throughout the paper, β' denotes a relative homology class in $H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z})$, whereas β denotes an absolute homology class in $H_2(\mathcal{X}; \mathbb{Z})$.) In this paper, we prove a mirror theorem regarding the disk potential $F_{0,1}^{\mathcal{X},(\mathcal{L},f)}$ when $\mathcal{X}$ is a semi-projective toric Calabi-Yau 3-orbifold. Mirror symmetry relates the A-model topological string theory on a Calabi-Yau 3-fold to the B-model topological string theory on the mirror Calabi-Yau 3-fold. The mirror of a semi-projective toric Calabi-Yau 3-fold is a Calabi-Yau hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^2$ defined by an equation $uv = H(x, y, q)$, where $(u, v) \in \mathbb{C}^2$, $(x, y) \in (\mathbb{C}^*)^2$ and q is the complex moduli parametrising the B-model. The function $H(x, y, q)$ is determined by both the combinatorial toric data of $\mathcal{X}$ and the framed brane $(\mathcal{L}, f)$. The affine curve $C_q = \{H(x, y, q) = 0\}$ in $(\mathbb{C}^*)^2$ is called the *mirror curve*. We can fix a labelling of the $\mathfrak{m}$ points with $x = 0$ on the mirror curve by the elements in $G_\tau^* = \mathrm{Hom}(G, \mathbb{C}^*) \cong \mathbb{Z}/\mathfrak{m}\mathbb{Z}$. For each $\eta \in G_\tau^*$, there is a small open neighbourhood U_η^ϵ in the (compactified) mirror curve of the $x = 0$ point labelled by η and a branch $(\log y)_{U_\eta^\epsilon}$ of $\log y$ defined on U_η^ϵ, where $y = y(x, q)$ is defined implicitly by the equation $H(x, y, q) = 0$. When $\mathcal{L}$ is an outer brane, the closure of the 1-dimensional orbit intersecting $\mathcal{L}$ contains a unique torus fixed (stacky) point $\mathfrak{p}_\sigma = \mathcal{B}G_\sigma$, where G_σ is the inertia group of $\mathfrak{p}_\sigma$. With the above convention, we state our open mirror theorem as follows.

Theorem 1.1. *Under the closed mirror map $\boldsymbol{\tau}_2 = \boldsymbol{\tau}_2(q)$ and the open mirror map $X = xe^{A(q)}$ (the explicit formula of $\boldsymbol{\tau}_2(q)$ and $A(q)$ will be given in Section 4),*

$$x\frac{\partial}{\partial x}\left(\sum_{\eta \in \mu_{\mathfrak{m}}^*} (\log y)_{U_\eta^\epsilon}(q, x)\phi_\eta\right) = \frac{\partial^2}{\partial x^2} F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, X),$$

where $\{\phi_\eta\}_{\eta \in G}$ is the canonical basis of $H_{\mathrm{CR}}^*(\mathcal{B}G_\tau; \mathbb{C})$.

Remark 1.2. The definition of the disk function $F_{0,1}^{\mathcal{X},(\mathcal{L},f)}$ and the formulation of the above Theorem 1.1 are slightly different from those in the first version of this paper in 2012 [34], but the above Theorem 1.1 implies [34, Theorem 1.1], which is used to prove an open version of Ruan's Crepant Resolution Conjecture for disk invariants of toric Calabi-Yau 3-orbifold relative to an effective outer Aganagic-Vafa brane [50].

1.4. Similar results for compact Lagrangian tori

There are other open GW invariants relative to different types of Lagrangian submanifolds. C.-H. Cho [15] and J. Solomon [70] define disk invariants of a compact symplectic manifold in real dimensions four and six relative to a Lagrangian submanifold that is the fixed locus of an anti-symplectic involution. The mirror theorem for disk invariants for the quintic 3-fold relative to the real quintic is conjectured in [72] and proved in [63]. It has been generalised to compact Calabi-Yau 3-folds that are projective complete intersections [64], where a mirror theorem for genus one open GW invariants (annulus invariants) is also proved.

Open orbifold GW invariants of compact toric orbifolds with respect to Lagrangian torus fibres of the moment map are defined in [24], which generalises the work of [38] on compact toric manifolds. The third author and collaborators prove mirror theorems on disk invariants in this context [20, 18]. The third author and collaborators also prove mirror theorems on disk invariants of toric Calabi-Yau manifolds/orbifolds (which must be noncompact) with respect to Lagrangian torus fibres of the Gross fibration [21, 19].

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1.5. Applications

The main theorem (Theorem 1.1) of this paper has several applications. Here we mention two important applications:

- As mentioned in Remark 1.2 above, Theorem 1.1 has been applied to prove an open version of Ruan's Crepant Resolution Conjecture for disk invariants of a toric Calabi-Yau 3-orbifold relative to an effective outer Aganagic-Vafa brane [50]. Using Theorem 1.1, S. Yu [73] proves an open version of Crepant Transformation Conjecture (and in particular the Crepant Resolution Conjecture) for disk invariants of a semi-projective toric Calabi-Yau 3-orbifold relative to a general (effective or ineffective, inner or outer) Aganagic-Vafa brane defined in Section 3.3 of this paper. This generalises the Open Crepant Resolution Conjecture (OCRC) for disk invariants of $[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$ relative to possibly ineffective Aganagic-Vafa branes proved in [12].
- Recently, the first two authors and Zong prove the BKMP Remodeling Conjecture for all semi-projective toric Calabi-Yau 3-orbifolds [36]. Theorem 1.1 is one of the key ingredients of this proof.

1.6. Overview of the paper

The rest of the paper is organised as follows. In Section 2, we review the necessary materials concerning smooth toric DM stacks. In Section 3, we apply localisation to relate open-closed GW invariants and descendant GW invariants of 3-dimensional smooth Calabi-Yau toric DM stacks. In Section 4, we prove a mirror theorem for orbifold disk invariants.

2. Smooth toric DM stacks

In this section, we follow the definitions in [47, Section 3.1], with slightly different notation. We work over $\mathbb{C}$.

2.1. Definition

Let N be a finitely generated abelian group, and let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We have a short exact sequence of (additive) abelian groups:

$$0 \longrightarrow N_{\mathrm{tor}} \longrightarrow N \longrightarrow \bar{N} = N/N_{\mathrm{tor}} \longrightarrow 0,$$

where N_{tor} is the subgroup of torsion elements in N. Then N_{tor} is a finite abelian group and $\bar{N} = \mathbb{Z}^n$, where $n = \dim_{\mathbb{R}} N_{\mathbb{R}}$. The natural projection $N \longrightarrow \bar{N}$ is denoted $b \mapsto \bar{b}$. A *smooth toric DM stack* is an extension of toric varieties [39, 7]. A smooth toric DM stack is given by the following data:

- $b_1, \dots, b_{r'} \in N$ that generate a subgroup of N of finite index, and
- a simplicial fan Σ in $N_{\mathbb{R}}$ such that the set of 1-cones is

  $$\{\rho_1, \dots, \rho_{r'}\},$$

  where $\rho_i = \mathbb{R}_{\geq 0}\bar{b}_i$, $i = 1, \dots, r'$.

The datum $\mathbf{\Sigma} = (\Sigma, (b_1, \dots, b_{r'}))$ is a *stacky fan* in the sense of [7]. The vectors $b_1, \dots, b_{r'}$ may or may not generate N; if they do not, we choose *additional* vectors $b_{r'+1}, \dots, b_r$ such that $b_1, \dots, b_r$ generate N. There is a surjective group homomorphism

$$\begin{aligned} \phi : \widetilde{N} := \oplus_{i=1}^{r} \mathbb{Z}\tilde{b}_i &\longrightarrow N, \\ \tilde{b}_i &\mapsto b_i. \end{aligned}$$

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Define $\mathbb{L} := \mathrm{Ker}(\phi) \cong \mathbb{Z}^k$, where $k := r - n$. Then we have the following short exact sequence of finitely generated abelian groups:

$$0 \to \mathbb{L} \xrightarrow{\psi} \widetilde{N} \xrightarrow{\phi} N \to 0. \tag{1}$$

Applying $- \otimes_{\mathbb{Z}} \mathbb{C}^*$ to equation (1), we obtain an exact sequence of abelian groups

$$1 \to K \to G \to \widetilde{\mathbb{T}} \to \mathbb{T} \to 1, \tag{2}$$

where

$$\begin{aligned}
&\mathbb{T} := N \otimes_{\mathbb{Z}} \mathbb{C}^* = \bar{N} \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n,\\
&\widetilde{\mathbb{T}} := \widetilde{N} \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^r,\\
&G := \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^k,\\
&K := \mathrm{Tor}_1^{\mathbb{Z}}(N, \mathbb{C}^*) \cong N_{\mathrm{tor}}.
\end{aligned}$$

The action of $\widetilde{\mathbb{T}}$ on itself extends to a $\widetilde{\mathbb{T}}$-action on $\mathbb{C}^r = \mathrm{Spec}\mathbb{C}[Z_1, \ldots, Z_r]$. The torus G acts on $\mathbb{C}^r$ via the group homomorphism $G \to \widetilde{\mathbb{T}}$ in equation (2), so $K \subset G$ acts on $\mathbb{C}^r$ trivially. The isomorphism $K \cong N_{\mathrm{tor}}$ is not canonical.

With the above preparation, we are now ready to define a smooth toric DM stack $\mathcal{X}$. Let

$$\mathcal{A} = \{I \subset \{1, \ldots, r\} : \sum_{i \notin I} \mathbb{R}_{\geq 0} \bar{b}_i \text{ is a cone of } \Sigma\}$$

be the set of anti-cones; note that $\{r'+1, \ldots, r\} \subset I$ for any anti-cone $I \subset \mathcal{A}$. Given $I \subset \{1, \ldots, r\}$, let $\mathbb{C}^I$ be the subvariety of $\mathbb{C}^r$ defined by the ideal in $\mathbb{C}[Z_1, \ldots, Z_r]$ generated by $\{Z_i \mid i \notin I\}$. Define the smooth toric DM stack $\mathcal{X}$ as the quotient stack

$$\mathcal{X} := [U_{\mathcal{A}}/G],$$

where

$$U_{\mathcal{A}} := \mathbb{C}^r \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I = \bigcap_{I \notin \mathcal{A}} \left(\mathbb{C}^r \setminus \mathbb{C}^I\right).$$

Note that for $i = r'+1, \ldots, r$, $\mathbb{R}_{\geq 0} b_i$ is not a cone in Σ, so $\{i\}' := \{1, \ldots, r\} \setminus \{i\} \notin \mathcal{A}$. Therefore,

$$U_{\mathcal{A}} \subset \bigcap_{i=r'+1}^{r} \left(\mathbb{C}^r \setminus \mathbb{C}^{\{i\}'}\right) = \mathbb{C}^{r'} \times (\mathbb{C}^*)^{r-r'}.$$

The stack $\mathcal{X}$ contains the DM torus $\mathcal{T} := [\widetilde{\mathbb{T}}/G]$ as a dense open subset, and the $\widetilde{\mathbb{T}}$-action on $U_{\mathcal{A}}$ descends to a $\mathcal{T}$-action on $\mathcal{X}$. The smooth toric DM stack $\mathcal{X}$ is a *toric orbifold* if the G-action on $\widetilde{\mathbb{T}}$ is free.

Let $G^{\mathrm{rig}} = G/K$. Then G^{rig} acts freely on $\widetilde{\mathbb{T}}$ and $\widetilde{\mathbb{T}}/G^{\mathrm{rig}} = \mathbb{T}$. The *rigidification* of the smooth toric DM stack $\mathcal{X}$ is the toric orbifold

$$\mathcal{X}^{\mathrm{rig}} = [U_{\mathcal{A}}/G^{\mathrm{rig}}].$$

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The coarse moduli space of the stack $\mathcal{X}$ is the simplical toric variety X_Σ defined by the simplicial fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$. By [37, Theorem I], the morphism $\mathcal{X} \to X_\Sigma$ factors canonically via toric morphisms

$$\mathcal{X} \longrightarrow \mathcal{X}^{\mathrm{rig}} \longrightarrow \mathcal{X}^{\mathrm{can}} \longrightarrow X_\Sigma, \tag{3}$$

where

- $\mathcal{X} \longrightarrow \mathcal{X}^{\mathrm{rig}}$ is a K-gerbe over $\mathcal{X}^{\mathrm{rig}}$;
- $\mathcal{X}^{\mathrm{rig}} \longrightarrow \mathcal{X}^{\mathrm{can}}$ is a fibred product of roots of toric divisors;
- $\mathcal{X}^{\mathrm{can}} \longrightarrow X_\Sigma$ is the minimal orbifold having X_Σ as coarse moduli space.

Restricting equation (3) to the open substack $\mathcal{T} \subset \mathcal{X}$, one obtains $\mathcal{T} \cong \mathbb{T} \times \mathcal{B}K \longrightarrow \mathbb{T} \longrightarrow \mathbb{T} \longrightarrow \mathbb{T}$, where $\mathbb{T} \times \mathcal{B}K \longrightarrow \mathbb{T}$ is the projection to the first factor and $\mathbb{T} \longrightarrow \mathbb{T}$ is the identity map.

Remark 2.1. The purpose of introducing additional vectors $b_{r'+1}, \ldots, b_r$ is to ensure that G is *connected*. The stacky fan $\mathbf{\Sigma}$ together with the extra vectors $b_{r'+1}, \ldots, b_r$ is an *extended stacky fan* in the sense of Jiang [48]. It follows from the definition that $\{r'+1, \ldots, r\} \subset I$ for any $I \in \mathcal{A}$.

Let M, $\widetilde{M}$, and $\mathbb{L}^\vee$ be the character lattices of the tori $\mathbb{T}$, $\widetilde{\mathbb{T}}$ and G, respectively:

$$\begin{aligned} M &= \mathrm{Hom}(N, \mathbb{Z}) = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*), \\ \widetilde{M} &= \mathrm{Hom}(\widetilde{N}, \mathbb{Z}) = \mathrm{Hom}(\widetilde{\mathbb{T}}, \mathbb{C}^*), \\ \mathbb{L}^\vee &= \mathrm{Hom}(\mathbb{L}, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{C}^*). \end{aligned}$$

Applying $\mathrm{Hom}(-, \mathbb{Z})$ to equation (1), we obtain the following exact sequence of (additive) abelian groups:

$$0 \longrightarrow M \xrightarrow{\phi^\vee} \widetilde{M} \xrightarrow{\psi^\vee} \mathbb{L}^\vee \longrightarrow \mathrm{Ext}^1(N, \mathbb{Z}) \longrightarrow 0. \tag{4}$$

Therefore, the group homomorphism $\psi^\vee : \widetilde{M} \longrightarrow \mathbb{L}^\vee$ is surjective if and only if $N_{\mathrm{tor}} = 0$.

We now consider a class of examples of 3-dimensional Calabi-Yau smooth toric DM stacks of the form $[\mathbb{C}^3/\mathbb{Z}_3]$. Let $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$ be the generator of $\mathbb{Z}_3$. Given $i, j, k \in \{0, 1, 2\}$ such that $i + j + k \in 3\mathbb{Z}$, we define $\mathcal{X}_{i,j,k}$ to be the quotient stack of the following $\mathbb{Z}_3$-action on $\mathbb{C}^3$:

$$\omega \cdot (Z_1, Z_2, Z_3) = (\omega^i Z_1, \omega^j Z_2, \omega^k Z_3).$$

In the following example, we consider

$$\mathcal{X}_{1,1,1}, \quad \mathcal{X}_{1,2,0} = [\mathbb{C}^2/\mathbb{Z}_3] \times \mathbb{C}, \quad \mathcal{X}_{0,0,0} = \mathbb{C}^3 \times \mathcal{B}\mathbb{Z}_3.$$

Example 2.2.

1. $\mathcal{X} = \mathcal{X}_{1,1,1}$ (see Figure 1). The toric data are given as follows:

$$\begin{gathered} N = \mathbb{Z}^3, \quad N_{\mathrm{tor}} = 0; \\ b_1 = (1,0,1), b_2 = (0,1,1), b_3 = (-1,-1,1), b_4 = (0,0,1); \\ r = 4, r' = 3, k = 1; \\ \Sigma = \{\text{the 3-cone spanned by } \{b_1, b_2, b_3\}, \text{ and its faces, and faces of faces, etc.}\}; \\ \mathcal{A} = \{I \subset \{1,2,3,4\} : 4 \in I\}; \\ \mathbb{L} \cong \mathbb{Z}, \quad \mathbb{L}^\vee \cong \mathbb{Z}; \end{gathered}$$

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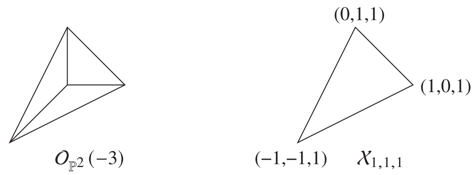

Figure 1. *$\mathcal{X}_{1,1,1}$ and its crepant resolution $\mathcal{O}_{\mathbb{P}^2}(-3)$.*

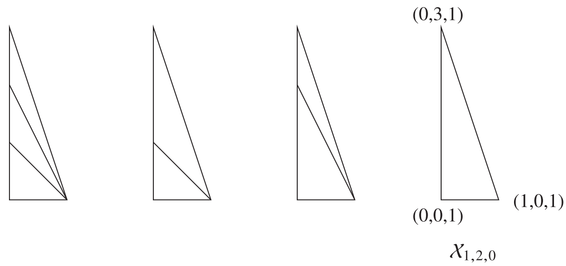

Figure 2. *$\mathcal{X}_{1,2,0}$ and its (partial) crepant resolutions.*

2. $\mathcal{X} = \mathcal{X}_{1,2,0}$, transversal A_2-singularity (see Figure 2). The toric data are given as follows:

$$N = \mathbb{Z}^3, \quad N_{\rm tor} = 0;$$
$$b_1 = (1,0,1), b_2 = (0,3,1), b_3 = (0,0,1), b_4 = (0,1,1), b_5 = (0,2,1);$$
$$r = 5, r' = 3, k = 2;$$
$$\Sigma = \{\text{the 3-cone spanned by } \{b_1, b_2, b_3\}, \text{ and its faces, and faces of faces, etc.}\};$$
$$\mathcal{A} = \{I \subset \{1,2,3,4,5\} : \{4,5\} \subset I\};$$
$$\mathbb{L} \cong \mathbb{Z}^2, \quad \mathbb{L}^\vee \cong \mathbb{Z}^2.$$

3. $\mathcal{X} = \mathcal{X}_{0,0,0}$. The toric data is given as follows:

$$N = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \quad N_{\rm tor} = \mathbb{Z}_3;$$
$$b_1 = (1,0,0,0), b_2 = (0,1,0,0), b_3 = (0,0,1,0), b_4 = (1,0,0,1);$$
$$r = 4, r' = 3, k = 1;$$
$$\Sigma = \{\text{the 3-cone spanned by } \{b_1, b_2, b_3\}, \text{ and its faces, and faces of faces, etc.}\};$$
$$\mathcal{A} = \{I \subset \{1,2,3,4\} : 4 \in I\};$$
$$\mathbb{L} \cong \mathbb{Z}, \quad \mathbb{L}^\vee \cong \mathbb{Z}.$$

2.2. Equivariant line bundles and torus-invariant Cartier divisors

A character $\chi \in \widetilde{M}$ gives a $\widetilde{\mathbb{T}}$-action on $\mathbb{C}^r \times \mathbb{C}$ by

$$(\tilde{t}_1, \ldots, \tilde{t}_r) \cdot (Z_1, \ldots, Z_r, u) = (\tilde{t}_1 Z_1, \ldots, \tilde{t}_r Z_r, \chi(\tilde{t}_1, \ldots, \tilde{t}_r) u),$$

where

$$(\tilde{t}_1, \ldots, \tilde{t}_r) \in \widetilde{\mathbb{T}} \cong (\mathbb{C}^*)^r, \quad (Z_1, \ldots, Z_r) \in \mathbb{C}^r, \quad u \in \mathbb{C}.$$

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Therefore $\mathbb{C}^r \times \mathbb{C}$ can be viewed as the total space of a $\widetilde{\mathbb{T}}$-equivariant line bundle $\widetilde{L}_\chi$ over $\mathbb{C}^r$. If

$$\chi(\tilde{t}_1, \ldots, \tilde{t}_r) = \prod_{i=1}^r \tilde{t}_i^{c_i},$$

where $c_1, \ldots, c_r \in \mathbb{Z}$, then

$$\widetilde{L}_\chi = \mathcal{O}_{\mathbb{C}^r}\Big(\sum_{i=1}^r c_i \widetilde{\mathcal{D}}_i\Big),$$

where $\widetilde{\mathcal{D}}_i$ is the $\widetilde{\mathbb{T}}$-divisor in $\mathbb{C}^r$ defined by $Z_i = 0$. We have

$$\widetilde{M} \cong \mathrm{Pic}_{\widetilde{\mathbb{T}}}(\mathbb{C}^r) \cong H^2_{\widetilde{\mathbb{T}}}(\mathbb{C}^r; \mathbb{Z}),$$

where the first isomorphism is given by $\chi \mapsto \widetilde{L}_\chi$ and the second isomorphism is given by the $\widetilde{\mathbb{T}}$-equivariant first Chern class $(c_1)_{\widetilde{\mathbb{T}}}$. Define

$$D_i^{\mathcal{T}} := (c_1)_{\widetilde{\mathbb{T}}}(\mathcal{O}_{\mathbb{C}^r}(\widetilde{\mathcal{D}}_i)) \in H^2_{\widetilde{\mathbb{T}}}(\mathbb{C}^r; \mathbb{Z}) \cong H^2_{\mathcal{T}}([\mathbb{C}^r/G]; \mathbb{Z}).$$

Then $\{D_1^{\mathcal{T}}, \ldots, D_r^{\mathcal{T}}\}$ is a $\mathbb{Z}$-basis of $H^2_{\widetilde{\mathbb{T}}}(\mathbb{C}^r; \mathbb{Z}) \cong \widetilde{M}$ dual to the $\mathbb{Z}$-basis $\{\tilde{b}_1, \ldots, \tilde{b}_r\}$ of $\widetilde{N}$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}_{\widetilde{\mathbb{T}}}(\mathbb{C}^r) \arrow[r, "\iota^*_{\mathcal{T}}"] \arrow[d, "(c_1)_{\widetilde{\mathbb{T}}}"'] & \mathrm{Pic}_{\widetilde{\mathbb{T}}}(U_{\mathcal{A}}) \arrow[r, "\cong"] \arrow[d, "(c_1)_{\widetilde{\mathbb{T}}}"'] & \mathrm{Pic}_{\mathcal{T}}(\mathcal{X}) \arrow[d, "(c_1)_{\mathcal{T}}"'] \\
H^2_{\widetilde{\mathbb{T}}}(\mathbb{C}^r; \mathbb{Z}) \arrow[r, "\iota^*_{\mathcal{T}}"] & H^2_{\widetilde{\mathbb{T}}}(U_{\mathcal{A}}; \mathbb{Z}) \arrow[r, "\cong"] & H^2_{\mathcal{T}}(\mathcal{X}; \mathbb{Z}),
\end{tikzcd}$$

where $\iota^*_{\mathcal{T}}$ is a surjective group homomorphism induced by the inclusion $\iota : U_{\mathcal{A}} \hookrightarrow \mathbb{C}^r$, and

$$\mathrm{Ker}(\iota^*_{\mathcal{T}}) = \bigoplus_{i=r'+1}^r \mathbb{Z} D_i^{\mathcal{T}}.$$

Therefore,

$$\mathrm{Pic}_{\mathcal{T}}(\mathcal{X}) \cong H^2_{\mathcal{T}}(\mathcal{X}; \mathbb{Z}) \cong \widetilde{M}/\oplus_{i=r'+1}^r \mathbb{Z} D_i^{\mathcal{T}}.$$

Let $\bar{D}_i^{\mathcal{T}} := \iota^*_{\mathcal{T}} D_i^{\mathcal{T}}$. Then

$$\bar{D}_i^{\mathcal{T}} = 0, \quad i = r'+1, \ldots, r,$$

and

$$H^2_{\mathcal{T}}(\mathcal{X}; \mathbb{Z}) = \bigoplus_{i=1}^{r'} \mathbb{Z} \bar{D}_i^{\mathcal{T}} \cong \mathbb{Z}^{r'}.$$

For $i = 1, \ldots, r'$, $\widetilde{\mathcal{D}}_i \cap U_{\mathcal{A}}$ is a $\widetilde{\mathbb{T}}$-divisor in $U_{\mathcal{A}}$, and it descends to a $\mathbb{T}$-divisor $\mathcal{D}_i$ in $\mathcal{X}$. We have

$$\bar{D}_i^{\mathcal{T}} = (c_1)_{\mathcal{T}}(\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)), \quad i = 1, \ldots, r'.$$

For $i = r'+1, \ldots, r$, $\widetilde{\mathcal{D}}_i \cap U_{\mathcal{A}}$ is empty, so it is the zero $\widetilde{\mathbb{T}}$-divisor.

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2.3. Line bundles and Cartier divisors

We have group isomorphisms

$$\mathbb{L}^\vee \cong \mathrm{Pic}_G(\mathbb{C}^r) \cong H^2_G(\mathbb{C}^r;\mathbb{Z}),$$

where the first isomorphism is given by $\chi \in \mathbb{L}^\vee = \mathrm{Hom}(G,\mathbb{C}^*) \mapsto \widetilde{L}_\chi$ and the second isomorphism is given by the G-equivariant first Chern class $(c_1)_G$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}_G(\mathbb{C}^r) \arrow[r, "\iota^*"] \arrow[d, "(c_1)_G"'] & \mathrm{Pic}_G(U_{\mathcal{A}}) \arrow[r, "\cong"] \arrow[d, "(c_1)_G"'] & \mathrm{Pic}(\mathcal{X}) \arrow[d, "c_1"'] \\
H^2_G(\mathbb{C}^r;\mathbb{Z}) \arrow[r, "\iota^*"] & H^2_G(U_{\mathcal{A}};\mathbb{Z}) \arrow[r, "\cong"] & H^2(\mathcal{X};\mathbb{Z}),
\end{tikzcd}$$

where ι^* is a surjective group homomorphism induced by the inclusion $\iota : U_{\mathcal{A}} \hookrightarrow \mathbb{C}^r$. The surjective map $H^2_G(\mathbb{C}^r;\mathbb{Z}) \to H^2(\mathcal{X};\mathbb{Z})$ is the restriction of the Kirwan map

$$\kappa : H^*_G(\mathbb{C}^r;\mathbb{Z}) \longrightarrow H^*(\mathcal{X};\mathbb{Z}).$$

Define

$$D_i := (c_1)_G(\mathcal{O}_{\mathbb{C}^r}(\widetilde{\mathcal{D}}_i)) \in H^2_G(\mathbb{C}^r;\mathbb{Z}) \cong H^2([\mathbb{C}^r/G];\mathbb{Z}).$$

Then

$$\mathrm{Ker}(\iota^*) = \bigoplus_{i=r'+1}^{r} \mathbb{Z}D_i.$$

Therefore,

$$\mathrm{Pic}(\mathcal{X}) \cong H^2(\mathcal{X};\mathbb{Z}) \cong \mathbb{L}^\vee/\oplus_{i=r'+1}^{r}\mathbb{Z}D_i.$$

Recall that $\psi^\vee : \widetilde{M} \longrightarrow \mathbb{L}^\vee$ is surjective if and only if $N_{\mathrm{tor}} = 0$. Let

$$\bar{D}_i = c_1(\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)) \in H^2(\mathcal{X};\mathbb{Z}), \quad i = 1,\dots,r.$$

The map

$$\bar{\psi}^\vee : \mathrm{Pic}_{\mathcal{T}}(\mathcal{X}) \cong H^2_{\mathcal{T}}(\mathcal{X};\mathbb{Z}) \to \mathrm{Pic}(\mathcal{X}) \cong H^2(\mathcal{X};\mathbb{Z}),$$

given by

$$\bar{D}_i^{\mathcal{T}} \mapsto \bar{D}_i, \quad i = 1,\dots,r',$$

is surjective if and only if $N_{\mathrm{tor}} = 0$. In general, $\mathrm{Coker}(\psi^\vee) \cong \mathrm{Coker}(\bar{\psi}^\vee)$ is a finite abelian group.

Pick a $\mathbb{Z}$-basis $\{e_1,\dots,e_k\}$ of $\mathbb{L} \cong \mathbb{Z}^k$, and let $\{e_1^\vee,\dots,e_k^\vee\}$ be the dual $\mathbb{Z}$-basis of $\mathbb{L}^\vee$. For each $a \in \{1,\dots,k\}$, we define a *charge vector*

$$l^{(a)} = (l_1^{(a)},\dots,l_r^{(a)}) \in \mathbb{Z}^r$$

by

$$\psi(e_a) = \sum_{i=1}^{r} l_i^{(a)} \tilde{b}_i,$$

where $\psi : \mathbb{L} \longrightarrow \widetilde{N}$ is the inclusion map. Then

$$D_i = \psi^\vee(D_i^{\mathcal{T}}) = \sum_{a=1}^{k} l_i^{(a)} e_a^\vee, \quad i = 1, \ldots, r,$$

and

$$\sum_{i=1}^{r} l_i^{(a)} b_i = \phi \circ \psi(e_a) = 0, \quad a = 1, \ldots, k.$$

Example 2.3. We use the notation in Example 2.2.

1. $\mathcal{X} = \mathcal{X}_{1,1,1}$.

$$D_1 = D_2 = D_3 = 1,\ D_4 = -3;$$
$$l^{(1)} = (1, 1, 1, -3);$$
$$\mathrm{Pic}_{\mathcal{T}}(\mathcal{X}) \cong \mathbb{Z}^3, \quad \mathrm{Pic}(\mathcal{X}) \cong \mathbb{Z}/3\mathbb{Z};$$

2. $\mathcal{X} = \mathcal{X}_{1,2,0}$.

$$D_1 = (0,0),\ D_2 = (0,1),\ D_3 = (1,0),\ D_4 = (-2,1),\ D_5 = (1,-2);$$
$$l^{(1)} = (0,0,1,-2,1),\ l^{(2)} = (0,1,0,1,-2);$$
$$\mathrm{Pic}_{\mathcal{T}}(\mathcal{X}) = \mathbb{Z}^3, \quad \mathrm{Pic}(\mathcal{X}) = \mathbb{Z}^2 \big/ \big(\mathbb{Z}(-2,1) \oplus \mathbb{Z}(1,-2)\big) \cong \mathbb{Z}/3\mathbb{Z}.$$

3. $\mathcal{X} = \mathcal{X}_{0,0,0}$.

$$D_1 = 3,\ D_2 = 0,\ D_3 = 0,\ D_4 = -3;$$
$$l^{(1)} = (3, 0, 0, -3);$$
$$\mathrm{Pic}_{\mathcal{T}}(\mathcal{X}) = \mathbb{Z}^3, \quad \mathrm{Pic}(\mathcal{X}) = \mathbb{Z}/3\mathbb{Z}.$$

2.4. Torus invariant subvarieties and their generic stabilisers

Let $\Sigma(d)$ be the set of d-dimensional cones. For each $\sigma \in \Sigma(d)$, we define

$$I_\sigma := \{i \in \{1, \ldots, r\} \mid \rho_i \not\subset \sigma\} \in \mathcal{A}, \quad I'_\sigma := \{1, \ldots, r\} \setminus I_\sigma.$$

Then $|I'_\sigma| = d$ and $|I_\sigma| = r - d$. Let $\tilde{V}(\sigma) \subset U_{\mathcal{A}}$ be the closed subvariety defined by the ideal of $\mathbb{C}[Z_1, \ldots, Z_r]$ generated by

$$\{Z_i = 0 \mid \rho_i \subset \sigma\} = \{Z_i = 0 \mid i \in I'_\sigma\}.$$

Then $\mathcal{V}(\sigma) := [\widetilde{V}(\sigma)/G]$ is an $(n-d)$-dimensional $\mathcal{T}$-invariant closed substack of $\mathcal{X} = [U_{\mathcal{A}}/G]$.

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The group homomorphism $G \cong (\mathbb{C}^*)^k \longrightarrow \widetilde{\mathbb{T}} \cong (\mathbb{C}^*)^r$ is given by

$$g \mapsto (\chi_1(g), \ldots, \chi_r(g)),$$

where $\chi_i \in \mathrm{Hom}(G, \mathbb{C}^*) = \mathbb{L}^\vee$ is given by

$$\chi_i(u_1, \ldots, u_k) = \prod_{a=1}^{k} u_a^{l_i^{(a)}}.$$

Let

$$G_\sigma := \{g \in G \mid g \cdot z = z \text{ for all } z \in \widetilde{V}(\sigma)\} = \bigcap_{i \in I_\sigma} \mathrm{Ker}(\chi_i).$$

Then G_σ is the generic stabiliser of $\mathcal{V}(\sigma)$. It is a finite subgroup of G. If $\tau \subset \sigma$, then $I_\sigma \subset I_\tau$, so $G_\tau \subset G_\sigma$. There are two special cases:

- Let $\{0\}$ be the unique 0-dimensional cone. Then $G_{\{0\}} = K$ is the generic stabiliser of $\mathcal{V}(\{0\}) = \mathcal{X}$.
- If $\sigma \in \Sigma(n)$, where $n = \dim_{\mathbb{C}} \mathcal{X}$, then $\mathfrak{p}_\sigma := \mathcal{V}(\sigma)$ is a $\mathcal{T}$-fixed point in $\mathcal{X}$, and $\mathfrak{p}_\sigma = \mathcal{B}G_\sigma$.

Example 2.4. We use the notation in Example 2.2. Let $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^3$ denote the 3-dimensional cone spanned by $\bar{b}_1, \bar{b}_2, \bar{b}_3$. For $j = 1, 2, 3$, let τ_j denote the 2-dimensional cone in $N_{\mathbb{R}}$ spanned by $\{\bar{b}_i : i \in \{1, 2, 3\} - \{j\}\}$.

1. $\mathcal{X} = \mathcal{X}_{1,1,1}$: $G_\sigma = \mathbb{Z}_3, \quad G_{\tau_1} = G_{\tau_2} = G_{\tau_3} = \{1\}$.
2. $\mathcal{X} = \mathcal{X}_{1,2,0}$: $G_\sigma = \mathbb{Z}_3 = G_{\tau_3}, \quad G_{\tau_1} = G_{\tau_2} = \{1\}$.
3. $\mathcal{X} = \mathcal{X}_{0,0,0}$: $G_\sigma = \mathbb{Z}_3 = G_{\tau_1} = G_{\tau_2} = G_{\tau_3}$.

Define the set of flags in Σ to be

$$F(\Sigma) = \{(\tau, \sigma) \in \Sigma(n-1) \times \Sigma(n) : \tau \subset \sigma\}.$$

Given $(\tau, \sigma) \in F(\Sigma)$, let $\mathfrak{l}_\tau := \mathcal{V}(\tau)$ be the 1-dimensional $\mathcal{T}$-invariant subvariety of $\mathcal{X}$. Then $\mathfrak{p}_\sigma$ is contained in $\mathfrak{l}_\tau$. There is a unique $i \in \{1, \ldots, r'\}$ such that $i \in I'_\sigma \setminus I'_\tau$. The representation of G_σ on the tangent line $T_{\mathfrak{p}_\sigma}\mathfrak{l}_\tau$ to $\mathfrak{l}_\tau$ at the stacky point $\mathfrak{p}_\sigma$ is given by $\chi_{(\tau,\sigma)} := \chi_i|_{G_\sigma} : G_\sigma \to \mathbb{C}^*$. The image $\chi_i(G_\sigma) \subset \mathbb{C}^*$ is a cyclic subgroup of $\mathbb{C}^*$; we define the order of this group to be $r(\tau, \sigma)$. Then there is a short exact sequence of finite abelian groups:

$$1 \to G_\tau \longrightarrow G_\sigma \xrightarrow{\chi_{(\tau,\sigma)}} \boldsymbol{\mu}_{r(\tau,\sigma)} \to 1,$$

where $\boldsymbol{\mu}_a = \{z \in \mathbb{C}^* \mid z^a = 1\}$ is the group of ath roots of unity.

2.5. The extended nef cone and the extended Mori cone

In this paragraph, $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Given a finitely generated abelian group Λ with $\Lambda/\Lambda_{\mathrm{tor}} \cong \mathbb{Z}^m$, define $\Lambda_{\mathbb{F}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{F} \cong \mathbb{F}^m$. We have the following short exact sequences of vector spaces:

$$0 \to \mathbb{L}_{\mathbb{F}} \to \widetilde{N}_{\mathbb{F}} \to N_{\mathbb{F}} \to 0,$$
$$0 \to M_{\mathbb{F}} \to \widetilde{M}_{\mathbb{F}} \to \mathbb{L}_{\mathbb{F}}^\vee \to 0.$$

We also have the following isomorphisms of vector spaces over $\mathbb{F}$:

$$H^2(\mathcal{X};\mathbb{F}) \cong H^2(X;\mathbb{F}) \cong \mathbb{L}_{\mathbb{F}}^\vee/\oplus_{i=r'+1}^r \mathbb{F}D_i,$$
$$H^2_{\mathcal{T}}(\mathcal{X};\mathbb{F}) \cong H^2_{\mathbb{T}}(X;\mathbb{F}) \cong \widetilde{M}_{\mathbb{F}}/\oplus_{i=r'+1}^r \mathbb{F}D_i^{\mathcal{T}},$$

where X is the coarse moduli space of $\mathcal{X}$.

From now on, we assume all the maximal cones in Σ are n-dimensional, where $n = \dim_{\mathbb{C}} \mathcal{X}$. Given a maximal cone $\sigma \in \Sigma(n)$, we define

$$\mathbb{K}_\sigma^\vee := \bigoplus_{i \in I_\sigma} \mathbb{Z}D_i.$$

Then $\mathbb{K}_\sigma^\vee$ is a sublattice of $\mathbb{L}^\vee$ of finite index. We define the *extended* σ-nef cone to be

$$\widetilde{\mathrm{Nef}}_\sigma = \sum_{i \in I_\sigma} \mathbb{R}_{\geq 0} D_i,$$

which is a k-dimensional cone in $\mathbb{L}_{\mathbb{R}}^\vee \cong \mathbb{R}^k$. The *extended nef cone* of the extended stacky fan $(\Sigma, b_1, \ldots, b_r)$ is

$$\widetilde{\mathrm{Nef}}_{\mathcal{X}} := \bigcap_{\sigma \in \Sigma(n)} \widetilde{\mathrm{Nef}}_\sigma.$$

The *extended* σ-Kähler cone $\widetilde{C}_\sigma$ is defined to be the interior of $\widetilde{\mathrm{Nef}}_\sigma$; the *extended Kähler cone* of $\mathcal{X}$, $\widetilde{C}_{\mathcal{X}}$, is defined to be the interior of the extended nef cone $\widetilde{\mathrm{Nef}}_{\mathcal{X}}$.

Let $\mathbb{K}_\sigma$ be the dual lattice of $\mathbb{K}_\sigma^\vee$; it can be viewed as an additive subgroup of $\mathbb{L}_{\mathbb{Q}}$:

$$\mathbb{K}_\sigma = \{\beta \in \mathbb{L}_{\mathbb{Q}} \mid \langle D, \beta \rangle \in \mathbb{Z} \ \forall D \in \mathbb{K}_\sigma^\vee\},$$

where $\langle -, - \rangle$ is the natural pairing between $\mathbb{L}_{\mathbb{Q}}^\vee$ and $\mathbb{L}_{\mathbb{Q}}$. Define

$$\mathbb{K} := \bigcup_{\sigma \in \Sigma(n)} \mathbb{K}_\sigma.$$

Then $\mathbb{K}$ is a subset (which is not necessarily a subgroup) of $\mathbb{L}_{\mathbb{Q}}$, and $\mathbb{L} \subset \mathbb{K}$.

We define the *extended* σ-Mori cone $\widetilde{\mathrm{NE}}_\sigma \subset \mathbb{L}_{\mathbb{R}}$ to be the dual cone of $\widetilde{\mathrm{Nef}}_\sigma \subset \mathbb{L}_{\mathbb{R}}^\vee$:

$$\widetilde{\mathrm{NE}}_\sigma = \{\beta \in \mathbb{L}_{\mathbb{R}} \mid \langle D, \beta \rangle \geq 0 \ \forall D \in \widetilde{\mathrm{Nef}}_\sigma\}.$$

It is a k-dimensional cone in $\mathbb{L}_{\mathbb{R}}$. The *extended Mori cone* of the extended stacky fan $(\Sigma, b_1, \ldots, b_r)$ is

$$\widetilde{\mathrm{NE}}_{\mathcal{X}} := \bigcup_{\sigma \in \Sigma(n)} \widetilde{\mathrm{NE}}_\sigma.$$

Finally, we define

$$\mathbb{K}_{\mathrm{eff},\sigma} := \mathbb{K}_\sigma \cap \widetilde{\mathrm{NE}}_\sigma, \quad \mathbb{K}_{\mathrm{eff}} := \mathbb{K} \cap \widetilde{\mathrm{NE}}(\mathcal{X}) = \bigcup_{\sigma \in \Sigma(n)} \mathbb{K}_{\mathrm{eff},\sigma}.$$

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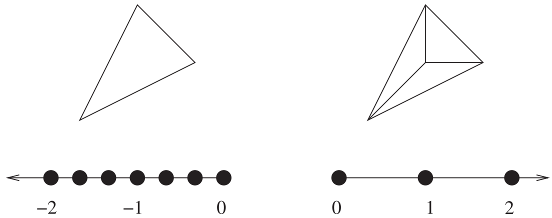

Figure 3. *$\mathbb{K}_{\mathrm{eff}}$ of $\mathcal{X}_{1,1,1}$ and its crepant resolution $\mathcal{O}_{\mathbb{P}^2}(-3)$.*

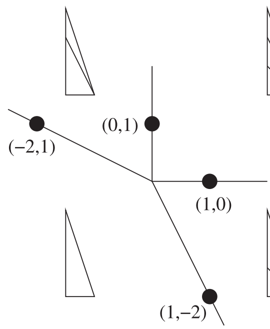

Figure 4. *The secondary fan of the crepant resolution of $\mathcal{X}_{1,2,0}$.*

Example 2.5.

1. $\mathcal{X} = \mathcal{X}_{1,1,1}$ (see Figure 3).

$$\mathbb{K}^\vee \cong 3\mathbb{Z}, \quad \widetilde{\mathrm{Nef}}_{\mathcal{X}} = \mathbb{R}_{\le 0};$$
$$\mathbb{K} \cong \frac{1}{3}\mathbb{Z}, \quad \widetilde{\mathrm{NE}}_{\mathcal{X}} = \mathbb{R}_{\le 0}, \quad \mathbb{K}_{\mathrm{eff}} = \frac{1}{3}\mathbb{Z}_{\le 0}.$$

2. $\mathcal{X} = \mathcal{X}_{1,2,0}$ (see Figures 4 and 5).

$$\mathbb{K}^\vee \cong \mathbb{Z}(-2,1) \oplus \mathbb{Z}(1,-2), \quad \widetilde{\mathrm{Nef}}_{\mathcal{X}} = \mathbb{R}_{\ge 0}(-2,1) + \mathbb{R}_{\ge 0}(1,-2);$$
$$\mathbb{K} \cong \mathbb{Z}\Big(-\frac{2}{3}, -\frac{1}{3}\Big) \oplus \mathbb{Z}\Big(-\frac{1}{3}, -\frac{2}{3}\Big), \quad \widetilde{\mathrm{NE}}_{\mathcal{X}} = \mathbb{R}_{\ge 0}\Big(-\frac{2}{3}, -\frac{1}{3}\Big) + \mathbb{R}_{\ge 0}\Big(-\frac{1}{3}, -\frac{2}{3}\Big),$$
$$\mathbb{K}_{\mathrm{eff}} = \mathbb{Z}_{\ge 0}\Big(-\frac{2}{3}, -\frac{1}{3}\Big) + \mathbb{Z}_{\ge 0}\Big(-\frac{1}{3}, -\frac{2}{3}\Big).$$

3. $\mathcal{X} = \mathcal{X}_{0,0,0}$.

$$\mathbb{K}^\vee \cong 3\mathbb{Z}, \quad \widetilde{\mathrm{Nef}}_{\mathcal{X}} = \mathbb{R}_{\le 0};$$
$$\mathbb{K} \cong \frac{1}{3}\mathbb{Z}, \quad \widetilde{\mathrm{NE}}_{\mathcal{X}} = \mathbb{R}_{\le 0}, \quad \mathbb{K}_{\mathrm{eff}} = \frac{1}{3}\mathbb{Z}_{\le 0}.$$

Assumption 2.6. *From now on, we make the following assumptions on $\mathcal{X}$.*

(a) *The coarse moduli space X_Σ of $\mathcal{X}$ is semi-projective.*
(b) *We may choose $b_{r'+1}, \dots, b_r$ such that $\hat{\rho} := D_1 + \cdots + D_r$ is contained in the closure of the extended Kähler cone $\widetilde{C}_{\mathcal{X}}$.*

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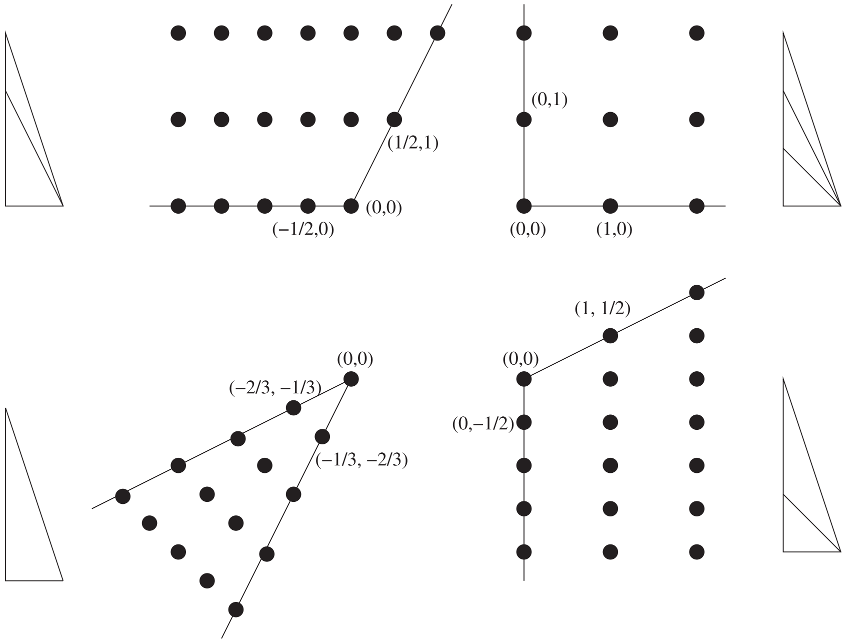

Figure 5. *$\mathbb{K}_{\text{eff}}$ of $\mathcal{X}_{1,2,0}$ and its (partial) crepant resolutions.*

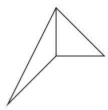

Figure 6. $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

Remark 2.7.

1. We make the above assumptions (a) and (b) so that the equivariant mirror theorem [27, Theorem 31] takes a particularly simple form. See Section 4.1 in this paper for the precise statement.
2. By [28, Proposition 14.4.1], X_Σ is semi-projective if and only if $|\Sigma|$ is equal to the cone spanned by $b_1, \ldots, b_r$. For example, the total space of $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ is a smooth toric Calabi-Yau 3-fold that is not semi-projective (see Figure 6).
3. When $\mathcal{X}$ is a Calabi-Yau smooth toric DM stack, Assumption (b) holds if its coarse moduli space X_Σ has a toric crepant resolution of singularities; see [47, Remark 3.4]. By [28, Proposition 11.4.19], any 3-dimensional Gorenstein toric variety X_Σ has a resolution of singularities $\phi : X_{\Sigma'} \to X_\Sigma$ such that ϕ is projective and crepant. So Assumption 2.6 (b) holds for any 3-dimensional Calabi-Yau smooth toric DM stacks.

2.6. Smooth toric DM stacks as symplectic quotients

Let $G_\mathbb{R} \cong U(1)^k$ be the maximal compact subgroup of $G \cong (\mathbb{C}^*)^k$. Then the Lie algebra of $G_\mathbb{R}$ is $\mathbb{L}_\mathbb{R}$. Let

$$\widetilde{\mu} : \mathbb{C}^r \to \mathbb{L}_\mathbb{R}^\vee = \bigoplus_{a=1}^{k} \mathbb{R} e_a^\vee$$

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be the moment map of the Hamiltonian $G_{\mathbb{R}}$-action on $\mathbb{C}^r$, equipped with the Kähler form

$$\sqrt{-1}\sum_{i=1}^{r} dZ_i \wedge d\bar{Z}_i.$$

Then

$$\widetilde{\mu}(Z_1,\dots,Z_r) = \sum_{i=1}^{r}\sum_{a=1}^{k} l_i^{(a)}|Z_i|^2 e_a^\vee.$$

If $\mathbf{r} = \sum_{a=1}^{k} r_a e_a^\vee$ is in the extended Kähler cone of $\mathcal{X}$, then

$$\mathcal{X} = [\widetilde{\mu}^{-1}(\mathbf{r})/G_{\mathbb{R}}].$$

The generic stabiliser K (which is a finite subgroup of $G \cong (\mathbb{C}^*)^k$) is contained in the maximal compact subgroup $G_{\mathbb{R}}$ of G. The quotient $G_{\mathbb{R}}^{\mathrm{rig}} := G_{\mathbb{R}}/K \cong U(1)^k$ is the maximal compact subgroup of $G^{\mathrm{rig}} = G/K \cong (\mathbb{C}^*)^k$, and

$$\mathcal{X}^{\mathrm{rig}} = [\widetilde{\mu}^{-1}(\mathbf{r})/G_{\mathbb{R}}^{\mathrm{rig}}]$$

as a symplectic quotient.

The real numbers $r_1,\dots,r_k$ are extended Kähler parameters. The symplectic structure $\omega(\mathbf{r})$ depends on $\mathbf{r}$. The map $\mathbf{r} \mapsto [\omega(\mathbf{r})]$ is given by $\mathbb{L}_{\mathbb{R}}^\vee \to H^2(\mathcal{X};\mathbb{R})$. Let $T_a = -r_a + \sqrt{-1}\theta_a$ be complexified extended Kähler parameters of $\mathcal{X}$.

2.7. The inertia stack and the Chen-Ruan orbifold cohomology

Given $\sigma \in \Sigma$, define

$$\mathrm{Box}(\sigma) := \left\{ v \in N : \bar{v} = \sum_{i\in I'_\sigma} c_i \bar{b}_i, \quad 0 \le c_i < 1 \right\}.$$

Then $N_{\mathrm{tor}} \subset \mathrm{Box}(\sigma) \subset N$. If σ is a d-dimensional cone, then the set $\{\sum_{i\in I'_\sigma} c_i\bar{b}_i : c_i \in \mathbb{R}, 0 \le c_i < 1\}$ is a fundamental domain of the action of $\bar{N}_\sigma = \oplus_{i\in I'_\sigma}\mathbb{Z}\bar{b}_i \cong \mathbb{Z}^d$ on $N_\sigma \otimes_{\mathbb{Z}} \mathbb{R} = \oplus_{i\in I'_\sigma}\mathbb{R}\bar{b}_i \cong \mathbb{R}^d$. If $\tau \subset \sigma$, then $I'_\tau \subset I'_\sigma$, so $\mathrm{Box}(\tau) \subset \mathrm{Box}(\sigma)$.

Let $\sigma \in \Sigma(n)$ be a maximal cone in Σ. We have a short exact sequence of abelian groups

$$0 \to \mathbb{K}_\sigma/\mathbb{L} \to \mathbb{L}_{\mathbb{R}}/\mathbb{L} \to \mathbb{L}_{\mathbb{R}}/\mathbb{K}_\sigma \to 0,$$

which can be identified with the following short exact sequence of multiplicative abelian groups

$$1 \to G_\sigma \to G_{\mathbb{R}} \to (G/G_\sigma)_{\mathbb{R}} \to 1,$$

where $G_{\mathbb{R}} \cong U(1)^k$ is the maximal compact subgroup of $G \cong (\mathbb{C}^*)^k$, and $(G/G_\sigma)_{\mathbb{R}} \cong U(1)^k$ is the maximal compact subgroup of $(G/G_\sigma) \cong (\mathbb{C}^*)^k$.

Given a real number x, we recall some standard notation: $\lfloor x\rfloor$ is the greatest integer less than or equal to x, $\lceil x\rceil$ is the least integer greater or equal to x and $\{x\} = x - \lfloor x\rfloor$ is the fractional part of x. Define $v : \mathbb{K}_\sigma \to N$ by

$$v(\beta) = \sum_{i=1}^{r} \lceil \langle D_i, \beta\rangle \rceil b_i.$$

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Then

$$\overline{v(\beta)} = \sum_{i \in I'_\sigma} \{-\langle D_i, \beta\rangle\} \bar{b}_i,$$

so $v(\beta) \in \mathrm{Box}(\sigma)$. Indeed, v induces a bijection $K_\sigma/\mathbb{L} \cong \mathrm{Box}(\sigma)$.

For any $\tau \in \Sigma$, there exists $\sigma \in \Sigma(n)$ such that $\tau \subset \sigma$. The bijection $G_\sigma \to \mathrm{Box}(\sigma)$ restricts to a bijection $G_\tau \to \mathrm{Box}(\tau)$.

Define

$$\mathrm{Box}(\mathbf{\Sigma}) := \bigcup_{\sigma \in \Sigma} \mathrm{Box}(\sigma) = \bigcup_{\sigma \in \Sigma(n)} \mathrm{Box}(\sigma).$$

Then $N_{\mathrm{tor}} \subset \mathrm{Box}(\mathbf{\Sigma}) \subset N$. There is a bijection $\mathbb{K}/\mathbb{L} \to \mathrm{Box}(\mathbf{\Sigma})$.

Given $v \in \mathrm{Box}(\sigma)$, where $\sigma \in \Sigma(d)$, define $c_i(v) \in [0,1) \cap \mathbb{Q}$ by

$$\bar{v} = \sum_{i \in I'_\sigma} c_i(v) \bar{b}_i.$$

Suppose that $k \in G_\sigma$ corresponds to $v \in \mathrm{Box}(\sigma)$ under the bijection $G_\sigma \cong \mathrm{Box}(\sigma)$. Then

$$\chi_i(k) = \begin{cases} 1, & i \in I_\sigma, \\ e^{2\pi\sqrt{-1}c_i(v)}, & i \in I'_\sigma. \end{cases}$$

Define

$$\mathrm{age}(k) = \mathrm{age}(v) = \sum_{i \notin I_\sigma} c_i(v).$$

Let $IU = \{(z,k) \in U_{\mathcal{A}} \times G \mid k \cdot z = z\}$, and let G act on IU by $h \cdot (z,k) = (h \cdot z, k)$. The inertia stack $\mathcal{IX}$ of $\mathcal{X}$ is defined to be the quotient stack

$$\mathcal{IX} := [IU/G].$$

Note that $(z = (Z_1, \dots, Z_r), k) \in IU$ if and only if

$$k \in \bigcup_{\sigma \in \Sigma} G_\sigma \text{ and } Z_i = 0 \text{ whenever } \chi_i(k) \neq 1.$$

So

$$IU = \bigcup_{v \in \mathrm{Box}(\mathbf{\Sigma})} U_v,$$

where

$$U_v := \{(Z_1, \dots, Z_m) \in U_{\mathcal{A}} : Z_i = 0 \text{ if } c_i(v) \neq 0\}.$$

The connected components of $\mathcal{IX}$ are

$$\{\mathcal{X}_v := [U_v/G] : v \in \mathrm{Box}(\mathbf{\Sigma})\}.$$

The involution $IU \to IU$, $(z,k) \mapsto (z, k^{-1})$ induces involutions $\mathrm{inv} : \mathcal{IX} \to \mathcal{IX}$ and $\mathrm{inv} : \mathrm{Box}(\mathbf{\Sigma}) \to \mathrm{Box}(\mathbf{\Sigma})$ such that $\mathrm{inv}(\mathcal{X}_v) = \mathcal{X}_{\mathrm{inv}(v)}$.

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In the remainder of this subsection, we consider rational cohomology and write $H^*(-)$ instead of $H^*(-;\mathbb{Q})$.

As a graded vector space over $\mathbb{Q}$ (and as the state-space of the relevant quantum theory in physics [74]), the Chen-Ruan orbifold cohomology [23] is defined to be

$$H^*_{\mathrm{CR}}(\mathcal{X}) = \bigoplus_{v\in \mathrm{Box}(\mathbf{\Sigma})} H^*(\mathcal{X}_v)[2\mathrm{age}(v)].$$

Let $\mathbf{1}_v$ be the unit in $H^*(\mathcal{X}_v)$. Then $\mathbf{1}_v \in H^{2\mathrm{age}(v)}_{\mathrm{CR}}(\mathcal{X})$. In particular,

$$H^0_{\mathrm{CR}}(\mathcal{X}) = \bigoplus_{v\in N_{\mathrm{tor}}} \mathbb{Q}\mathbf{1}_v.$$

Suppose that $\mathcal{X}$ is a *proper* toric DM stack. Then the orbifold Poincaré pairing on $H^*_{\mathrm{CR}}(\mathcal{X})$ is defined as

$$(\alpha,\beta) := \int_{\mathcal{I}\mathcal{X}} \alpha \cup \mathrm{inv}^*(\beta). \tag{5}$$

We also have an equivariant pairing on $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$:

$$(\alpha,\beta)_{\mathbb{T}} := \int_{\mathcal{I}\mathcal{X}_{\mathbb{T}}} \alpha \cup \mathrm{inv}^*(\beta), \tag{6}$$

where

$$\int_{\mathcal{I}\mathcal{X}_{\mathbb{T}}} : H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \to H^*_{\mathbb{T}}(\mathrm{point}) = H^*(B\mathbb{T})$$

is the equivariant pushforward to a point. When $\mathcal{X}$ is not proper, equation (5) is not defined, but we can still define via equation (6) an equivariant pairing $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \to \mathcal{Q}_{\mathbb{T}}$, where $\mathcal{Q}_{\mathbb{T}}$ is the fractional field of the ring $H^*(B\mathbb{T})$.

Example 2.8.

1. $\mathcal{X} = \mathcal{X}_{1,1,1}$.

$$N = \mathbb{Z}^3, \quad \mathrm{Box}(\mathbf{\Sigma}) = \{(0,0,0),(0,0,1),(0,0,2)\};$$
$$H^0_{\mathrm{CR}}(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,0,0)},\ H^2_{\mathrm{CR}}(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,0,1)},\ H^4_{\mathrm{CR}}(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,0,2)}.$$

2. $\mathcal{X} = \mathcal{X}_{1,2,0}$.

$$N = \mathbb{Z}^3, \quad \mathrm{Box}(\mathbf{\Sigma}) = \{(0,0,0),(0,2,1),(0,1,1)\};$$
$$H^0_{\mathrm{CR}}(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,0,0)}, \quad H^2_{\mathrm{CR}}(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,2,1)} \oplus \mathbb{Q}\mathbf{1}_{(0,1,1)}.$$

3. $\mathcal{X} = \mathcal{X}_{0,0,0}$.

$$N = \mathbb{Z}^3 \oplus \mathbb{Z}_3, \quad \mathrm{Box}(\mathbf{\Sigma}) = N_{\mathrm{tor}} = \mathbb{Z}_3 = \{0,1,2\};$$
$$H^0_{\mathrm{CR}}(\mathcal{X}) = \mathbb{Q}\mathbf{1}_0 \oplus \mathbb{Q}\mathbf{1}_1 \oplus \mathbb{Q}\mathbf{1}_2.$$

3. All genus open-closed Gromov-Witten invariants

In this section, $\mathcal{X}$ is a 3-dimensional Calabi-Yau smooth Deligne-Mumford stack.

3.1. Rigidification

The rigidification $\mathcal{X}^{\mathrm{rig}}$ of $\mathcal{X}$ is a toric Calabi-Yau 3-orbifold. The Calabi-Yau condition implies $\mathcal{X}^{\mathrm{rig}} = \mathcal{X}^{\mathrm{can}}$, where $\mathcal{X}^{\mathrm{can}}$ is determined by the simplicial fan Σ and then by choosing each b_i to be the primitive generator of each 1-cone (compare to equation (3) in Section 2.1). Let $\mathcal{T}'$ (respectively, $\mathbb{T}'$) be the subtorus of $\mathcal{T} \cong (\mathbb{C}^*)^3 \times \mathcal{B}K$ (respectively, $\mathbb{T} \cong (\mathbb{C}^*)^3$) preserving the Calabi-Yau 3-form on $\mathcal{X}$ (respectively, $\mathcal{X}^{\mathrm{rig}}$). Then $\mathbb{T}' \cong (\mathbb{C}^*)^2$ and $\mathcal{T}' \cong (\mathbb{C}^*)^2 \times \mathcal{B}K$. There is a primitive $\mathsf{u}_3 \in M = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \cong \mathbb{Z}^3$ such that $\mathrm{Ker}(\mathsf{u}_3) = \mathbb{T}'$. Define $M' := M/\langle \mathsf{u}_3 \rangle \cong \mathbb{Z}^2$. Then $\bar{N}' := \mathsf{u}_3^{\perp} = \{\mathsf{v} \in \bar{N} : \langle \mathsf{u}_3, \mathsf{v}\rangle = 0\}$ is the dual lattice of $M' = \mathrm{Hom}(\mathbb{T}', \mathbb{C}^*)$.

The simplicial fan Σ is determined by a convex polytope $\Delta \subset N'_{\mathbb{R}} = \bar{N}' \otimes_{\mathbb{Z}} \mathbb{R}$ together with a triangulation of Δ, where all the vertices are in the lattice $\bar{N}$. The fan Σ is a cone over this triangulation in $N'_{1,\mathbb{R}} \subset N_{\mathbb{R}}$, where $N'_{1,\mathbb{R}} = \{\mathsf{v} \in N_{\mathbb{R}} : \langle \mathsf{u}_3, \mathsf{v}\rangle = 1\}$.

3.2. Toric graph

Let $\mathbb{T}_{\mathbb{R}} \cong U(1)^3$ (respectively, $\mathbb{T}'_{\mathbb{R}} \cong U(1)^2$) be the maximal compact subgroup of $\mathbb{T} \cong (\mathbb{C}^*)^3$ (respectively, $\mathbb{T}' \cong (\mathbb{C}^*)^2$), and we choose an $\mathbf{r}$ in the extended Kähler cone. The $\mathbb{T}$-action on $\mathcal{X}^{\mathrm{rig}}$ restricts to a Hamiltonian $\mathbb{T}_{\mathbb{R}}$-action on the Kähler orbifold $(\mathcal{X}^{\mathrm{rig}}, \omega(\mathbf{r}))$. Since $M_{\mathbb{R}}$ (respectively, $M'_{\mathbb{R}}$) is canonically identified with the dual of the Lie algebra of $\mathbb{T}_{\mathbb{R}}$ (respectively, $\mathbb{T}'_{\mathbb{R}}$), the Kähler form $\omega(\mathbf{r})$ determines a moment map $\mu_{\mathbb{T}_{\mathbb{R}}} : \mathcal{X}^{\mathrm{rig}} \longrightarrow M_{\mathbb{R}}$ up to translation by a vector in $M_{\mathbb{R}}$. The image $\mu_{\mathbb{T}_{\mathbb{R}}}(\mathcal{X}^{\mathrm{rig}})$ is a convex polyhedron. The moment map $\mu_{\mathbb{T}'_{\mathbb{R}}} : \mathcal{X}^{\mathrm{rig}} \longrightarrow M'_{\mathbb{R}}$ is the composition $\pi \circ \mu_{T_{\mathbb{R}}}$, where $\pi : M_{\mathbb{R}} \cong \mathbb{R}^3 \to M'_{\mathbb{R}} \cong \mathbb{R}^2$ is the projection. The map $\mu_{\mathbb{T}'_{\mathbb{R}}}$ is surjective. Let $\mathcal{X}_1^{\mathrm{rig}} \subset \mathcal{X}^{\mathrm{rig}}$ be the union of 0-dimensional and 1-dimensional $\mathbb{T}$-orbits in $\mathcal{X}^{\mathrm{rig}}$. The toric graph is defined by $\Gamma := \mu_{T'_{\mathbb{R}}}(\mathcal{X}_1^{\mathrm{rig}}) \subset M'_{\mathbb{R}} \cong \mathbb{R}^2$. It is determined by the Kähler class $[\omega(\mathbf{r})] \in H^2(\mathcal{X}^{\mathrm{rig}}; \mathbb{R}) = H^2(X_\Sigma; \mathbb{R})$ up to translation by a vector in $M'_{\mathbb{R}}$. The vertices (respectively, edges) of Γ are in one-to-one correspondence to 3-dimensional (respectively, 2-dimensional) cones in Σ. Conversely, the Kähler class $[\omega(\mathbf{r})] \in H^2(\mathcal{X}^{\mathrm{rig}}; \mathbb{R})$ is determined by the toric graph.

Pulling back under the map $\mathcal{X} \longrightarrow \mathcal{X}^{\mathrm{rig}}$ defines a one-to-one correspondence between Kähler forms/classes on $\mathcal{X}$ and on its rigidification $\mathcal{X}^{\mathrm{rig}}$.

3.3. Aganagic-Vafa A-branes

In [6], Aganagic-Vafa introduced a class of Lagrangian submanifolds of semi-projective smooth toric Calabi-Yau 3-folds. In this section, we generalise this construction and define Aganagic-Vafa A-branes in a general 3-dimensional Calabi-Yau smooth toric DM stack with semi-projective coarse moduli space.

Let $\mathcal{X} = [\widetilde{\mu}^{-1}(\mathbf{r})/G_{\mathbb{R}}]$ be a 3-dimensional Calabi-Yau smooth toric DM stack, where

$$\mathbf{r} = \sum_{a=1}^{k} r_a e_a^{\vee} \in \widetilde{C}(\mathcal{X}) \subset \mathbb{L}_{\mathbb{R}}^{\vee},$$

and $\widetilde{\mu}^{-1}(\mathbf{r}) \subset \mathbb{C}^{k+3}$ is defined by the following equations:

$$\sum_{i=1}^{k+3} l_i^{(a)} |X_i|^2 = r_a, \quad a = 1, \ldots, k.$$

Write $X_i = \rho_i e^{\sqrt{-1}\phi_i}$, where $\rho_i = |X_i|$. An Aganagic-Vafa brane is a Lagrangian suborbifold of $\mathcal{X}$ of the form

$$\mathcal{L} = [\widetilde{L}/G_{\mathbb{R}}],$$

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where

$$\widetilde{L} = \left\{ (X_1, \dots, X_{k+3}) \in \widetilde{\mu}^{-1}(\mathbf{r}) : \sum_{i=1}^{k+3} \hat{l}_i^1 |X_i|^2 = c_1, \sum_{i=1}^{k+3} \hat{l}_i^2 |X_i|^2 = c_2, \sum_{i=1}^{k+3} \phi_i = \text{const} \right\}$$

for some $\hat{l}_i^\alpha \in \mathbb{Z}$, $\sum_{i=1}^{k+3} \hat{l}_i^\alpha = 0$, $\alpha = 1, 2$. Note that the action of $G_\mathbb{R}$ on $\mathbb{C}^{k+3}$ preserves the subsets $\widetilde{\mu}^{-1}(\mathbf{r})$ and $\widetilde{L}$. If we view $\mathcal{X} = [\widetilde{\mu}^{-1}(\mathbf{r})/G_\mathbb{R}]$ as a Lie groupoid (and in particular a category), then $\mathcal{L} = [\widetilde{L}/G_\mathbb{R}]$ is a full subcategory.

An Aganagic-Vafa brane $\mathcal{L}$ intersects a unique 1-dimensional orbit $\mathfrak{o}_\tau \cong \mathbb{C}^* \times \mathcal{B}G_\tau$ along $\mathcal{S}_\tau := \mathcal{L} \cap \mathfrak{o}_\tau \cong S^1 \times \mathcal{B}G_\tau$. The inclusion $\mathcal{S}_\tau \subset \mathcal{L}$ is a homotopic equivalence, so the fundamental group of $\mathcal{L}$ is

$$\pi_1(\mathcal{L}) \cong \pi_1(S^1 \times \mathcal{B}G_\tau) \cong \mathbb{Z} \times G_\tau.$$

In particular, it is abelian, so it is isomorphic to its abelianisation $H_1(\mathcal{L}; \mathbb{Z})$.

If $(\tau, \sigma) \in F(\Sigma)$, then there is an inclusion $\iota^{(\tau,\sigma)} : \mathcal{S}_\tau \hookrightarrow \mathcal{X}_\sigma = [\mathbb{C}^3/G_\sigma]$ that induces

$$\iota_*^{(\tau,\sigma)} : \pi_1(\mathcal{S}_\tau) \cong \mathbb{Z} \times G_\tau \to \pi_1(\mathcal{X}_\sigma) \cong G_\sigma.$$

3.4. Moduli spaces of stable maps to $(\mathcal{X}, \mathcal{L})$

In [49], Katz-Liu introduced stable maps to a symplectic manifold with Lagrangian boundary conditions at all genera; the domain of such a map is a prestable bordered Riemann surface: that is, a smooth or nodal bordered Riemann surface. (See also [55], [38].) In [24, Section 2], Cho-Poddar define stable maps to a symplectic *orbifold* $\mathcal{X}$ with Lagrangian boundary conditions, under the assumption that the Lagrangian suborbifold $\mathcal{L}$ does not contain any stacky points (so that $\mathcal{L}$ is indeed a smooth manifold); the domain of such a map is a prestable bordered *orbifold* Riemann surface in the sense of [24, Section 2]: that is, a smooth or nodal bordered orbifold Riemann surface, where a stacky point is either an interior marked point or an interior node.

In general, $\mathcal{L}$ is a suborbifold that contains stacky points. To obtain compactness of the moduli spaces when $\mathcal{X}$ and $\mathcal{L}$ are compact, one needs to allow orbifold structures at boundary marked points and boundary nodes. In the present paper, $\mathcal{L}$ may contain stacky points, but we do not need to allow orbifold structures on the boundary of the domain for the following two reasons:

(i) Our enumerative problem only requires interior insertions, so we do not need to introduce any boundary marked points.
(ii) In our case, $\mathcal{X}$ and $\mathcal{L}$ are noncompact, and we will define and compute open GW invariants by torus localisation on moduli space of stable maps $\mathcal{X}$ with boundaries in $\mathcal{L}$. If a stable map represents a torus fixed point in the moduli space, then any node in the domain must be mapped to a torus fixed (scheme or stacky) point in $\mathcal{X}$, but $\mathcal{L}$ does not contain any torus fixed point, so the domain does not contain any boundary nodes.

Let $(\Sigma, x_1, \dots, x_n)$ be a prestable bordered orbifold Riemann surface with n interior marked point. Then the coarse moduli space $(\bar{\Sigma}, \bar{x}_1, \dots, \bar{x}_n)$ is a prestable bordered Riemann surface with n interior marked points, defined in [49, Section 3.6] and [55, Section 3.2]. We define the topological type (g, h) of Σ to be the topological type of $\bar{\Sigma}$ (see [55, Section 3.2]).

Let $(\Sigma, \partial\Sigma)$ be a prestable bordered orbifold Riemann surface of type (g, h), and let $\partial\Sigma = R_1 \cup \cdots \cup R_h$ be union of connected components. Each connected component is a circle that contains no orbifold points. A (bordered) *prestable map* to the pair $(\mathcal{X}, \mathcal{L})$ is a map $u : (\Sigma, \partial\Sigma) \to (\mathcal{X}, \mathcal{L})$, where Σ is a prestable bordered orbifold Riemann surface such that $u \circ \nu : \hat{\Sigma} \to \mathcal{X}$ is holomorphic, where $\nu : \hat{\Sigma} \to \Sigma$ is the normalisation (so $\hat{\Sigma}$ is a possibly disconnected smooth bordered orbifold Riemann surface); a prestable map to $(\mathcal{X}, \mathcal{L})$ is *stable* if its automorphism group is finite. The topological type of a stable map u is given by the degree $\beta' = \bar{u}_*[\bar{\Sigma}] \in H_2(X, L; \mathbb{Z})$ (where X and L are the coarse

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moduli spaces of $\mathcal{X}$ and $\mathcal{L}$ respectively, and $\bar{u} : \bar{\Sigma} \to X$ is the map between coarse moduli spaces) and $\vec{\mu}_i = u_*[R_i] = (\mu_i, \lambda_i) \in H_1(\mathcal{L};\mathbb{Z}) \cong \mathbb{Z} \times G_\tau$ (where $\mu_i \in \mathbb{Z}$ is the winding number and $\lambda_i \in G_\tau$ is the monodromy). Given $\beta' \in H_2(X, L;\mathbb{Z})$ and

$$\vec{\mu} = ((\mu_1, \lambda_1), \ldots, (\mu_h, \lambda_h)) \in H_1(\mathcal{L};\mathbb{Z})^h.$$

Let $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ be the moduli space of stable maps of type (g, h), degree β', winding numbers $\mu_i \in \mathbb{Z}$ and monodromies $\lambda_i \in G_\tau$, with n interior marked points.

3.5. The tangent-obstruction complex and the virtual dimension

Similar to [49, Section 4.2], the tangent space $\mathcal{T}^1_\xi$ and the obstruction space $\mathcal{T}^2_\xi$ at a moduli point

$$\xi = [u : ((\Sigma, x_1, \ldots, x_n), \partial\Sigma) \to (\mathcal{X}, \mathcal{L})] \in \overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$$

fit into the following exact sequence of real vector spaces:

$$\begin{aligned} 0 \to &\mathrm{Aut}((\Sigma, x_1, \ldots, x_n), \partial\Sigma) \to H^0(\Sigma, \partial\Sigma, u^*T_{\mathcal{X}}, (u|_{\partial\Sigma})^*T_{\mathcal{L}}) \to \mathcal{T}^1_\xi \\ \to &\mathrm{Def}((\Sigma, x_1, \ldots, x_n), \partial\Sigma) \to H^1(\Sigma, \partial\Sigma, u^*T_{\mathcal{X}}, (u|_{\partial\Sigma})^*T_{\mathcal{L}}) \to \mathcal{T}^2_\xi, \end{aligned} \tag{7}$$

where

- $\mathrm{Aut}((\Sigma, x_1, \ldots, x_n), \partial\Sigma)$ is the space of infinitesimal automorphism of the domain $((\Sigma, x_1, \ldots, x_n), \partial\Sigma)$ and is equal to $H^0(\Sigma, \partial\Sigma, T_\Sigma(-\sum_{j=1}^n x_j), T_{\partial\Sigma})$ when Σ is a smooth bordered orbifold Riemann surface;
- $\mathrm{Def}((\Sigma, x_1, \ldots, x_n), \partial\Sigma)$ is the space of infinitesimal deformations of the domain, and is equal to $H^1(\Sigma, \partial\Sigma, T_\Sigma(-\sum_{j=1}^n x_i), T_{\partial\Sigma})$ when Σ is a smooth bordered orbifold Riemann surface;
- $H^0(\Sigma, \partial\Sigma, u^*T_{\mathcal{X}}, (u|_{\partial\Sigma})^*T_{\mathcal{L}})$ is the space of infinitesimal deformation of the map for a fixed domain;
- $H^1(\Sigma, \partial\Sigma, u^*T_{\mathcal{X}}, (u|_{\partial\Sigma})^*T_{\mathcal{L}})$ is the space of obstructions to deforming the map for a fixed domain.

Globally on the moduli space $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L}, \beta', \vec{\mu})$, there is an exact sequence of sheaves

$$0 \to B_1 \to B_2 \to \mathcal{T}^1 \to B_4 \to B_5 \to \mathcal{T}^2 \to 0 \tag{8}$$

whose fibre at the moduli point ξ is equation (7).

Let $\mathfrak{M}_{(g,h),n}$ be the moduli of prestable bordered orbifold Riemann surfaces of type (g, h) with n interior marked point. Then $\mathfrak{M}_{(g,h),n}$ is a differentiable stack (with corners) of real dimension

$$3(2g - 2 + h) + 2n = \dim_{\mathbb{R}} \mathrm{Def}((\Sigma, x_1, \ldots, x_n), \partial\Sigma) - \dim_{\mathbb{R}} \mathrm{Aut}((\Sigma, x_1, \ldots, x_n), \partial\Sigma). \tag{9}$$

There are evaluation maps (at interior marked points)

$$\mathrm{ev}_j : \overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu}) \to \mathcal{I}\mathcal{X}, \quad j = 1, \ldots, n.$$

Given $\vec{v} = (v_1, \ldots, v_n)$, where $v_1, \ldots, v_n \in \mathrm{Box}(\mathbf{\Sigma})$, define

$$\overline{\mathcal{M}}_{(g,h),\vec{v}}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu}) := \bigcap_{j=1}^n \mathrm{ev}_j^{-1}(\mathcal{X}_{v_j}).$$

Suppose that $\xi \in \overline{\mathcal{M}}_{(g,h),\vec{v}}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$. By the Riemann-Roch theorem for prestable bordered orbifold Riemann surface (which can be derived by combining the proof of the Riemann-Roch theorem for prestable bordered Riemann surfaces and prestable orbifold closed Riemann surfaces),

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$$\begin{aligned}&\dim_{\mathbb{R}} H^0(\Sigma, \partial\Sigma, u^*T_{\mathcal{X}}, (u|_{\partial\Sigma})^*T_{\mathcal{L}}) - \dim_{\mathbb{R}} H^1(\Sigma, \partial\Sigma, u^*T_{\mathcal{X}}, (u|_{\partial\Sigma})^*T_{\mathcal{L}})\\ &\quad = 3(2-2g-h) - 2\sum_{j=1}^{n} \mathrm{age}(v_j).\end{aligned} \tag{10}$$

The above equation (10) is the relative virtual dimension of $\overline{\mathcal{M}}_{(g,h),\vec{v}}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu}) \to \mathfrak{M}_{(g,h),n}$ that sends a stable map to its domain. The virtual (real) dimension of $\overline{\mathcal{M}}_{(g,h),\vec{v}}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ is equal to

$$\dim_{\mathbb{R}} \mathcal{T}^1_{\xi} - \dim_{\mathbb{R}} \mathcal{T}^2_{\xi} = 2\sum_{j=1}^{n}(1 - \mathrm{age}(v_j)),$$

where $\mathrm{age}(v_j) \in \{0, 1, 2\}$.

3.6. Torus action and equivariant invariants

Let $\mathbb{T}'_{\mathbb{R}} \cong U(1)^2$ be the maximal compact subgroup of $\mathbb{T}' \cong (\mathbb{C}^*)^2$. For any $t \in \mathbb{T}'_{\mathbb{R}}$, the map $\phi_t : \mathcal{X} \to \mathcal{X}$ given by $x \mapsto t \cdot x$ is an automorphism of the smooth toric DM stack $\mathcal{X}$, and $\phi_t(\mathcal{L}) = \mathcal{L}$, so $\mathbb{T}'_{\mathbb{R}}$ acts on the moduli spaces $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$; here we use the notion of group actions on stacks in [65]. Let $F \subset \overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ be the substack of $\mathbb{T}'_{\mathbb{R}}$ fixed points. The restriction of the exact sequence equation (8) to the substack F is the direct sum of two exact sequences

$$0 \to B_1^f \to B_2^f \to \mathcal{T}^{1,f} \to B_4^f \to B_5^f \to \mathcal{T}^{2,f} \to 0, \tag{11}$$

$$0 \to B_1^m \to B_2^m \to \mathcal{T}^{1,m} \to B_4^m \to B_5^m \to \mathcal{T}^{2,m} \to 0, \tag{12}$$

where equation (11) is the subcomplex fixed by the torus action. The virtual tangent bundle $\mathcal{T}_F^{\mathrm{vir}}$ of F is

$$\mathcal{T}_F^{\mathrm{vir}} = \mathcal{T}^{1,f} - \mathcal{T}^{2,f}$$

whose ranks can be different on different connected components of F. We will see that each connected component of F is a compact orbifold and that $\mathcal{T}_F^{\mathrm{vir}}$ is equal to the tangent bundle $\mathcal{T}_F$ of F. So

$$[F]^{\mathrm{vir}} = [F].$$

The virtual normal bundle N^{vir} of F in $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ is

$$N^{\mathrm{vir}} = \mathcal{T}^{1,m} - \mathcal{T}^{2,m}.$$

Given $\gamma_1, \ldots, \gamma_n \in H^*_{\mathbb{T}',\mathrm{CR}}(\mathcal{X}; \mathbb{Q}) = H^*_{\mathbb{T}'_{\mathbb{R}},\mathrm{CR}}(\mathcal{X}, \mathbb{Q})$, we define

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{\mathcal{X},\mathcal{L},\mathbb{T}'_{\mathbb{R}}}_{g,\beta',\vec{\mu}} := \int_{[F]^{\mathrm{vir}}} \frac{\prod_{j=1}^{n} (\mathrm{ev}_j^* \gamma_i)|_F}{e_{\mathbb{T}'_{\mathbb{R}}}(N^{\mathrm{vir}})} \in \mathcal{Q}_{\mathbb{T}'_{\mathbb{R}}}, \tag{13}$$

where $\mathcal{Q}_{\mathbb{T}'_{\mathbb{R}}}$ is the fractional field of $H^*_{\mathbb{T}'_{\mathbb{R}}}(\mathrm{point}; \mathbb{Q})$, and

$$\frac{1}{e_{\mathbb{T}'_{\mathbb{R}}}(N_F^{\mathrm{vir}})} = \frac{e_{\mathbb{T}'_{\mathbb{R}}}(\mathcal{T}^{2,m})}{e_{\mathbb{T}'_{\mathbb{R}}}(\mathcal{T}^{1,m})} = \frac{e_{\mathbb{T}'_{\mathbb{R}}}(B_1^m) e_{\mathbb{T}'_{\mathbb{R}}}(B_5^m)}{e_{\mathbb{T}'_{\mathbb{R}}}(B_2^m) e_{\mathbb{T}'_{\mathbb{R}}}(B_4^m)}.$$

More precisely, the definition in equation (13) also requires an orientation on the virtual tangent bundle $\mathcal{T}^1 - \mathcal{T}^2$, which we will specify later.

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3.7. Tangent weights: the 3-torus, the Calabi-Yau 2-torus, and the framing 1-torus

Let $\mathfrak{o}_\tau \cong \mathbb{C}^* \times \mathcal{B}G_\tau$ be the unique 1-dimensional $\mathbb{T}$-orbit that intersects the Aganagic-Vafa A-brane $\mathcal{L}$, where $\tau \in \Sigma(2)$, as before. Let $\mathfrak{l}_\tau$ be the closure of $\mathfrak{o}_\tau$, and let ℓ_τ be the coarse moduli of $\mathfrak{l}_\tau$. Then ℓ_τ is either $\mathbb{P}^1$ or $\mathbb{C}$.

Definition 3.1. We say $\mathcal{L}$ is an *inner* brane if $\ell_\tau \cong \mathbb{P}^1$; we say $\mathcal{L}$ is an *outer* brane if $\ell_\tau \cong \mathbb{C}$.

If $\mathcal{L}$ is an outer brane, let $\sigma \in \Sigma(3)$ be the unique 3-cone such that $(\tau, \sigma) \in F(\Sigma)$; if $\mathcal{L}$ is an inner brane, we choose $\sigma \in \Sigma(3)$ such that $(\tau, \sigma) \in F(\Sigma)$, and let $\sigma_- \in \Sigma(3)$ denote the other choice, so that $\mathfrak{p}_\sigma$ and $\mathfrak{p}_{\sigma_-}$ are the two torus fixed points in $\mathfrak{l}_\tau$. For inner branes, we also denote $\sigma_+ = \sigma$.

By permuting $b_1, \dots, b_r$ if necessary, we may assume that $I'_\sigma = \{1,2,3\}$, and $(\tau_1, \sigma) = (\tau, \sigma)$, (τ_2, σ) and (τ_3, σ) are three flags in the toric graph in the counterclockwise direction such that

$$I'_{\tau_1} = \{2,3\}, \quad I'_{\tau_2} = \{3,1\}, \quad I'_{\tau_3} = \{1,2\}.$$

Here we fixed an orientation of $\mathbb{R}^2$. If $\mathcal{L}$ is an inner brane, we assume in addition $I'_{\sigma_-} = \{2,3,4\}$.

Recall from Section 2.4 that for any flag $(\tau, \sigma) \in F(\Sigma)$, $\chi_{(\tau,\sigma)} \in \mathrm{Hom}(G_\sigma, \mathbb{C}^*)$ is the character of the 1-dimensional G_σ representation $T_{\mathfrak{p}_\sigma}\mathfrak{l}_\tau$. Let

$$\mathfrak{r} := r(\tau,\sigma) = |G_\sigma / G_\tau|, \quad \mathfrak{m} := |G_\tau / K|.$$

Then we have the following two short exact sequences of finite abelian groups:

$$1 \to G_\tau \longrightarrow G_\sigma \xrightarrow{\chi_{(\tau,\sigma)}} \mu_{\mathfrak{r}} \to 1, \quad 1 \to K \longrightarrow G_\tau \xrightarrow{\chi_{(\tau_3,\sigma)}} \mu_{\mathfrak{m}} \to 1.$$

Note that for any $\lambda \in G_\tau$, $\chi_{(\tau,\sigma)}(\lambda) = 1$ and $\chi_{(\tau_2,\sigma)}(\lambda)\chi_{(\tau_3,\sigma)}(\lambda) = 1$. Let $\bar{\lambda}$ denote the unique element in $\{0, 1, \dots, \mathfrak{m}-1\}$ such that

$$\chi_3(\lambda) = e^{2\pi\sqrt{-1}\bar{\lambda}/\mathfrak{m}}.$$

Let $\mathsf{u}_3 \in M$ be defined as in Section 3.1, so that $\langle \mathsf{u}_3, \bar{b}_i \rangle = 1$. We may choose a $\mathbb{Z}$-basis $\{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3\}$ of $\bar{N}$ such that $\langle \mathsf{u}_3, \mathsf{v}_i \rangle = \delta_{i,3}$, and

$$\bar{b}_1 = \mathfrak{r}\mathsf{v}_1 - \mathfrak{s}\mathsf{v}_2 + \mathsf{v}_3, \quad \bar{b}_2 = \mathfrak{m}\mathsf{v}_2 + \mathsf{v}_3, \quad \bar{b}_3 = \mathsf{v}_3.$$

Moreover, the choice $(\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3)$ is unique if we require $\mathfrak{s} \in \{0, 1, \dots, \mathfrak{r}-1\}$. Let $\{\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3\}$ be the $\mathbb{Z}$-basis of M that is dual to the $\mathbb{Z}$-basis $\{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_2\}$ of $\bar{N}$. Let $\{\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3\}$ be the $\mathbb{Q}$-basis of $M_{\mathbb{Q}}$ that is dual to the $\mathbb{Q}$-basis $\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}$ of $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Then

$$\mathsf{w}_1 = \frac{1}{\mathfrak{r}}\mathsf{u}_1, \; \mathsf{w}_2 = \frac{\mathfrak{s}}{\mathfrak{r}\mathfrak{m}}\mathsf{u}_1 + \frac{1}{\mathfrak{m}}\mathsf{u}_2, \; \mathsf{w}_3 = -\frac{\mathfrak{s}+\mathfrak{m}}{\mathfrak{r}\mathfrak{m}}\mathsf{u}_1 - \frac{1}{\mathfrak{m}}\mathsf{u}_2 + \mathsf{u}_3.$$

Moreover, for $i \in \{1,2,3\}$,

$$\mathsf{w}_i = e_{\mathcal{T}}(T_{\mathfrak{p}_\sigma}\mathfrak{l}_{\tau_i}) = e_{\mathcal{T}}(\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i))\Big|_{\mathfrak{p}_\sigma}.$$

The inclusion $\mathbb{T}' \subset \mathbb{T}$ induces the following surjective ring homomorphism

$$H^*(\mathcal{B}\mathbb{T}; R) = R[\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3] \longrightarrow H^*(\mathcal{B}\mathbb{T}'; R) = R[\mathsf{u}'_1, \mathsf{u}'_2], \quad \mathsf{u}_1 \mapsto \mathsf{u}'_1, \; \mathsf{u}_2 \mapsto \mathsf{u}'_2, \; \mathsf{u}_3 \mapsto 0, \tag{14}$$

where $R = \mathbb{Z}$ or $\mathbb{Q}$.

Given a *framing* that is an integer $f \in \mathbb{Z}$, let $\mathbb{T}_f \subset \mathbb{T}'$ be the kernel of the character $\mathsf{u}_2' - f\mathsf{u}_1' \in \mathrm{Hom}(\mathbb{T}';\mathbb{C}^*)$. Then $\mathbb{T}_f \cong \mathbb{C}^*$ is a 1-dimensional subtorus of the Calabi-Yau torus $\mathbb{T}'$. The inclusion $\mathbb{T}_f \subset \mathbb{T}'$ induces a surjective ring homomorphism

$$H^*(\mathcal{B}\mathbb{T}';R) = R[\mathsf{u}_1',\mathsf{u}_2'] \longrightarrow H^*(\mathcal{B}\mathbb{T}_f;R) = R[\mathsf{u}], \quad \mathsf{u}_1' \mapsto \mathsf{u},\ \mathsf{u}_2' \mapsto f\mathsf{u}, \tag{15}$$

where $R = \mathbb{Z}$ or $\mathbb{Q}$. For $i = 1,2,3$, let w_i' denote the image of w_i under the ring homomorphism equation (14), and let w_i^f denote the image of w_i' under the ring homomorphism in (15). Then

$$\mathsf{w}_1' = \frac{1}{\mathfrak{r}}\mathsf{u}_1',\ \mathsf{w}_2' = \frac{\mathfrak{s}}{\mathfrak{r}\mathfrak{m}}\mathsf{u}_1' + \frac{1}{\mathfrak{m}}\mathsf{u}_2',\ \mathsf{w}_3' = -\frac{\mathfrak{s}+\mathfrak{m}}{\mathfrak{r}\mathfrak{m}}\mathsf{u}_1' - \frac{1}{\mathfrak{m}}\mathsf{u}_2' = -\mathsf{w}_1' - \mathsf{w}_2' \in H^2(\mathcal{B}\mathbb{T}') = \mathbb{Q}\mathsf{u}_1' \oplus \mathbb{Q}\mathsf{u}_2', \tag{16}$$

and $\mathsf{w}_i^f = w_i\mathsf{u}$, where $w_i \in \mathbb{Q}$ are given by

$$w_1 = \frac{1}{\mathfrak{r}}, \quad w_2 = \frac{\mathfrak{s}+\mathfrak{r}f}{\mathfrak{r}\mathfrak{m}}, \quad w_3 = -w_1 - w_2 = \frac{-\mathfrak{m}-\mathfrak{s}-\mathfrak{r}f}{\mathfrak{r}\mathfrak{m}}.$$

3.8. *Disk factors as equivariant open GW invariants*

A framed Aganagic-Vafa Lagrangian brane is a pair $(\mathcal{L}, f)$, where $\mathcal{L}$ is an Aganagic-Vafa brane together with a choice of a flag $(\tau,\sigma) \in F(\Sigma)$ such that $\mathfrak{o}_\tau$ is the unique 1-dimensional orbit intersecting $\mathcal{L}$ and a choice of framing $f \in \mathbb{Z}$. Given a framed Aganagic-Vafa Lagrangian brane $(\mathcal{L}, f)$, we choose an isomorphism $\pi_1(\mathcal{L}) \cong \mathbb{Z}\times G_\tau$ such that if $h = \iota_*^{(\tau,\sigma)}(d_0,\lambda)$ (where $\iota_*^{(\tau,\sigma)}$ is defined in Section 3.3); then

$$\chi_{(\tau_1,\sigma)}(h) = e^{2\pi\sqrt{-1}d_0 w_1}, \quad \chi_{(\tau_2,\sigma)}(h) = e^{2\pi\sqrt{-1}d_0(w_2 - \frac{\bar{\lambda}}{\mathfrak{m}})}, \quad \chi_{(\tau_3,\sigma)}(h) = e^{2\pi\sqrt{-1}d_0(w_3 + \frac{\bar{\lambda}}{\mathfrak{m}})}.$$

Let ℓ_τ be the coarse moduli of $\mathfrak{l}_\tau$, as before. Let $p_\sigma \in \ell_\tau$ be the coarse moduli of $\mathfrak{p}_\sigma \cong \mathcal{B}G_\sigma$, and let $S_\tau := L \cap \ell_\tau \cong S^1$ be the coarse moduli of $\mathcal{S}_\tau = \mathcal{L} \cap \mathfrak{l}_\tau \cong S^1 \times \mathcal{B}G_\tau$.

3.8.1. $(\mathcal{L}, f)$ is a framed outer brane

In this case $\ell_\tau = \mathbb{C}$. Let $D \subset \ell_\tau$ be the disk that contains p_σ with boundary S_τ, oriented by the complex structure on ℓ_τ, and let $b = [D] \in H_2(X, L;\mathbb{Z})$. Given $(d_0,\lambda) \in H_1(\mathcal{L};\mathbb{Z}) \cong \mathbb{Z}\times G_\tau$, where $d_0 > 0$, define

$$\overline{\mathcal{M}}(d_0,\lambda) := \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X},\mathcal{L} \mid d_0 b, (d_0,\lambda)).$$

The virtual real dimension of $\overline{\mathcal{M}}(d_0,\lambda)$ is $2(1-\mathrm{age}(h(d_0,\lambda)))$, where $h(d_0,\lambda) := \iota_*^{(\tau,\sigma)}(d_0,\lambda) \in G_\sigma$. Define the disk factor

$$D_{d_0,\lambda} := \langle \mathbf{1}_{h(d_0,\lambda)} \rangle^{\mathcal{X},\mathcal{L},\mathbb{T}_\mathbb{R}'}_{0,d_0 b,(d_0,\lambda)},$$

which is a rational function in $\mathsf{w}_1', \mathsf{w}_2'$, homogeneous of degree $\mathrm{age}(h(d_0,\lambda)) - 1$. The disk factor is computed in [11] when G_σ is cyclic and in [66, Section 3.3] for general G_σ. In our notation, the formula in [66, Section 3.3] says[2]

$$D_{d_0,\lambda} = \left(\frac{\mathfrak{r}\mathsf{w}_1'}{d_0}\right)^{\mathrm{age}(h(d_0,\lambda))-1} \frac{1}{d_0|G_\tau|} \cdot \frac{\prod_{a=1}^{\lfloor \frac{d_0}{\mathfrak{r}} \rfloor + \mathrm{age}(h(d_0,\lambda))-1} \left(\frac{d_0\mathsf{w}_2'}{\mathfrak{r}\mathsf{w}_1'} + a - c_2(h(d_0,\lambda))\right)}{\lfloor \frac{d_0}{\mathfrak{r}} \rfloor !}, \tag{17}$$

[2]The disk function in [66, Section 3.3] and our disk factor are the same when $h(d_0,\lambda) \neq 0$. When $h(d_0,\lambda) = 0$, the disk function is $\langle\ \rangle^{\mathcal{X},\mathcal{L}}_{\dots}$ (no insertion), while the disk factor is $\langle 1 \rangle^{\mathcal{X},\mathcal{L}}_{\dots}$ (one insertion of 1), so there is an additional factor of $(\frac{\mathfrak{r}}{\mathsf{w}_1})^{\delta_{0,h(d_0,\lambda)}}$ in the disk function in [66, Section 3.3].

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where $c_i(\cdot) \in \mathbb{Q} \cap [0,1)$ is defined in Section 2.7. More explicitly,

$$c_2(h(d_0,\lambda)) = \left\langle d_0 w_2 - \frac{\bar{\lambda}}{\mathfrak{m}} \right\rangle.$$

3.8.2. $(\mathcal{L}, f)$ **is a framed inner brane**

In this case $\ell_\tau \cong \mathbb{P}^1$. It contains two torus fixed points $p_+ = p_\sigma$ and $p_- = p_{\sigma_-}$, where $\sigma_- \in \Sigma(3)$. The circle S_τ is the intersection of two disks D_+ and D_- that contain p_+ and p_-, respectively. Let

$$b = [D] \in H_2(X, L; \mathbb{Z}), \quad \alpha = [\ell_\tau] \in H_2(X; \mathbb{Z}).$$

Then $[D_-] = \alpha - b \in H_2(X, L; \mathbb{Z})$. Given $(d_0, \lambda) \in H_1(\mathcal{L}; \mathbb{Z}) \cong \mathbb{Z} \times G_\tau$, where $d_0 \neq 0$, we define

$$\overline{\mathcal{M}}(d_0,\lambda) := \begin{cases} \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X}, \mathcal{L} \mid d_0 b, (d_0,\lambda)), & d_0 > 0, \\ \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X}, \mathcal{L} \mid -d_0(\alpha - b), (d_0,\lambda)), & d_0 < 0. \end{cases}$$

Then

$$\text{virtual dimension of } \overline{\mathcal{M}}(d_0,\lambda) = \begin{cases} 1 - \mathrm{age}(h^+(d_0,\lambda)), & d_0 > 0, \\ 1 - \mathrm{age}(h^-(d_0,\lambda)), & d_0 < 0, \end{cases}$$

where $h^\pm(d_0,\lambda) = \iota_*^{(\tau,\sigma_\pm)}(d_0,\lambda) \in G_{\sigma_\pm}$. Define

$$D_{d_0,\lambda} := \begin{cases} \langle 1_{h^+(d_0,\lambda)} \rangle^{\mathcal{X},\mathcal{L},\mathbb{T}'_{\mathbb{R}}}_{0,d_0 b,(d_0,\lambda)}, & d_0 > 0, \\ \\ \langle 1_{h^-(d_0,\lambda)} \rangle^{\mathcal{X},\mathcal{L},\mathbb{T}'_{\mathbb{R}}}_{0,-d_0(\alpha-b),(d_0,\lambda)}, & d_0 < 0. \end{cases}.$$

Then $D_{d_0,\lambda}$ is a rational function in $\mathsf{u}_1', \mathsf{u}_2'$, homogeneous of degree $\mathrm{age}(h^\pm(d_0,\lambda)) - 1$ if $\pm d_0 > 0$.

More precisely, the disk factor $D_{d_0,\lambda}$ is defined up to a sign depending on choice of orientation of $\overline{\mathcal{M}}(d_0,\lambda)$, which will be clarified in Section 3.11 by relative GW invariants.

3.9. *Normal bundle to* $\mathfrak{l}_\tau$

Let $\mathcal{L}$ be an inner brane so that $\mathfrak{l}_\tau$ is a proper smooth toric DM curve. Let $\hat{\mathfrak{l}}_\tau$ be the image of $\mathfrak{l}_\tau$ under the morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{rig}}$. We have

$$\mathfrak{l}_\tau \longrightarrow \hat{\mathfrak{l}}_\tau \longrightarrow \mathfrak{l}_\tau^{\mathrm{rig}} \longrightarrow \ell_\tau \cong \mathbb{P}^1,$$

where $\mathfrak{l}_\tau \to \hat{\mathfrak{l}}_\tau$ is a K-banded gerbe, $\hat{\mathfrak{l}}_\tau \to \mathfrak{l}_\tau^{\mathrm{rig}}$ is a $\boldsymbol{\mu}_{\mathfrak{m}}$-banded gerbe and $\mathfrak{l}_\tau \to \mathfrak{l}_\tau^{\mathrm{rig}}$ is a G_τ-banded gerbe. The normal bundle $\mathfrak{l}_\tau$ in $\mathcal{X}$ is a direct sum of two $\mathbb{T}$-equivariant line bundles over $\mathfrak{l}_\tau$:

$$N_{\mathfrak{l}_\tau/\mathcal{X}} = L_2 \oplus L_3,$$

where $L_2 = \mathcal{O}_{\mathcal{X}}(\mathcal{D}_2)\big|_{\mathfrak{l}_\tau}$ and $L_3 = \mathcal{O}_{\mathcal{X}}(\mathcal{D}_3)\big|_{\mathfrak{l}_\tau}$. The total space of $N_{\mathfrak{l}_\tau/\mathcal{X}}$ is a smooth toric DM stack that is isomorphic to the open substack $\mathcal{Y} := \mathcal{X}_\sigma \cup \mathcal{X}_{\sigma_-}$ of $\mathcal{X}$. Let $\hat{\mathcal{D}}_i$ be the image of $\mathcal{D}_i$ under $\mathcal{X} \to \mathcal{X}^{\mathrm{rig}}$. Then

$$N_{\hat{\mathfrak{l}}_\tau/\mathcal{X}^{\mathrm{rig}}} = \hat{L}_2 \oplus \hat{L}_3,$$

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where $\hat{L}_2 = \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}(\hat{\mathcal{D}}_2)\big|_{\hat{\mathfrak{l}}_\tau}$ and $\hat{L}_3 = \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}(\hat{\mathcal{D}}_3)\big|_{\hat{\mathfrak{l}}_\tau}$. The total space of $N_{\hat{\mathfrak{l}}_\tau/\mathcal{X}^{\mathrm{rig}}}$ is a toric orbifold that is isomorphic to the open substack $\mathcal{Y}^{\mathrm{rig}} = \mathcal{X}^{\mathrm{rig}}_\sigma \cup \mathcal{X}^{\mathrm{rig}}_{\sigma_-}$ of the toric orbifold $\mathcal{X}^{\mathrm{rig}}$.

Let Σ_0 be the simplicial fan in $N_{\mathbb{R}}$ consisting of σ, σ_- and their subcones. The stacky fan of $\mathcal{Y}^{\mathrm{rig}}$ is given by $(\Sigma_0, (\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4))$, where

$$\bar{b}_1 = \mathfrak{r}\mathsf{v}_1 - \mathfrak{s}\mathsf{v}_2 + \mathsf{v}_3, \quad \bar{b}_2 = \mathfrak{m}\mathsf{v}_2 + \mathsf{v}_3, \quad \bar{b}_3 = \mathsf{v}_3, \quad \bar{b}_4 = -\mathfrak{r}_-\mathsf{v}_1 + c\mathsf{v}_2 + \mathsf{v}_3,$$

where c is some integer. For inner branes, we also denote $\mathfrak{r}_+ = \mathfrak{r}$. We have $\mathcal{Y} = [U/G_0]$, where

$$U = \{(Z_1, Z_2, Z_3, Z_4) \in \mathbb{C}^4 : (Z_1, Z_4) \neq (0,0)\},$$
$$G_0 = \{(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3, \tilde{t}_4) \in (\mathbb{C}^*)^4 : \tilde{t}_1^{\mathfrak{r}}(\tilde{t}_4)^{-\mathfrak{r}_-} = (\tilde{t}_1)^{-\mathfrak{s}}\tilde{t}_2^{\mathfrak{m}}\tilde{t}_4^{c} = \tilde{t}_1\tilde{t}_2\tilde{t}_3\tilde{t}_4 = 1\}.$$

We have a short exact sequence of abelian groups:

$$1 \to \boldsymbol{\mu}_{\mathfrak{m}} \longrightarrow G_0 \stackrel{\chi_1 \times \chi_4}{\longrightarrow} G_{\mathfrak{r},\mathfrak{r}_-} \to 1,$$

where $G_{\mathfrak{r},\mathfrak{r}_-} = \{(\tilde{t}_1, \tilde{t}_4) \in (\mathbb{C}^*)^2 : \tilde{t}_1^{\mathfrak{r}}(\tilde{t}_4)^{-\mathfrak{r}_-} = 1\}$ and $\chi_i(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3, \tilde{t}_4) = \tilde{t}_i$. The subgroup $\boldsymbol{\mu}_{\mathfrak{m}}$ of G_0 acts trivially on $V = \{(Z_1, Z_2, Z_3, Z_4) \in U_{\mathcal{A}} : Z_2 = Z_3 = 0\}$, so the G_0-action on V factors through a $G_{\mathfrak{r},\mathfrak{r}_-}$-action on V, and

$$\hat{\mathfrak{l}}_\tau = [V/G_0], \quad \mathfrak{l}_\tau^{\mathrm{rig}} = [V/G_{\mathfrak{r},\mathfrak{r}_-}] \cong \mathcal{F}_{\mathfrak{r},\mathfrak{r}_-},$$

where $\mathcal{F}_{\mathfrak{r},\mathfrak{r}_-}$ denotes the football obtained by glueing $[\mathbb{C}/\boldsymbol{\mu}_{\mathfrak{r}}]$ and $[\mathbb{C}/\boldsymbol{\mu}_{\mathfrak{r}_-}]$ along $[\mathbb{C}^*/\boldsymbol{\mu}_{\mathfrak{r}}] \cong [\mathbb{C}^*/\boldsymbol{\mu}_{\mathfrak{r}_-}] \cong \mathbb{C}^*$. The two torus fixed points in $\mathfrak{l}_\tau^{\mathrm{rig}}$ are

$$\mathfrak{p}_x = [(\{(0,0,0)\} \times \mathbb{C}^*)/G_{\mathfrak{r},\mathfrak{r}_-}] \cong \mathcal{B}\boldsymbol{\mu}_{\mathfrak{r}}, \quad \mathfrak{p}_y = [(\mathbb{C}^* \times \{(0,0,0)\})/G_{\mathfrak{r},\mathfrak{r}_-}] \cong \mathcal{B}\boldsymbol{\mu}_{\mathfrak{r}_-},$$

and $\mathfrak{l}_\tau^{\mathrm{rig}} - \{\mathfrak{p}_x, \mathfrak{p}_y\} \cong \mathbb{C}^*$. We have a surjective group homomorphism $\mathbb{Z}\oplus\mathbb{Z} \to \mathrm{Pic}(\mathfrak{l}_\tau^{\mathrm{rig}})$ sending (n_x, n_y) to $\mathcal{O}_{\mathfrak{l}_\tau^{\mathrm{rig}}}(n_x\mathfrak{p}_x + n_y\mathfrak{p}_y)$; the kernel is $\mathbb{Z}(\mathfrak{r}, -\mathfrak{r}_-)$.

Let $\mathcal{O}(-1)$ denote the tautological line bundle over $\mathcal{B}\mathbb{C}^*$ associated with the fundamental representation $\mathbb{C}^* \to GL(1,\mathbb{C})$, $t \mapsto t$. Given a line bundle L over a DM stack $\mathcal{Z}$ and a positive integer $\mathfrak{m}$, let $\sqrt[\mathfrak{m}]{L/\mathcal{Z}}$ denote the following fibre product (compare to [14, Definition 2.2.6]):

$$\begin{tikzcd}
\sqrt[\mathfrak{m}]{L/\mathcal{Z}} = \mathcal{Z} \times_{\mathcal{B}\mathbb{C}^*} \mathcal{B}\mathbb{C}^* \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{B}\mathbb{C}^* \arrow[d, "\otimes\mathfrak{m}"] \\
\mathcal{Z} \arrow[r, "\phi_L"] & \mathcal{B}\mathbb{C}^*
\end{tikzcd}$$

where the morphism $\phi_L : \mathcal{Z} \longrightarrow \mathcal{B}\mathbb{C}^*$ is defined by L (so that $\phi_L^*\mathcal{O}(-1) = L$), and $\mathcal{B}\mathbb{C}^* \to \mathcal{B}\mathbb{C}^*$ is induced by the $\mathfrak{m}$th power map from $\mathbb{C}^*$ to itself. Then $p_1 : \sqrt[\mathfrak{m}]{L/\mathcal{Z}} \to \mathcal{Z}$ is a $\boldsymbol{\mu}_{\mathfrak{m}}$-banded gerbe. Let $\sqrt[\mathfrak{m}]{L} := p_2^*\mathcal{O}(-1) \in \mathrm{Pic}(\sqrt[\mathfrak{m}]{L/\mathcal{Z}})$. Then $(\sqrt[\mathfrak{m}]{L})^{\otimes\mathfrak{m}} = p_1^*L$: that is, $\sqrt[\mathfrak{m}]{L}$ is an $\mathfrak{m}$th root of p_1^*L.

It is straightforward to check that

- $\hat{\mathfrak{l}}_\tau$ is isomorphic to $\sqrt[\mathfrak{m}]{\mathcal{O}_{\mathfrak{l}_\tau^{\mathrm{rig}}}(\mathfrak{s}\mathfrak{p}_x - c\mathfrak{p}_y)/\mathfrak{l}_\tau^{\mathrm{rig}}}$ as a $\boldsymbol{\mu}_{\mathfrak{m}}$-banded gerbe over $\mathfrak{l}_\tau^{\mathrm{rig}} \cong \mathcal{F}_{\mathfrak{r},\mathfrak{r}_-}$, and
- $\hat{L}_2 \cong \sqrt[\mathfrak{m}]{\mathcal{O}_{\mathfrak{l}_\tau^{\mathrm{rig}}}(\mathfrak{s}\mathfrak{p}_x - c\mathfrak{p}_y)}$, $\hat{L}_3 = \hat{L}_2^{-1} \otimes p_1^*\mathcal{O}_{\mathfrak{l}_\tau^{\mathrm{rig}}}(-\mathfrak{p}_x - \mathfrak{p}_y)$, where $p_1 : \hat{\mathfrak{l}}_\tau \to \mathfrak{l}_\tau^{\mathrm{rig}}$ and $\mathcal{O}_{\mathfrak{l}_\tau^{\mathrm{rig}}}(-\mathfrak{p}_x - \mathfrak{p}_y)$ is the cotangent bundle of $\mathfrak{l}_\tau^{\mathrm{rig}}$.

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3.10. Degeneration

Let $\Sigma_1 = \{\{0\}, \mathbb{R}_{\geq 0}\mathsf{v}_1, \mathbb{R}_{\geq 0}(-\mathsf{v}_1)\}$ be the complete 1-dimensional fan in $\mathbb{R}\mathsf{v}_1 \cong \mathbb{R}$, and let $\Sigma_2 = \{\{0\}, \mathbb{R}_{\geq 0}\mathsf{v}_4, \mathbb{R}_{\geq 0}(-\mathsf{v}_4)\}$ be the complete 1-dimensional fan in $\mathbb{R}\mathsf{v}_4 \cong \mathbb{R}$. Then $X_{\Sigma_1} = X_{\Sigma_2} = \mathbb{P}^1$. The stacky fan $(\Sigma_1, (\mathfrak{r}\mathsf{v}_1, -\mathfrak{r}_-\mathsf{v}_1))$ defines the 1-dimensional toric orbifold $\mathcal{F}_{\mathfrak{r},-\mathfrak{r}_-}$, and the stacky fan

$$\mathbf{\Sigma}_\square = (\Sigma_1 \times \Sigma_2, (b'_1 = \mathfrak{r}\mathsf{v}_1, b'_2 = -\mathfrak{r}_-\mathsf{v}_1, b'_3 = \mathsf{v}_4, b'_4 = -\mathsf{v}_4))$$

defines the 2-dimensional toric orbifold $\mathcal{F}_{\mathfrak{r},\mathfrak{r}_-} \times \mathbb{P}^1$. The 1-dimensional cones in the fan $\Sigma_1 \times \Sigma_2$ are $\{\rho_i = \mathbb{R}_{\geq 0}b'_i : 1 \leq i \leq 4\}$. Let Σ' be the fan obtained by adding a 1-dimensional cone $\rho_5 = \mathbb{R}_{\geq 0}b'_5$, where $b'_5 = -\mathsf{v}_1 - \mathsf{v}_4$. Let $\mathcal{S}'$ be the 2-dimensional toric orbifold defined by the stacky fan $\mathbf{\Sigma}' = (\Sigma', (b'_1, b'_2, b'_3, b'_4, b'_5))$, and let $\mathfrak{l}'_i = \mathcal{V}(\rho_i) \subset \mathcal{S}'$ be 1-dimensional closed toric substack associated with the ray ρ_i. The morphism $\mathbf{\Sigma}' \to \mathbf{\Sigma}_\square$ of stacky fans induces a morphism $\nu : \mathcal{S}' \to \mathcal{F}_{\mathfrak{r},\mathfrak{r}_-} \times \mathbb{P}^1$ of toric orbifolds; ν contracts the divisor $\mathfrak{l}'_5$ to the torus fixed point $[0,1] \times [0,1] \cong \mathcal{B}\boldsymbol{\mu}_{\mathfrak{r}_-}$ in $\mathcal{F}_{\mathfrak{r},\mathfrak{r}_-} \times \mathbb{P}^1$. Let $p : \mathcal{F}_{\mathfrak{r},\mathfrak{r}_-} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection to the second factor. The composition $\pi' := p \circ \nu$ is a flat morphism, and

$$(\pi')^{-1}([0,1]) = \mathfrak{l}'_3, \quad (\pi')^{-1}([1,0]) = \mathfrak{l}'_4 \cup \mathfrak{l}'_5,$$

where $\mathfrak{l}'_3 \cong \mathcal{F}_{\mathfrak{r},\mathfrak{r}_-}$, $\mathfrak{l}'_4 \cong \mathcal{F}_{\mathfrak{r},1}$, and $\mathfrak{l}'_5 \cong \mathcal{F}_{1,\mathfrak{r}_-}$. The torus fixed points in $\mathcal{S}'$ are

$$\mathfrak{p}_x^0 = \mathfrak{l}'_1 \cap \mathfrak{l}'_3 \cong \mathcal{B}\boldsymbol{\mu}_{\mathfrak{r}}, \ \mathfrak{p}_y^0 = \mathfrak{l}'_2 \cap \mathfrak{l}'_3 \cong \mathcal{B}\boldsymbol{\mu}_{\mathfrak{r}_-}, \ \mathfrak{p}_x^\infty = \mathfrak{l}'_1 \cap \mathfrak{l}'_4 \cong \mathcal{B}\boldsymbol{\mu}_{\mathfrak{r}}, \ \mathfrak{p}_y^\infty = \mathfrak{l}'_2 \cap \mathfrak{l}'_5 \cong \mathcal{B}\boldsymbol{\mu}_{\mathfrak{r}_-}, \ p_z = \mathfrak{l}'_4 \cap \mathfrak{l}'_5,$$

where p_z is a scheme point.

Given any $f \in \mathbb{Z}$, define

$$\hat{\mathcal{S}} = \sqrt[\mathfrak{m}]{\mathcal{O}_{\mathcal{S}'}(\mathfrak{s}\mathfrak{l}'_1 - c\mathfrak{l}'_2 + f\mathfrak{l}'_5)/\mathcal{S}'},$$

which is a $\boldsymbol{\mu}_{\mathfrak{m}}$-banded gerbe over $\mathcal{S}'$, and let $\hat{q} : \hat{\mathcal{S}} \to \mathcal{S}' = \mathcal{S}^{\mathrm{rig}}$ be the morphism to the rigidification. Define $\pi := \hat{q} \circ \pi' : \mathcal{S} \to \mathbb{P}^1$, and let $\hat{\mathfrak{l}}_i \subset \mathcal{S}$ be the divisor that corresponds to $\mathfrak{l}'_i \subset \mathcal{S}'$ under $\hat{q} : \hat{\mathcal{S}} \to \mathcal{S}'$. Then $\hat{q}_i := \hat{q}|_{\mathfrak{l}_i} : \hat{\mathfrak{l}}_i \to \mathfrak{l}'_i = \hat{\mathfrak{l}}_i^{\mathrm{rig}}$ is a $\boldsymbol{\mu}_{\mathfrak{m}}$-banded gerbe. We have

$$\pi^{-1}([0,1]) = \hat{\mathfrak{l}}_3 \cong \hat{\mathfrak{l}}_\tau \quad \pi^{-1}([1,0]) = \hat{\mathfrak{l}}_4 \cup \hat{\mathfrak{l}}_5.$$

Define $\tilde{L}_2, \tilde{L}_3 \in \mathrm{Pic}(\hat{\mathcal{S}})$ by

$$\tilde{L}_2 := \sqrt[\mathfrak{m}]{\mathcal{O}_{\mathcal{S}'}(\mathfrak{s}\mathfrak{l}'_1 - c\mathfrak{l}'_2 + f\mathfrak{l}'_5)} \quad \tilde{L}_3 := \tilde{L}_2^{-1} \otimes q^*\mathcal{O}_{\mathcal{S}'}(-\mathfrak{l}'_1 - \mathfrak{l}'_2).$$

Then

$$\tilde{L}_2\big|_{\hat{\mathfrak{l}}_3} = \sqrt[\mathfrak{m}]{\mathcal{O}_{\mathfrak{l}'_3}(\mathfrak{s}\mathfrak{p}_x^0 - c\mathfrak{p}_y^0)} \cong \hat{L}_2, \quad \tilde{L}_3\big|_{\hat{\mathfrak{l}}_3} = \sqrt[\mathfrak{m}]{\mathcal{O}_{\mathfrak{l}'_3}(-\mathfrak{s}\mathfrak{p}_x^0 + c\mathfrak{p}_y^0)} \otimes \hat{q}_3^*\mathcal{O}_{\mathfrak{l}'_3}(-\mathfrak{p}_x^0 - \mathfrak{p}_y^0) \cong \hat{L}_3.$$

For $i \in \{2,3\}$, define $\hat{L}_i^+ = \tilde{L}_i\big|_{\hat{\mathfrak{l}}_4}$ and $\hat{L}_i^- = \tilde{L}_i\big|_{\hat{\mathfrak{l}}_5}$. Then

$$\hat{L}_2^+ = \sqrt[\mathfrak{m}]{\mathcal{O}_{\mathfrak{l}'_4}(\mathfrak{s}\mathfrak{p}_x^\infty + fp_z)}, \quad \hat{L}_3^+ = \sqrt[\mathfrak{m}]{\mathcal{O}_{\mathfrak{l}'_4}(-\mathfrak{s}\mathfrak{p}_x^\infty - fp_z)} \otimes \hat{q}_4^*\mathcal{O}_{\mathfrak{l}'_4}(-\mathfrak{p}_x^\infty),$$
$$\hat{L}_2^- = \sqrt[\mathfrak{m}]{\mathcal{O}_{\mathfrak{l}'_5}(-c\mathfrak{p}_y^\infty - fp_z)}, \ \hat{L}_3^- = \sqrt[\mathfrak{m}]{\mathcal{O}_{\mathfrak{l}'_5}(c\mathfrak{p}_y^\infty + fp_z)} \otimes \hat{q}_5^*\mathcal{O}_{\mathfrak{l}'_5}(-\mathfrak{p}_y^\infty).$$

To summarise:

- $\hat{\mathcal{S}}$ is a degeneration from $\hat{\mathfrak{l}}_\tau$ to a nodal DM curve $\hat{\mathfrak{l}}_4 \cup \hat{\mathfrak{l}}_5$, and $\mathcal{S}' = \hat{\mathcal{S}}^{\mathrm{rig}}$ is a degeneration from the football $\mathfrak{l}_\tau^{\mathrm{rig}} \cong \mathcal{F}_{\mathfrak{r},\mathfrak{r}_-}$ to the nodal DM curve $\mathfrak{l}'_4 \cup \mathfrak{l}'_5$.

- For $i = 2, 3$, the line bundle $\tilde{L}_i$ on $\hat{\mathcal{S}}$ defines a degeneration of the line bundle $\hat{L}_i \to \hat{\mathfrak{l}}_\tau$ to a line bundle on $\hat{\mathfrak{l}}_4 \cup \hat{\mathfrak{l}}_5$ that restricts to $\hat{L}_i^+$ on $\hat{\mathfrak{l}}_4$ and $\hat{L}_i^-$ on $\hat{\mathfrak{l}}_5$.
- $\hat{\mathcal{S}}$, $\tilde{L}_i$, $\hat{\mathfrak{l}}_4$, $\hat{\mathfrak{l}}_5$, and $\hat{L}_i^\pm$ depend on f, while $\mathcal{S}'$, $\mathfrak{l}'_4$ and $\mathfrak{l}'_5$ do not.

Moreover, the total space of $\tilde{L}_2 \oplus \tilde{L}_3 \to \hat{\mathcal{S}}$ is a 4-dimensional toric orbifold $\hat{\mathcal{W}}$ defined by a stacky fan

$$(\Sigma_f, (\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4, \mathsf{v}_4, -\mathsf{v}_4, -\mathsf{v}_1 - f\mathsf{v}_2 - \mathsf{v}_4)), \tag{18}$$

where Σ_f is a simplicial fan in $\bigoplus_{i=1}^4 \mathbb{R}\mathsf{v}_i$, and $(\bar{b}_1, \ldots, -\mathsf{v}_1 - f\mathsf{v}_2 - \mathsf{v}_4)$ is a 7-tuple of vectors in $\bar{N} \oplus \mathbb{Z}\mathsf{v}_4 \cong \mathbb{Z}^4$. Let $\mathcal{W}$ be the 4-dimensional smooth toric DM stack defined by the the stacky fan

$$(\Sigma_f, (b_1, b_2, b_3, b_4, \mathsf{v}_4, -\mathsf{v}_4, -\tilde{\mathsf{v}}_1 - f\tilde{\mathsf{v}}_2 - \mathsf{v}_4)), \tag{19}$$

where $\tilde{\mathsf{v}}_1, \tilde{\mathsf{v}}_2 \in N$ are lifts of $\mathsf{v}_1, \mathsf{v}_2 \in \bar{N}$, so that $(b_1, \ldots, -\tilde{\mathsf{v}}_1 - f\tilde{\mathsf{v}}_2 - \mathsf{v}_4)$ is a 7-tuple of elements in $N \oplus \mathbb{Z}\mathsf{v}_4$. Then $\mathcal{W}$ is a K-banded gerbe over $\hat{\mathcal{W}} = \mathcal{W}^{\mathrm{rig}}$ and is a degeneration from the total space $\mathcal{Y}$ of $N_{\mathfrak{l}_\tau/\mathcal{X}}$ to the total space $\mathcal{Y}_\infty$ of a direct sum $L_2^\infty \oplus L_3^\infty$ of line bundles over a nodal DM curve $\mathfrak{l}_+ \cup \mathfrak{l}_-$; $\mathfrak{l}_+ \cup \mathfrak{l}_-$ is a K-banded gerbe over $\hat{\mathfrak{l}}_4 \cup \hat{\mathfrak{l}}_5$ and a G_τ-banded gerbe over $\mathfrak{l}'_4 \cup \mathfrak{l}'_5$. For $i = 2, 3$, let $L_i^\pm = L_i^\infty|_{\mathfrak{l}_\pm}$. Then $L_i^\pm$ is the pullback of $\hat{L}_i^\pm$. Let $\mathfrak{p}_0 \cong \mathcal{B}G_\tau$ be the node that is the intersection of $\mathfrak{l}_+$ and $\mathfrak{l}_-$, and let $\mathfrak{p}_\pm$ be the unique torus fixed point in $\mathfrak{l}_\pm - \{\mathfrak{p}_0\}$. Then $\mathfrak{p}_+ \cong \mathcal{B}G_\sigma$ and $\mathfrak{p}_- \cong \mathcal{B}G_{\sigma_-}$. Define $\mathbb{T}'$ weights

$$\mathsf{w}_1^\pm := (c_1)_{\mathbb{T}'}(T_{\mathfrak{p}_\pm}\mathfrak{l}^\pm), \quad \mathsf{w}_2^\pm := (c_1)_{\mathbb{T}'}(L_2)|_{\mathfrak{p}_\pm}, \quad \mathsf{w}_3^\pm := (c_1)_{\mathbb{T}'}(L_3)|_{\mathfrak{p}_\pm} \in H^2(\mathcal{B}\mathbb{T}') = \mathbb{Q}\mathsf{u}'_1 \oplus \mathbb{Q}\mathsf{u}'_2.$$

Then $\mathsf{w}_i^+ = \mathsf{w}'_i$ is given by equation (16), $\mathsf{w}_1^\pm + \mathsf{w}_2^\pm + \mathsf{w}_3^\pm = 0$, and

$$\mathsf{w}_1^- = -\frac{1}{\mathfrak{r}_-}\mathsf{u}'_1, \quad \mathsf{w}_2^- = \frac{c}{\mathfrak{r}_-\mathfrak{m}}\mathsf{u}'_1 + \frac{1}{\mathfrak{m}}\mathsf{u}'_2, \quad \mathsf{w}_3^- = \frac{-c+\mathfrak{m}}{\mathfrak{r}_-\mathfrak{m}}\mathsf{u}'_1 - \frac{1}{\mathfrak{m}}\mathsf{u}'_2. \tag{20}$$

We also have

$$(c_1)_{\mathbb{T}'}(T_{\mathfrak{p}_0}\mathfrak{l}_\pm) = \mp\mathsf{u}'_1, \quad (c_1)_{\mathbb{T}'}(L_2^\pm)_{\mathfrak{p}_0} = \frac{\mathsf{u}'_2 - f\mathsf{u}'_1}{\mathfrak{m}} = -(c_1)_{\mathbb{T}'}(L_3^\pm)_{\mathfrak{p}_0}.$$

The above weights are summarised in Figure 7 below.

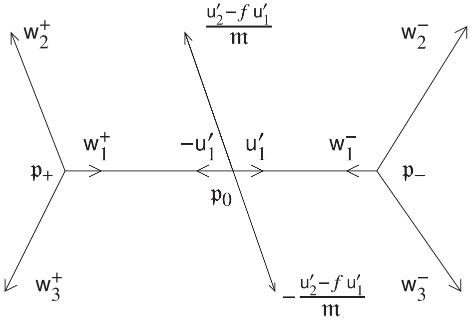

Figure 7. *Degenerated $N_{\mathfrak{l}_\tau/\mathcal{X}}$ and the $\mathbb{T}'$-weights.*

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3.11. Disk factors as equivariant relative GW invariants

When $d_0 > 0$, let $\overline{\mathcal{M}}_{0,1}(\mathfrak{l}_+/\mathfrak{p}_0,(d_0,\lambda))$ be the moduli space of relative maps to $(\mathfrak{l}_+,\mathfrak{p}_0)$ with the relative condition (d_0,λ), where $\lambda \in G_\tau$ [1]. A relative stable map to $(\mathfrak{l}_+,\mathfrak{p}_0)$ is a morphism to $\mathfrak{l}_+[m]$ that is the union of $\mathfrak{l}_+$ and a chain of m copies of $\mathbb{P}^1 \times \mathcal{B}G_\lambda$. Let $\mathcal{M}_{0,1}(\mathfrak{l}_+/\mathfrak{p}_0,(d_0,\lambda)) \subset \overline{\mathcal{M}}_{0,1}(\mathfrak{l}_+/\mathfrak{p}_0,(d_0,\lambda))$ be the open substack where the target is $\mathfrak{l}_+[0]=\mathfrak{l}_+$. The tangent space $\mathcal{T}^1_\xi$ and the obstruction space $\mathcal{T}^2_\xi$ at a moduli point $\xi = [u:(\mathcal{C},x,y)\to(\mathfrak{l}_+,\mathfrak{p}_0)]$ in $\mathcal{M}_{0,1}(\mathfrak{l}_+/\mathfrak{p}_0,(d_0,\lambda))$ (where $u^{-1}(\mathfrak{p}_0)=d_0y$ as Cartier divisors) fit into the following exact sequence of complex vector spaces:

$$\begin{aligned}0 \to &\mathrm{Ext}^0(\Omega_{\mathcal{C}}(x+y),\mathcal{O}_{\mathcal{C}}) \to H^0(\mathcal{C},u^*(T_{\mathfrak{l}_+}(-\mathfrak{p}_0))) \to \mathcal{T}^1_\xi\\ \to &\mathrm{Ext}^1(\Omega_{\mathcal{C}}(x+y),\mathcal{O}_{\mathcal{C}}) \to H^1(\mathcal{C},u^*(T_{\mathfrak{l}_+}(-\mathfrak{p}_0))) \to \mathcal{T}^2_\xi.\end{aligned} \tag{21}$$

Globally on $\mathcal{M}_{0,1}(\mathfrak{l}_+/\mathfrak{p}_0,(d_0,\lambda))$, there is an exact sequence of sheaves

$$0 \to B_1 \to B_2 \to \mathcal{T}^1 \to B_4 \to B_5 \to \mathcal{T}^2 \to 0 \tag{22}$$

whose fibre at the moduli point ξ is equation (21).

Let $\pi:\mathcal{U}_+ \to \mathcal{M}_{0,1}(\mathfrak{l}_+/\mathfrak{p}_0,(d_0,\lambda))$ be the universal domain curve, and let $F_+:\mathcal{U}_+\to\mathfrak{l}_+$ be the evaluation map. We define

$$V^+_{0,1} := R^\bullet\pi_*F_+^*(L_2^+\oplus L_3^+) \in K_{\mathbb{T}'}\big(\mathcal{M}_{0,1}(\mathfrak{l}_+/\mathfrak{p}_0,(d_0,\lambda))\big),$$

where $R^\bullet\pi_*$ is the K-theoretic push-forward.

For $d_0>0$, we define

$$\begin{aligned}D_{d_0,\lambda} &= \langle 1_{h^+(d_0,\lambda)}\rangle^{\mathcal{X},\mathcal{L},T'_{\mathbb{R}}}_{0,d_0b,(d_0,\lambda)}\\ &= \int_{[\mathcal{M}_{0,1}(\mathfrak{l}_+,\mathfrak{p}_0,(d_0,\lambda))]^{\mathrm{vir}}} \mathrm{ev}^*(1_{h^+(d_0,\lambda)})e_{\mathbb{T}'}(V^+_{0,1}).\end{aligned}$$

When $d_0<0$, let $\overline{\mathcal{M}}_{0,1}(\mathfrak{l}_-/\mathfrak{p}_0,(-d_0,\lambda^{-1}))$ be the moduli space of relative stable maps to $(\mathfrak{l}_-,\mathfrak{p}_0)$ with relative condition $(-d_0,\lambda^{-1})$, and let $\mathcal{M}_{0,1}(\mathfrak{l}_-/\mathfrak{p}_0,(-d_0,\lambda^{-1}))$ be the open substack where the target is $\mathfrak{l}_-$. Let $\pi:\mathcal{U}_-\to\mathcal{M}_{0,1}(\mathfrak{l}_-/\mathfrak{p}_0,(-d_0,\lambda^{-1}))$ be the universal domain curve, and let $F_-:\mathcal{U}_-\to\mathfrak{l}_-$ be the evaluation map. We define

$$V^-_{0,1} = R^\bullet\pi_*F_-^*(L_2^-\oplus L_3^-) \in K_{\mathbb{T}'}\Big(\mathcal{M}_{0,1}(\mathfrak{l}_-/\mathfrak{p}_0,(-d_0,\lambda^{-1}))\Big)$$

and define

$$\begin{aligned}D_{d_0,\lambda} &= \langle 1_{h^-(d_0,\lambda)}\rangle^{\mathcal{X},\mathcal{L},T'_{\mathbb{R}}}_{0,-d_0(\alpha-b),(d_0,\lambda))}\\ &= \int_{[\mathcal{M}_{0,1}(\mathfrak{l}_-,\mathfrak{p}_0,(-d_0,\lambda^{-1}))]^{\mathrm{vir}}} \mathrm{ev}^*(1_{h^-(d_0,\lambda)})e_{\mathbb{T}'}(V^-_{0,1}).\end{aligned}$$

Let $u:(\mathcal{C},x,y)\to\mathfrak{l}_+$ be a relative stable map that represents a point in $\mathcal{M}$. Suppose that u is fixed by the torus action. Recall that $c_i: G_\sigma \to [0,1)\cap\mathbb{Q}$ is defined by $\chi_i(k)=\exp(2\pi\sqrt{-1}c_i(k))$. In the computation below, let $k^\pm = h^\pm(d_0,\lambda)$. For $j=1,2,3$, let $\epsilon_j = c_j(k^+)$. Then $\epsilon_1 = \langle \frac{d_0}{\mathfrak{r}_+}\rangle$. We have the following weights

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$$\mathrm{ch}_{\mathbb{T}'}\big(H^0(\mathcal{C},u^*L_1^+)\big)=\sum_{a=0}^{\lfloor d_0 w_1\rfloor} e^{a\frac{\mathsf{u}_1'}{d_0}},\quad \mathrm{ch}_{\mathbb{T}'}\big(H^1(\mathcal{C},u^*L_1^+)\big)=0,$$

$$\mathrm{ch}_{\mathbb{T}'}\big(H^0(\mathcal{C},u^*L_2^+\otimes\mathcal{O}_y)\big)=\delta_{\langle d_0 w_2-\epsilon_2\rangle,0}e^{\frac{\mathsf{u}_2'-f\mathsf{u}_1'}{\mathfrak{m}}},\quad \mathrm{ch}_{\mathbb{T}'}\big(H^1(\mathcal{C},u^*L_2^+\otimes\mathcal{O}_y)\big)=0,$$

$$\mathrm{ch}_{\mathbb{T}'}\big(H^0(\mathcal{C},u^*L_2^+)\big)=\begin{cases}\displaystyle\sum_{a=-\lfloor d_0 w_2-\epsilon_2\rfloor}^{0} e^{\mathsf{w}_2'+(a-\epsilon_2)\frac{\mathsf{u}_1'}{d_0}}, & f\ge 0,\\ 0, & f<0,\end{cases}$$

$$\mathrm{ch}_{\mathbb{T}'}\big(H^1(\mathcal{C},u^*L_2^+)\big)=\begin{cases}0, & f\ge 0,\\ \displaystyle\sum_{a=1}^{-\lfloor d_0 w_2-\epsilon_2\rfloor-1} e^{\mathsf{w}_2'+(a-\epsilon_2)\frac{\mathsf{u}_1'}{d_0}}, & f<0\end{cases}$$

$$\mathrm{ch}_{\mathbb{T}'}\big(H^0(\mathcal{C},u^*L_3^+)\big)=\begin{cases}\displaystyle\sum_{a=-\lfloor d_0 w_3-\epsilon_3\rfloor}^{0} e^{\mathsf{w}_3'+(a-\epsilon_3)\frac{\mathsf{u}_1'}{d_0}}, & f<0,\\ 0, & f\ge 0,\end{cases}$$

$$\mathrm{ch}_{\mathbb{T}'}\big(H^1(\mathcal{C},u^*L_3^+)\big)=\begin{cases}0, & f<0,\\ \displaystyle\sum_{a=1}^{-\lfloor d_0 w_3-\epsilon_3\rfloor-1} e^{\mathsf{w}_3'+(a-\epsilon_3)\frac{\mathsf{u}_1'}{d_0}}, & f\ge 0.\end{cases}$$

and the following identities

$$\sum_{a=-\lfloor w_3 d_0-\epsilon_3\rfloor}^{0} e^{\mathsf{w}_3'+(a-\epsilon_3)\frac{\mathsf{u}_1'}{d_0}}=\sum_{a=d_0 w_1+\epsilon_2+\epsilon_3}^{-\lfloor d_0 w_2-\epsilon_2\rfloor-1+\delta_{\langle d_0 w_2-\epsilon_2\rangle,0}} e^{-\mathsf{w}_2'+(\epsilon_2-a)\frac{\mathsf{u}_1'}{d_0}}$$

$$\sum_{a=1}^{-\lfloor w_3 d_0+1-\epsilon_3\rfloor} e^{\mathsf{w}_3'+(a-\epsilon_3)\frac{\mathsf{u}_1'}{d_0}}=\sum_{a=-\lfloor d_0 w_2-\epsilon_2\rfloor+\delta_{\langle d_0 w_2-\epsilon_2\rangle}}^{\frac{d_0}{s_1^+}+\epsilon_2+\epsilon_3-1} e^{-\mathsf{w}_2'+(\epsilon_2-a)\frac{\mathsf{u}_1'}{d_0}}$$

$$d_0 w_1+\epsilon_2+\epsilon_3=\left\lfloor\frac{d_0}{s_1^+}\right\rfloor+\mathrm{age}(k^+).$$

The $\mathbb{T}'$-equivariant Euler classes are

$$e_{\mathbb{T}'}(B_1^m)=1,$$

$$e_{\mathbb{T}'}(B_2^m)=\lfloor d_0 w_1\rfloor!\left(\frac{\mathsf{u}_1'}{d_0}\right)^{\lfloor d_0 w_1\rfloor},$$

$$\frac{e_{\mathbb{T}'}(B_5^m)}{e_{\mathbb{T}'}(B_4^m)}=(-1)^{\lceil d_0 w_2-\epsilon_2\rceil+\lfloor d_0 w_1\rfloor+\mathrm{age}(k^+)-1}\prod_{a=1}^{\lfloor d_0 w_1\rfloor+\mathrm{age}(k^+)-1}\left(\mathsf{w}_2'+(a-\epsilon_2)\frac{\mathsf{u}_1'}{d_0}\right).$$

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Since $|\mathrm{Aut}(f)| = d_0|G_\tau|$, by localisation,

$$
\begin{aligned}
D_{d_0,\lambda} = D(d_0,k^+,k^-) &= \frac{1}{|\mathrm{Aut}(f)|}\frac{e_{\mathbb{T}'}(B_1^m)e_{\mathbb{T}'}(B_5^m)}{e_{\mathbb{T}'}(B_2^m)e_{\mathbb{T}'}(B_4^m)}\\
&=\frac{1}{d_0|G_\tau|}\frac{\prod_{a=1}^{\lfloor d_0w_1\rfloor+\mathrm{age}(k^+)-1}(\mathsf{w}_2'+(a-\epsilon_2)\frac{\mathsf{u}_1'}{d_0})}{\lfloor d_0w_1\rfloor!(\frac{\mathsf{u}_1'}{d_0})^{\lfloor d_0w_1\rfloor}}\cdot(-1)^{\lceil d_0w_2-\epsilon_2\rceil+\lfloor d_0w_1\rfloor+\mathrm{age}(k^+)-1}\\
&=(-1)^{\lceil d_0w_2-\epsilon_2\rceil+\lfloor d_0w_1\rfloor+\mathrm{age}(k^+)-1}\left(\frac{\mathsf{u}_1'}{d_0}\right)^{\mathrm{age}(k^+)-1}\cdot\frac{1}{d_0|G_\tau|}\frac{\prod_{a=1}^{\lfloor d_0w_1\rfloor+\mathrm{age}(k^+)-1}(\frac{d_0\mathsf{w}_2'}{s_1^+\mathsf{w}_1'}+a-\epsilon_2)}{\lfloor d_0w_1\rfloor!}\\
&=-(-1)^{\lfloor d_0w_3+\frac{\bar\lambda}{\mathfrak{m}}\rfloor}\left(\frac{\mathsf{u}_1'}{d_0}\right)^{\mathrm{age}(k^+)-1}\cdot\frac{1}{d_0|G_\tau|}\frac{\prod_{a=1}^{\lfloor d_0w_1\rfloor+\mathrm{age}(k^+)-1}(\frac{d_0\mathsf{w}_2'}{s_1^+\mathsf{w}_1'}+a-\epsilon_2)}{\lfloor d_0w_1\rfloor!}
\end{aligned}
$$

Let $\mathsf{u} := \iota_f^*\mathsf{u}_1' \in H^2(\mathbb{T}_f;\mathbb{Z})$. Then $H^*(\mathbb{T}_f;\mathbb{Q}) = \mathbb{Q}[\mathsf{u}]$ and $\iota_f^*\mathsf{u}_2' = f\mathsf{u}$. Define $D_{d_0,\lambda,f} = \iota_f^*D_{d_0,\lambda}$. Hence when $d_0 > 0$,

$$
\begin{aligned}
&D_{d_0,\lambda,f}\\
&=-(-1)^{\lfloor d_0w_3+\frac{\bar\lambda}{\mathfrak{m}}\rfloor}\left(\frac{\mathsf{u}}{d_0}\right)^{\mathrm{age}(h^+(d_0,\lambda))-1}\cdot\frac{1}{d_0|G_\tau|}\frac{\prod_{a=1}^{\lfloor d_0w_1\rfloor+\mathrm{age}(h^+(d_0,\lambda))-1}(d_0w_2+a-c_2(h^+(d_0,\lambda)))}{\lfloor d_0w_1\rfloor!}.
\end{aligned}
$$

If $d_0 < 0$, similar computation shows (notice $\overline{\lambda^{-1}} = (1-\delta_{\bar\lambda,0})(\mathfrak{m}-\bar\lambda) \in \{0,\ldots,\mathfrak{m}-1\}$)

$$
\begin{aligned}
D_{d_0,\lambda,f} &= -(-1)^{\lfloor d_0w_2^-+\left(1-\frac{\bar\lambda}{m}-\delta_{\bar\lambda,0}\right)\rfloor}\left(\frac{\mathsf{u}}{d_0}\right)^{\mathrm{age}(h^-(d_0,\lambda))-1}\\
&\quad\cdot\frac{1}{-d_0|G_\tau|}\frac{\prod_{a=1}^{\lfloor d_0w_1^-\rfloor+\mathrm{age}(h^-(d_0,\lambda))-1}(d_0w_3^--c_3(h^-(d_0,\lambda))+a)}{\lfloor d_0w_1^-\rfloor!}.
\end{aligned}
$$

If $\mathcal{L}$ is an outer brane, it is the same as $d_0 > 0$. Define

$$
D_{d_0,\lambda,f} = -(-1)^{\lfloor d_0w_3+\frac{\bar\lambda}{\mathfrak{m}}\rfloor}(\frac{\mathsf{u}}{d_0})^{\mathrm{age}(h(d_0,\lambda))-1}\cdot\frac{1}{d_0|G_\tau|}\frac{\prod_{a=1}^{\lfloor d_0w_1\rfloor+\mathrm{age}(h(d_0,\lambda))-1}(d_0w_2+a-c_2(h(d_0,\lambda)))}{\lfloor d_0w_1\rfloor!}.
$$

3.12. Open-closed GW invariants and descendant GW invariants

For any torus fixed point $\mathfrak{p}_\sigma$ of $\mathcal{X}$, where $\sigma \in \Sigma(3)$, we have

$$
H^*_{\mathrm{CR}}(\mathfrak{p}_\sigma) = \bigoplus_{k\in G_\sigma}\mathbb{Q}\mathbf{1}_k,\quad H^*_{\mathrm{CR},\mathbb{T}'}(\mathfrak{p}_\sigma) = \bigoplus_{k\in G_\sigma}\mathbb{Q}[\mathsf{w}_1',\mathsf{w}_2']\mathbf{1}_k.
$$

The inclusion $\iota_\sigma : \mathfrak{p}_\sigma \hookrightarrow \mathcal{X}$ induces

$$
\iota_{\sigma*} : H^*_{\mathrm{CR},\mathbb{T}'}(\mathfrak{p}_\sigma) = H^*_{\mathbb{T}'}(\mathcal{I}\mathfrak{p}_\sigma) \to H^*_{\mathrm{CR},\mathbb{T}'}(\mathcal{X}) = H^*_{\mathbb{T}'}(\mathcal{I}\mathcal{X}).
$$

Define

$$
\phi_{\sigma,k} = \iota_{\sigma*}\mathbf{1}_k \in H^*_{\mathrm{CR},\mathbb{T}'}(\mathcal{X}),\ \phi^f_{\sigma,k} = \iota_f^*\phi_{\sigma,k} \in H^*_{\mathrm{CR},\mathbb{T}_f}(\mathcal{X}).
$$

Proposition 3.2 (framed inner brane). *Suppose that $(\mathcal{L},f)$ is a framed inner brane and*

$$
\vec\mu = ((\mu_1,\lambda_j),\ldots,(\mu_h,\lambda_h)),
$$

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where $(\mu_j,\lambda_j)\in H_1(\mathcal{L};\mathbb{Z})\cong\mathbb{Z}\times\mathcal{B}G_\tau$. *Let* $J_\pm=\{j\in\{1,\dots,h\}:\pm\mu_j>0\}$, *and let* $k_j^\pm=h^\pm(\mu_j,\lambda_j)$. *Then*

$$\langle\gamma_1,\dots,\gamma_n\rangle^{\mathcal{X},(\mathcal{L},f)}_{g,\beta',\vec{\mu}}=\prod_{j=1}^h D'_{\mu_j,\lambda_j,f}$$
$$\cdot\int_{[\overline{\mathcal{M}}_{g,n+h}(\mathcal{X},\beta)]^{\mathrm{vir}}}\frac{\left(\prod_{i=1}^n\mathrm{ev}_i^*\gamma_i\prod_{j\in J_+}\mathrm{ev}_{n+j}^*\phi^f_{\sigma_+,(h^+(d_0,\lambda_j))^{-1}}\prod_{j\in J_-}\mathrm{ev}_{n+j}^*\phi^f_{\sigma_-,(h^-(d_0,\lambda_j))^{-1}}\right)}{\prod_{j=1}^h\frac{\mathsf{u}}{\mu_j}\left(\frac{\mathsf{u}}{\mu_j}-\bar{\psi}_{n+j}\right)},$$

where

$$\beta\in H_2(\mathcal{X}),\quad \beta'=\beta+\Big(\sum_{j\in J_+}\mu_j\Big)b-\Big(\sum_{j\in J_-}\mu_j\Big)(\alpha-b)\in H_2(\mathcal{X},\mathcal{L}).$$

Proof. There exists $\beta\in H_2(\mathcal{X})$ such that

$$\beta'=\beta+\Big(\sum_{j\in J_+}\mu_j\Big)b+\sum_{j\in J_-}(-\mu_j)(\alpha-b).$$

Let $\langle k_j^+\rangle$ be the cyclic subgroup generated by k_j^+, and let r_j be the cardinality of $\langle k_j^+\rangle$ for $j\in J_+$ and r_j be the cardinality of $\langle k_j^-\rangle$ for $j\in J_-$.

We have

$$\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X},\mathcal{L}\mid\beta',\vec{\mu})^{\mathbb{T}'_{\mathbb{R}}}=\bigcup_{\Gamma\in G_{g,n}(\mathcal{X},\mathcal{L}|\beta',\vec{\mu})}F_\Gamma$$
$$\overline{\mathcal{M}}_{g,n+h}(\mathcal{X},\beta)^{\mathbb{T}'_{\mathbb{R}}}=\overline{\mathcal{M}}_{g,n+h}(\mathcal{X},\beta)^{\mathbb{T}'}=\bigcup_{\hat{\Gamma}\in G_{g,n+h}(\mathcal{X},\beta)}F_{\hat{\Gamma}}.$$

In the remaining part of this subsection, we use the following abbreviations:

$$\mathcal{M}=\overline{\mathcal{M}}_{g,n}(\mathcal{X},\mathcal{L}\mid\beta',\vec{\mu}),\quad \hat{\mathcal{M}}=\overline{\mathcal{M}}_{g,n+h}(\mathcal{X},\beta),$$
$$\mathcal{M}_j=\begin{cases}\overline{\mathcal{M}}_{(0,1),1}(\mathcal{X},\mathcal{L}\mid\mu_jb,(\mu_j,\lambda)), & j\in J_+\\ \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X},\mathcal{L}\mid-\mu_j(\alpha-b),(\mu_j,\lambda)), & j\in J_-\end{cases}$$
$$\mathcal{G}=G_{g,n}(\mathcal{X},\mathcal{L}\mid\beta',\vec{\mu}),\quad \hat{\mathcal{G}}=G_{g,n+h}(\mathcal{X},\beta)$$
$$\mathbf{x}=(x_1,\dots,x_n),\quad \mathbf{y}=(y_1,\dots,y_h).$$

Given $u:(\Sigma,\mathbf{x},\partial\Sigma)\to(\mathcal{X},\mathcal{L})$ that represents a point $\xi\in\mathcal{M}^{\mathbb{T}'_{\mathbb{R}}}$, we have

$$\Sigma=\mathcal{C}\cup\bigcup_{j=1}^h D_j,$$

where $\mathcal{C}$ is an orbicurve of genus g, $x_1,\dots,x_n\in\mathcal{C}$, $D_j=[\{z\in\mathbb{C}\mid |z|\le 1\}/\mathbb{Z}_{r_j}]$, $\mathcal{C}$ and D_j intersect at $y_j=B\mathbb{Z}_{r_j}$. Let $u_j=u|_{D_j}$ and $\hat{u}=u|_{\mathcal{C}}$. Then

1. For $j=1,\dots,h$, $u_j:(D_j,\partial D_j)\to(\mathcal{X},\mathcal{L})$ represents a point in $\mathcal{M}_j^{\mathbb{T}'_{\mathbb{R}}}$.
2. $\hat{u}:(\mathcal{C},\mathbf{x},\mathbf{y})\to\mathcal{X}$ represents a point $\hat{\xi}\in\hat{\mathcal{M}}^{\mathbb{T}'}$, and $\hat{u}(\mathbf{y}_j)=[\mathfrak{p}_\pm,(k_j^\pm)^{-1}]\in\mathcal{I}\mathfrak{p}_\pm\subset\mathcal{I}\mathcal{X}$ if $j\in J_\pm$.

Let $x_{n+j} = y_j$. Let F_Γ be the connected component of $\mathcal{M}^{\mathbb{T}'}$ associated to the decorated graph $\Gamma \in \mathcal{G}$, and let $F_{\hat{\Gamma}}$ be the connected component of $\hat{\mathcal{M}}^{\mathbb{T}'_{\mathbb{R}}}$ associated to the decorated graph $\hat{\Gamma} \in \mathcal{G}$. Then for any $\Gamma \in \mathcal{G}$ there exists $\hat{\Gamma} \in \hat{\mathcal{G}}$ such that

$$\mathrm{ev}_{n+j}(F_{\hat{\Gamma}}) = (\mathfrak{p}_\pm, (k_j^\pm)^{-1}) \in \mathcal{I}\mathfrak{p}_\pm \subset \mathcal{I}\mathcal{X}$$

if $j \in J_\pm$, and F_Γ can be identified with $F_{\hat{\Gamma}}$ up to a finite morphism. More precisely,

$$\begin{aligned} [F_\Gamma]^{\mathrm{vir}} &= \prod_{j \in J_+} \frac{|G_{\sigma_+}|}{r_j |\mathrm{Aut}(u_j)|} \prod_{j \in J_-} \frac{|G_{\sigma_-}|}{r_j |\mathrm{Aut}(u_j)|} [F_{\hat{\Gamma}}]^{\mathrm{vir}} \\ &= \prod_{j \in J_+} \frac{s_1^+}{r_j \mu_j} \prod_{j \in J_-} \frac{s_1^-}{-r_j \mu_j} [F_{\hat{\Gamma}}]^{\mathrm{vir}}. \end{aligned}$$

We have

$$\frac{1}{e_{\mathbb{T}'_{\mathbb{R}}}(N_\Gamma^{\mathrm{vir}})} = \frac{e_{\mathbb{T}'_{\mathbb{R}}}(B_1^m) e_{\mathbb{T}'_{\mathbb{R}}}(B_5^m)}{e_{\mathbb{T}'_{\mathbb{R}}}(B_2^m) e_{\mathbb{T}'_{\mathbb{R}}}(B_4^m)}, \qquad \frac{1}{e_{\mathbb{T}'_{\mathbb{R}}}(N_{\hat{\Gamma}}^{\mathrm{vir}})} = \frac{e_{\mathbb{T}'_{\mathbb{R}}}(\hat{B}_1^m) e_{\mathbb{T}'_{\mathbb{R}}}(\hat{B}_5^m)}{e_{\mathbb{T}'_{\mathbb{R}}}(\hat{B}_2^m) e_{\mathbb{T}'_{\mathbb{R}}}(\hat{B}_4^m)},$$

where

$$\begin{aligned} & e_{\mathbb{T}'_{\mathbb{R}}}(B_1^m) = e_{\mathbb{T}'_{\mathbb{R}}}(\hat{B}_1^m), \\ & e_{\mathbb{T}'_{\mathbb{R}}}(B_4^m) = e_{\mathbb{T}'_{\mathbb{R}}}(\hat{B}_4^m) \prod_{j \in J_+} \left(\frac{\mathsf{u}_1'}{r_j \mu_j} - \frac{\bar{\psi}_j}{r_j} \right) \prod_{j \in J_-} \left(\frac{\mathsf{u}_1'}{r_j \mu_j} - \frac{\bar{\psi}_j}{r_j} \right) \end{aligned}$$

For $k = 0, 1$ and $j = 1, \ldots, h$, let

$$H^k(D_j) = H^k\big(D_j, \partial D_j, u_j^* T\mathcal{X}, (u_j|_{\partial D_j})^* T\mathcal{L}\big).$$

Then there is a long exact sequence

$$\begin{aligned} 0 \to B_2 \to \hat{B}_2 \oplus \bigoplus_{j=1}^{h} H^0(D_j) \to \bigoplus_{j \in J_+} (T_{\mathfrak{p}_+}\mathcal{X})^{k_j^+} \oplus \bigoplus_{j \in J_-} (T_{\mathfrak{p}_-}\mathcal{X})^{k_j^-} \\ \to B_5 \to \hat{B}_5 \oplus \bigoplus_{j=1}^{h} H^1(D_j) \to 0, \end{aligned}$$

where $(T_{\mathfrak{p}_\pm}\mathcal{X})^{k_j^\pm}$ denote the $k_j^\pm$-invariant part of $T_{\mathfrak{p}_\pm}\mathcal{X}$. Note that

$$\begin{aligned} (T_{\mathfrak{p}_\pm}\mathcal{X})^{k_j^\pm} &= T_{(\mathfrak{p}_\pm, k_j^\pm)}\mathcal{I}\mathcal{X} = T_{(\mathfrak{p}_\pm, k_j^{-1})}\mathcal{I}\mathcal{X} \\ \frac{e_{\mathbb{T}'_{\mathbb{R}}}(H^1(D_j)^m)}{e_{\mathbb{T}'_{\mathbb{R}}}(H^0(D_j)^m)} &= |\mu_j||G_\tau| D_{\mu_j, \lambda_j}. \end{aligned}$$

Let

$$\begin{aligned} e_{\mathbb{T}_f}(N_\Gamma^{\mathrm{vir}}) &:= \iota_f^* e_{\mathbb{T}'_{\mathbb{R}}}(N_\Gamma^{\mathrm{vir}}) = e_{\mathbb{T}'_{\mathbb{R}}}(N_\Gamma^{\mathrm{vir}})\Big|_{\mathsf{u}_1' = \mathsf{u}, \mathsf{u}_2' = f\,\mathsf{u}}, \\ e_{\mathbb{T}_f}(N_{\hat{\Gamma}}^{\mathrm{vir}}) &:= \iota_f^* e_{\mathbb{T}'_{\mathbb{R}}}(N_{\hat{\Gamma}}^{\mathrm{vir}}) = e_{\mathbb{T}'_{\mathbb{R}}}(N_{\hat{\Gamma}}^{\mathrm{vir}})\Big|_{\mathsf{u}_1' = \mathsf{u}, \mathsf{u}_2' = f\,\mathsf{u}}. \end{aligned}$$

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Given $\gamma_1, \dots, \gamma_n \in H^*_{\mathrm{CR},\mathbb{T}_f}(\mathcal{X})$, we define

$$\langle \gamma_1, \dots, \gamma_h \rangle^{\mathcal{X},(\mathcal{L},f)}_{\beta',((\mu_1,\lambda_1),\dots,(\mu_h,\lambda_h))} := \int_{[F_\Gamma]^{\mathrm{vir}}} \frac{\big(\prod_{i=1}^n \mathrm{ev}_i^*\gamma_i\big)|_{F_\Gamma}}{e_{\mathbb{T}_f}(N_\Gamma^{\mathrm{vir}})}$$

$$= \prod_{j\in J_+} \frac{\mathfrak{r}_+}{r_j\mu_j} \prod_{j\in J_-} \frac{\mathfrak{r}_-}{-r_j\mu_j} \prod_{j=1}^h (|\mu_j||G_\tau|D_{\mu_j,\lambda_j,f})$$

$$\cdot \int_{[F_{\hat\Gamma}]^{\mathrm{vir}}} \frac{\big(\prod_{i=1}^n \mathrm{ev}_i^*\gamma_i \prod_{j\in J_+} \mathrm{ev}_{n+j}^*\phi_{\sigma_+,(k_j^+)^{-1}} \prod_{j\in J_-} \mathrm{ev}_{n+j}^*\phi^f_{\sigma_-,(k_j^-)^{-1}}\big)|_{F_\Gamma}}{\prod_{j\in J_+}(\frac{\mathsf{u}_1'}{r_j\mu_j} - \frac{\bar\psi_{n+j}}{r_j}) \prod_{j\in J_-}(\frac{\mathsf{u}_1'}{r_j\mu_j} - \frac{\bar\psi_{n+j}}{r_j}) e_{\mathbb{T}_f}(N_{\hat\Gamma}^{\mathrm{vir}})}$$

$$= \prod_{j=1}^h D'_{\mu_j,\lambda_j,f} \cdot \int_{[F_{\hat\Gamma}]^{\mathrm{vir}}} \frac{\big(\prod_{i=1}^n \mathrm{ev}_i^*\gamma_i \prod_{j\in J_+} \mathrm{ev}_{n+j}^*\phi^f_{\sigma_+,(k_j^+)^{-1}} \prod_{j\in J_-} \mathrm{ev}_{n+j}^*\phi^f_{\sigma_-,(k_j^-)^{-1}}\big)|_{F_\Gamma}}{\prod_{j=1}^h \frac{\mathsf{u}}{\mu_j}(\frac{\mathsf{u}}{\mu_j} - \bar\psi_{n+j}) e_{\mathbb{T}_f}(N_{\hat\Gamma}^{\mathrm{vir}})}$$

where

$$D'_{d_0,\lambda,f} = \begin{cases} -(-1)^{\lfloor d_0 w_3^+ + \frac{\bar\lambda}{\mathfrak{m}} \rfloor} \left(\frac{\mathsf{u}}{d_0}\right)^{\mathrm{age}(k^+)} \cdot \frac{\mathfrak{r}_+}{d_0} \cdot \frac{\prod_{a=1}^{\lfloor d_0 w_1^+\rfloor + \mathrm{age}(k^+)-1}(d_0 w_2^+ + a - c_2(k^+))}{\lfloor d_0 w_1^+ \rfloor !}, & d_0 > 0, \\ -(-1)^{\lfloor d_0 w_2^- + (1-\frac{\bar\lambda}{\mathfrak{m}} - \delta_{\bar\lambda,0}) \rfloor} \left(\frac{\mathsf{u}}{d_0}\right)^{\mathrm{age}(k^-)} \cdot \frac{\mathfrak{r}_-}{-d_0} \cdot \frac{\prod_{a=1}^{\lfloor d_0 w_1^-\rfloor + \mathrm{age}(k^-)-1}(d_0 w_3^- - c_3(k^-) + a)}{\lfloor d_0 w_1^- \rfloor !}, & d_0 < 0. \end{cases}$$

□

Suppose that $(\mathcal{L}, f)$ is a framed outer brane. Define

$$D'_{d_0,\lambda,f} = -(-1)^{\lfloor d_0 w_3 + \frac{\bar\lambda}{\mathfrak{m}} \rfloor} \left(\frac{\mathsf{u}}{d_0}\right)^{\mathrm{age}(k)} \cdot \frac{\mathfrak{r}}{d_0} \cdot \frac{\prod_{a=1}^{\lfloor d_0 w_1\rfloor + \mathrm{age}(k)-1}(d_0 w_2 + a - c_2(k))}{\lfloor d_0 w_1 \rfloor !},$$

where $k = h(d_0, \lambda)$. By the $d_0 > 0$ part of the proof of Proposition 3.2, we obtain:

Proposition 3.3 (framed outer brane). *Suppose that $(\mathcal{L}, f)$ is a framed inner brane and $\vec\mu = ((\mu_1, \lambda_1), \dots, (\mu_h, \lambda_h))$, where $(\mu_j, \lambda_j) \in H_1(\mathcal{L}; \mathbb{Z})$. Then*

$$\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},(\mathcal{L},f)}_{g,\beta',\vec\mu} = \prod_{j=1}^h D'_{\mu_j,\lambda,f} \cdot \int_{[\overline{\mathcal{M}}_{g,n+h}(\mathcal{X},\beta)]^{\mathrm{vir}}} \frac{\big(\prod_{i=1}^n \mathrm{ev}_i^*\gamma_i \prod_{j=1}^h \mathrm{ev}_{n+j}^*\phi^f_{\sigma,(h(d_0,\lambda))^{-1}}\big)}{\prod_{j=1}^h \frac{\mathsf{u}}{\mu_j}(\frac{\mathsf{u}}{\mu_j} - \bar\psi_{n+j})},$$

where

$$\beta \in H_2(\mathcal{X}), \quad \beta' = \beta + \Big(\sum_{j=1}^h \mu_j\Big) b.$$

3.13. Generating functions of open-closed GW invariants

From now on, we assume the generic stabiliser K is trivial so that $\mathcal{X} = \mathcal{X}^{\mathrm{rig}}$ is a toric Calabi-Yau 3-orbifold. Then

$$\chi_3 : G_\tau \longrightarrow \boldsymbol{\mu}_{\mathfrak{m}}, \quad \lambda \mapsto e^{2\pi\sqrt{-1}\bar\lambda/\mathfrak{m}}$$

is a group isomorphism. We have $\bar{N} = N$ and $\bar{b}_i = \bar{b}_i$. In particular,

$$b_1 = \mathfrak{r}\mathsf{v}_1 - \mathfrak{s}\mathsf{v}_2 + \mathsf{v}_3, \quad b_2 = \mathfrak{m}\mathsf{v}_2 + \mathsf{v}_3, \quad b_3 = \mathsf{v}_3.$$

There exists $m_a, n_a \in \mathbb{Z}$, such that

$$b_{3+a} = m_a\mathsf{v}_1 + n_a\mathsf{v}_2 + \mathsf{v}_3, \quad a = 1, \dots, k.$$

Introduce variables $\{X_j \mid j = 1, \dots, h\}$, and let

$$\boldsymbol{\tau}_2 = \sum_{i=1}^{m} \tau_i u_i,$$

where $u_1, \dots, u_m$ form a basis of $H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$. We choose $\mathbb{T}'$-equivariant lifting of $\boldsymbol{\tau}_2$ as follows: for each $u_i \in H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$, we choose the unique $\mathbb{T}'$-equivariant lifting $u_i^{\mathbb{T}'} \in H^2_{\mathrm{CR},\mathbb{T}'}(\mathcal{X};\mathbb{Q})$ such that $\iota^*_\sigma u_i^{\mathbb{T}'} = 0 \in H^2_{\mathrm{CR},\mathbb{T}'}(\mathfrak{p}_\sigma;\mathbb{Q})$, where $\iota^*_\sigma : H^2_{\mathrm{CR},\mathbb{T}'}(\mathcal{X};\mathbb{Q}) \to H^2_{\mathrm{CR},\mathbb{T}'}(\mathfrak{p}_\sigma;\mathbb{Q})$ is induced by the inclusion map $\iota_\sigma : \mathfrak{p}_\sigma \to \mathcal{X}$.

We define $\xi_0 := e^{-\pi\sqrt{-1}/\mathfrak{m}}$. If $(\mathcal{L}, f)$ is a framed outer brane, define

$$\begin{aligned} &F^{\mathcal{X},(\mathcal{L},f)}_{g,h}(\boldsymbol{\tau}_2, Q^b, X_1, \dots, X_h) \\ &= \sum_{\beta', n\geq 0} \sum_{(\mu_j,\lambda_j)\in H_1(\mathcal{L};\mathbb{Z})} \frac{\langle (\iota_f^* \boldsymbol{\tau}_2)^n \rangle^{\mathcal{X},(\mathcal{L},f)}_{g,\beta,(\mu_1,\lambda_1),\dots,(\mu_h,\lambda_h)}}{n!} \prod_{j=1}^{h} (Q^b X_j)^{\mu_j} \cdot \xi_0^{\bar{\lambda}_1} \mathbf{1}_{\lambda_1^{-1}} \otimes \cdots \otimes \xi_0^{\bar{\lambda}_h} \mathbf{1}_{\lambda_h^{-1}}, \end{aligned} \tag{23}$$

which is a function that takes values in $H^*_{\mathrm{CR}}(\mathcal{B}G_\tau;\mathbb{C})^{\otimes h}$, where

$$H^*_{\mathrm{CR}}(\mathcal{B}G_\tau;\mathbb{C}) = \bigoplus_{\lambda\in G_\tau} \mathbb{C}\mathbf{1}_\lambda.$$

When $\lambda = 1$ is the identity element of G_τ, $\mathbf{1}_1 = 1$ is the unit of $H^*_{\mathrm{CR}}(\mathcal{B}G_\tau;\mathbb{C})$.

If $(\mathcal{L}, f)$ is a framed inner brane, define

$$\begin{aligned} F^{\mathcal{X},(\mathcal{L},f)}_{g,h}(\boldsymbol{\tau}_2, Q^b, X_1, \dots, X_h) = \sum_{\beta', n\geq 0} \sum_{(\mu_j,\lambda_j)\in H_1(\mathcal{L};\mathbb{Z})} \frac{\langle (\iota_f^* \boldsymbol{\tau}_2)^n \rangle^{\mathcal{X},(\mathcal{L},f)}_{g,\beta,(\mu_1,\lambda_1),\dots,(\mu_h,\lambda_h)}}{n!} \\ \cdot \prod_{\substack{j\in\{1,\dots,h\}\\ \mu_j>0}} (Q^b X_j)^{\mu_j} \prod_{\substack{j\in\{1,\dots,h\}\\ \mu_j<0}} (Q^{b-\alpha} X_j)^{\mu_j} \cdot \xi_0^{\bar{\lambda}_1} \mathbf{1}_{\lambda_1^{-1}} \otimes \cdots \otimes \xi_0^{\bar{\lambda}_h} \mathbf{1}_{\lambda_h^{-1}}, \end{aligned} \tag{24}$$

which is a function that takes values in $H^*_{\mathrm{CR}}(\mathcal{B}G_\tau;\mathbb{C})^{\otimes h}$.

3.14. The equivariant J-function and the disk potential

Let $\{u_i\}_{i=1}^N$ be a homogeneous basis of $H^*_{\mathbb{T},\mathrm{CR}}(\mathcal{X};\mathbb{Q})$ and $\{u^i\}_{i=1}^N$ be its dual basis. Define

$$\boldsymbol{\tau} = \sum_{i=1}^{N} \tau_i u_i = \boldsymbol{\tau}_0 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_{>2},$$

where

$$\boldsymbol{\tau}_0 \in H^0_{\mathbb{T},\mathrm{CR}}(\mathcal{X};\mathbb{C}), \quad \boldsymbol{\tau}_2 \in H^2_{\mathbb{T},\mathrm{CR}}(\mathcal{X};\mathbb{C}), \quad \boldsymbol{\tau}_{>2} \in H^{>2}_{\mathbb{T},\mathrm{CR}}(\mathcal{X};\mathbb{C}).$$

The J-function [71, 26, 40] is a $H^*_{\mathbb{T},\mathrm{CR}}(\mathcal{X})$-valued function:

$$J(\boldsymbol{\tau}, z) := 1 + \sum_{\beta\geq 0, n\geq 0} \frac{1}{n!} \sum_{i=1}^{N} \left\langle 1, \boldsymbol{\tau}^n, \frac{u_i}{z-\bar{\psi}} \right\rangle^{\mathcal{X},\mathbb{T}}_{0,\beta} u^i.$$

Then

$$\iota_\sigma^* J(\boldsymbol{\tau}, z)\Big|_{\mathsf{u}_1=\mathsf{u},\, \mathsf{u}_2=f\,\mathsf{u},\, \mathsf{u}_3=0} = \sum_{k\in G_\sigma} J^f_{\sigma,k}(\boldsymbol{\tau}, z)\mathbf{1}_k,$$

where

$$J^f_{\sigma,k}(\boldsymbol{\tau}, z) = \delta_{k,1} + \sum_{\beta\geq 0, n\geq 0} \frac{1}{n!} \sum_{i=1}^{N} \left\langle 1, (\iota_f^*\boldsymbol{\tau})^n, \frac{|G_\sigma|\phi^f_{\sigma,k^{-1}}}{z-\bar{\psi}} \right\rangle^{\mathcal{X},\mathbb{T}_f}_{0,\beta}.$$

As a special case of Proposition 3.3,

$$\begin{aligned}
\langle \gamma_1,\ldots,\gamma_n\rangle^{\mathcal{X},(\mathcal{L},f)}_{0,\beta+d_0 b,(d_0,k)} &= D'_{d_0,\lambda,f} \int_{[\overline{\mathcal{M}}_{0,n+1}(\mathcal{X},\beta)]^{\mathrm{vir}}} \frac{\left(\prod_{i=1}^n \mathrm{ev}_i^*\gamma_i \cup \mathrm{ev}_{n+1}^*\phi^f_{(h^+(d_0,\lambda))^{-1}}\right)}{\frac{\mathsf{u}}{d_0}\left(\frac{\mathsf{u}}{d_0} - \bar{\psi}_{n+1}\right)} \\
&= D'_{d_0,\lambda,f} \left\langle 1, \gamma_1,\ldots,\gamma_n, \frac{\phi^f_{(h^+(d_0,\lambda))^{-1}}}{\frac{\mathsf{u}}{d_0} - \bar{\psi}} \right\rangle^{\mathcal{X}}_{0,\beta};
\end{aligned}$$

$$\begin{aligned}
F^{\mathcal{X},(\mathcal{L},f)}_{0,1}(\boldsymbol{\tau}_2, Q^b, X_1) &= \sum_{\beta,n\geq 0} \sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z})} \frac{1}{n!} \langle (\iota_f^*\boldsymbol{\tau}_2)^n \rangle^{\mathcal{X},(\mathcal{L},f)}_{0,\beta+d_0 b,(d_0,\lambda)} (Q^b X)^{d_0} \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}} \\
&= \frac{1}{|G_\sigma|} \sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z})} (Q^b X_1)^{d_0} D'_{d_0,\lambda,f} J^f_{\sigma,h^+(d_0,\lambda)}\left(\boldsymbol{\tau}_2, \frac{\mathsf{u}}{d_0}\right) \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}}.
\end{aligned}$$

Proposition 3.4. *Let $X = Q^b X_1$. If $(\mathcal{L}, f)$ is a framed outer brane, then*

$$F^{\mathcal{X},(\mathcal{L},f)}_{0,1}(\boldsymbol{\tau}_2, X) = \frac{1}{|G_\sigma|} \sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z})} X^{d_0} D'_{d_0,\lambda,f} J^f_{\sigma,h^+(d_0,\lambda)}\left(\boldsymbol{\tau}_2, \frac{\mathsf{u}}{d_0}\right) \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}}.$$

If $(\mathcal{L}, f)$ is a framed inner brane, then

$$\begin{aligned}
F^{\mathcal{X},(\mathcal{L},f)}_{0,1}(\boldsymbol{\tau}_2, Q, X) &= \frac{1}{|G_\sigma|} \sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z}), d_0>0} X^{d_0} D'_{d_0,\lambda,f} J^f_{\sigma_+,h^+(d_0,\lambda)}\left(\boldsymbol{\tau}_2, \frac{\mathsf{u}}{d_0}\right) \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}} \\
&+ \frac{1}{|G_\sigma|} \sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z}), d_0<0} X^{d_0} Q^{-d_0\alpha} D'_{d_0,\lambda,f} J^f_{\sigma_-,h^-(d_0,\lambda)}\left(\boldsymbol{\tau}_2, \frac{\mathsf{u}}{d_0}\right) \cdot \frac{\mathfrak{r}_+}{\mathfrak{r}_-} \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}}.
\end{aligned}$$

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4. Mirror symmetry for the disk amplitudes

4.1. The equivariant I-function and the equivariant mirror theorem

We choose $p_1, \ldots, p_k \in \mathbb{L}^\vee \cap \widetilde{\mathrm{Nef}}_{\mathcal{X}}$ such that

- $\{p_1, \ldots, p_k\}$ is a $\mathbb{Q}$-basis of $\mathbb{L}^\vee_{\mathbb{Q}}$.
- $\{\bar{p}_1, \ldots, \bar{p}_{k'}\}$ is a $\mathbb{Q}$-basis of $H^2(\mathcal{X}; \mathbb{Q})$.
- $p_a = D_{3+a}$ for $a = k' + 1, \ldots, k$.

We define *charges* $m_i^{(a)} \in \mathbb{Q}$ by $D_i = \sum_{a=1}^k m_i^{(a)} p_a$.

Let $q_0', q_1, \ldots, q_k$ be $k+1$ formal variables, and define $q^\beta = q_1^{\langle p_1, \beta\rangle} \cdots q_k^{\langle p_k, \beta\rangle}$ for $\beta \in \mathbb{K}$. We take the equivariant lifting $\bar{p}_a^{\mathcal{T}} \in H^2_{\mathbb{T}}(\mathcal{X}; \mathbb{Q})$ of $\bar{p}_a \in H^2(\mathcal{X}; \mathbb{Q})$. The equivariant I-function is an $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$-valued power series defined as follows [47]:

$$I(q_0', q, z) = e^{\frac{\log q_0' + \sum_{a=1}^{k'} \bar{p}_a^{\mathcal{T}} \log q_a}{z}} \sum_{\beta \in \mathbb{K}_{\mathrm{eff}}} q^\beta \prod_{i=1}^{r'} \frac{\prod_{m=\lceil \langle D_i, \beta\rangle \rceil}^{\infty} (\bar{D}_i^{\mathcal{T}} + (\langle D_i, \beta\rangle - m)z)}{\prod_{m=0}^{\infty} (\bar{D}_i^{\mathcal{T}} + (\langle D_i, \beta\rangle - m)z)} \cdot \prod_{i=r'+1}^{r} \frac{\prod_{m=\lceil \langle D_i, \beta\rangle \rceil}^{\infty} (\langle D_i, \beta\rangle - m)z}{\prod_{m=0}^{\infty} (\langle D_i, \beta\rangle - m)z} \mathbf{1}_{v(\beta)},$$

where $q^\beta = \prod_{a=1}^k q_a^{\langle p_a, \beta\rangle}$. Note that $\langle p_a, \beta\rangle \geq 0$ for $\beta \in \mathbb{K}_{\mathrm{eff}}$. The equivariant I-function can be rewritten as

$$I(q_0', q, z) = e^{\frac{\log q_0' + \sum_{a=1}^{k'} \bar{p}_a^{\mathcal{T}} \log q_a}{z}} \sum_{\beta \in \mathbb{K}_{\mathrm{eff}}} \frac{q^\beta}{z^{\langle \hat{\rho}, \beta\rangle + \mathrm{age}(v(\beta))}} \prod_{i=1}^{r'} \frac{\prod_{m=\lceil \langle D_i, \beta\rangle \rceil}^{\infty} (\frac{\bar{D}_i^{\mathcal{T}}}{z} + \langle D_i, \beta\rangle - m)}{\prod_{m=0}^{\infty} (\frac{\bar{D}_i^{\mathcal{T}}}{z} + \langle D_i, \beta\rangle - m)} \cdot \prod_{i=r'+1}^{r} \frac{\prod_{m=\lceil \langle D_i, \beta\rangle \rceil}^{\infty} (\langle D_i, \beta\rangle - m)}{\prod_{m=0}^{\infty} (\langle D_i, \beta\rangle - m)} \mathbf{1}_{v(\beta)},$$

where $\hat{\rho} = D_1 + \cdots + D_r \in \widetilde{C}_{\mathcal{X}}$.

Since $\mathcal{X}$ is a Calabi-Yau orbifold, $\mathrm{age}(v)$ is an integer for any $v \in \mathrm{Box}(\mathbf{\Sigma})$. Then

$$H^{\leq 2}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) = H^0_{\mathrm{CR}.\mathbb{T}}(\mathcal{X}) \oplus H^2_{\mathrm{CR},\mathbb{T}}(\mathcal{X}).$$

Let $\mathcal{Q} = \mathbb{Q}(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3)$ be the fractional field of $H^*_{\mathbb{T}}(\mathrm{point}; \mathbb{Q})$:

$$H^0_{\mathrm{CR},\mathbb{T}}(\mathcal{X}; \mathcal{Q}) = \mathcal{Q}\mathbf{1},$$
$$H^2_{\mathrm{CR},\mathbb{T}}(\mathcal{X}; \mathcal{Q}) = \bigoplus_{a=1}^{k'} \mathcal{Q}\bar{p}_a^{\mathcal{T}} \oplus \bigoplus_{\substack{v \in \mathrm{Box}(\mathbf{\Sigma}) \\ \mathrm{age}(v)=1}} \mathcal{Q}\mathbf{1}_v.$$

Recall that the embedding of the stacky fixed point $\mathfrak{p}_\sigma$ is $\iota_\sigma : \mathfrak{p}_\sigma \to \mathcal{X}$. We choose the lifting $\bar{p}_a^{\mathcal{T}}$ such that $\iota_\sigma^* \bar{p}_a^{\mathcal{T}} = 0$.

For $i = 1, \ldots, r$, we will define $\Omega_i \subset \mathbb{K}_{\mathrm{eff}} - \{0\}$ and $A_i(q)$ supported on Ω_i. We observe that if $\beta \in \mathbb{K}_{\mathrm{eff}}$ and $v(\beta) = 0$, then $\langle D_i, \beta\rangle \in \mathbb{Z}$ for $i = 1, \ldots, r$.

- For $i = 1, \ldots, r'$, let

$$\Omega_i = \left\{\beta \in \mathbb{K}_{\mathrm{eff}} : v(\beta) = 0, \langle D_i, \beta\rangle < 0 \text{ and } \langle D_j, \beta\rangle \geq 0 \text{ for } j \in \{1, \ldots, r\} - \{i\}\right\}.$$

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Then $\Omega_i \subset \{\beta \in \mathbb{K}_{\text{eff}} : v(\beta) = 0, \beta \neq 0\}$. We define

$$A_i(q) := \sum_{\beta \in \Omega_i} q^\beta \frac{(-1)^{-\langle D_i, \beta \rangle - 1}(-\langle D_i, \beta \rangle - 1)!}{\prod_{j \in \{1,\ldots,r\}-\{i\}} \langle D_j, \beta \rangle !}.$$

◦ For $i = r' + 1, \ldots, r$, let

$$\Omega_i := \{\beta \in \mathbb{K}_{\text{eff}} : v(\beta) = b_i, \langle D_j, \beta \rangle \notin \mathbb{Z}_{<0} \text{ for } j = 1, \ldots, r\},$$

and define

$$A_i(q) = \sum_{\beta \in \Omega_i} q^\beta \prod_{j=1}^{r} \frac{\prod_{m=\lceil \langle D_j, \beta \rangle \rceil}^{\infty} (\langle D_j, \beta \rangle - m)}{\prod_{m=0}^{\infty} (\langle D_j, \beta \rangle - m)}.$$

Let σ be the smallest cone containing b_i. Then

$$b_i = \sum_{j \in I'_\sigma} c_j(b_i) b_j,$$

where $c_j(b_i) \in (0, 1)$ and $\sum_{j \in I'_\sigma} c_j(b_i) = 1$. There exists a unique $D_i^\vee \in \mathbb{L}_\mathbb{Q}$ such that

$$\langle D_j, D_i^\vee \rangle = \begin{cases} 1, & j = i, \\ -c_j(b_i), & j \in I'_\sigma, \\ 0, & j \in I_\sigma - \{i\}. \end{cases}$$

Then

$$A_i(q) = q^{D_i^\vee} + \text{ higher order terms}$$

$$I(q'_0, q, z) = 1 + \frac{1}{z}(\log q'_0 \mathbf{1} + \sum_{a=1}^{k'} \log(q_a) \bar{p}_a^{\mathcal{T}} + \sum_{i=1}^{r'} A_i(q) \bar{D}_i^{\mathcal{T}} + \sum_{i=r'+1}^{r} A_i(q) \mathbf{1}_{b_i}) + o(z^{-1}).$$

For $i = 1, \ldots, r'$,

$$\bar{\mathcal{D}}_i^{\mathcal{T}} = \sum_{a=1}^{k'} m_i^{(a)} \bar{p}_a^{\mathcal{T}} + \lambda_i,$$

where $\lambda_i \in H^2(B\mathbb{T}; \mathbb{Q})$. Let $S_a(q) := \sum_{i=1}^{r'} m_i^{(a)} A_i(q)$. Then

$$I(q'_0, q, z) = 1 + \frac{1}{z}((\log q'_0 + \sum_{i=1}^{r'} \lambda_i A_i(q)) \mathbf{1} + \sum_{a=1}^{k'} (\log(q_a) + S_a(q)) \bar{p}_a^{\mathcal{T}} + \sum_{i=r'+1}^{r} A_i(q) \mathbf{1}_{b_i}) + o(z^{-1}).$$

Recall that the $\mathbb{T}$-equivariant J-function for $\mathcal{X}$ is

$$J(\boldsymbol{\tau}, z) = 1 + \sum_{\beta \geq 0, n \geq 0} \sum_{i=1}^{N} \frac{1}{n!} \langle 1, \boldsymbol{\tau}^n, \frac{u_i}{z - \bar{\psi}} \rangle_{0,\beta}^{\mathcal{X},\mathbb{T}} u^i,$$

where $\{u_i\}_{i=1}^N$ is an $H^*(B\mathbb{T})$-basis of $H^*_\mathbb{T}(\mathcal{X}; \mathbb{Q})$ and $\{u^i\}_{i=1}^N$ is the dual basis. Assume $u_0 = \mathbf{1}$, $u_a = \bar{p}_a^{\mathcal{T}}$ for $a = 1, \ldots, k'$ and $u_a = \mathbf{1}_{b_{a+3}}$ for $a = k' + 1, \ldots, k$. The mirror theorem for toric orbifolds [27] implies the following theorem.

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Theorem 4.1 (Coates-Corti-Iritani-Tseng [27]). *If the toric orbifold $\mathcal{X}$ satisfies Assumption 2.6, then*

$$e^{\frac{\tau_0(q_0',q)}{z}} J(\boldsymbol{\tau}_2(q), z) = I(q_0', q, z),$$

where the equivariant closed mirror map $(q_0', q) \mapsto \tau_0(q_0', q)1 + \boldsymbol{\tau}_2(q)$ is determined by the first-order term in the asymptotic expansion of the I-function

$$I(q_0', q, z) = 1 + \frac{\tau_0(q_0', q)1 + \boldsymbol{\tau}_2(q)}{z} + o(z^{-1}).$$

More explicitly, the equivariant closed mirror map is given by

$$\tau_0 = \log(q_0') + \sum_{i=1}^{r'} \lambda_i A_i(q),$$
$$\tau_a = \begin{cases} \log(q_a) + S_a(q), & 1 \le a \le k', \\ A_{a-3}(q), & k'+1 \le a \le k. \end{cases} \tag{25}$$

4.2. The pullback of the disk potential under the mirror map

By Proposition 3.4, if $(\mathcal{L}, f)$ is a framed outer brane, then

$$F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, Q, X) = \frac{1}{|G_\sigma|} \sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z})} X^{d_0} D'_{d_0,\lambda,f} J^f_{\sigma,h(d_0,\lambda)}\Big(\boldsymbol{\tau}_2, \frac{\mathsf{u}}{d_0}\Big)\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}}.$$

Let $F^{\mathcal{X},(\mathcal{L},f)}(q,X)$ be the pullback of $F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, Q, X)$ under the closed mirror map.

By Proposition 3.4, if $(\mathcal{L}, f)$ is a framed inner brane, then

$$\begin{aligned} &F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, Q, X) \\ &= \frac{1}{|G_\sigma|} \sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z}), d_0>0} X^{d_0} D'_{d_0,\lambda,f} J^f_{\sigma_+,h^+(d_0,\lambda)}\Big(\boldsymbol{\tau}_2, \frac{\mathsf{u}}{d_0}\Big)\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}} \\ &\quad + \frac{1}{|G_\sigma|} \sum_{(d_0,k^+,k^-)\in H_{\tau,\sigma_+,\sigma_-}, d_0<0} X^{d_0} Q^{-d_0\alpha} D'_{d_0,\lambda,f} J^f_{\sigma_-,h^-(d_0,\lambda)}\Big(\boldsymbol{\tau}_2, \frac{\mathsf{u}}{d_0}\Big)\cdot \frac{\mathfrak{r}_+}{\mathfrak{r}_-}\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}}. \end{aligned}$$

Given $\sigma \in \Sigma(3)$, $k \in G_\sigma$, and $f \in \mathbb{Z}$, define $I^f_{\sigma,k}(q,z)$ by

$$\iota_\sigma^* I(q,z)\Big|_{\mathsf{u}_1=\mathsf{u},\, \mathsf{u}_2=f\,\mathsf{u},\, \mathsf{u}_3=0} = \sum_{k\in G_\sigma} I^f_{\sigma,k}(q,z)\mathbf{1}_k.$$

Since a toric Calabi-Yau orbifold satisfies the weak Fano condition, by the equivariant mirror theorem (Theorem 4.1), we may write $F^{\mathcal{X},(\mathcal{L},f)}(q,X)$ in terms of $I^f_{\sigma,k}(q,z)$ in case of an outer brane and in terms of $I^f_{\sigma_+,k^+}(q,z)$ and $I^f_{\sigma_-,k^-}(q,z)$ in case of an inner brane.

Lemma 4.2. *If $(\mathcal{L}, f)$ is a framed outer brane, then*

$$F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(q, X) = \frac{1}{|G_\sigma|} \sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z})} X^{d_0} D'_{d_0,\lambda,f}\, e^{\frac{-d_0\tau_0(q)}{\mathsf{u}}} I^f_{\sigma,h(d_0,\lambda)}\Big(q, \frac{\mathsf{u}}{d_0}\Big)\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}}. \tag{26}$$

If $(\mathcal{L}, f)$ is a framed inner brane, then

$$\begin{aligned}
&F_{0,1}^{\mathcal{X},(\mathcal{L},f^+,f^-)}(q,X)\\
&= \frac{1}{|G_\sigma|}\sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z}),d_0>0} X^{d_0} D'_{d_0,\lambda,f}\, e^{\frac{-d_0\tau_0(q)}{\mathsf{u}}} I^f_{\sigma_+,h^+(d_0,\lambda)}\Big(q,\frac{\mathsf{u}}{d_0}\Big)\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}}\\
&\quad+\frac{1}{|G_\sigma|}\sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z}),d_0<0} X^{d_0} Q^{-d_0\alpha} D'_{d_0,\lambda,f}\, e^{\frac{-d_0\tau_0(q)}{\mathsf{u}}} I^f_{\sigma_-,h^-(d_0,\lambda)}\Big(q,\frac{\mathsf{u}}{d_0}\Big)\cdot\frac{\mathfrak{r}_+}{\mathfrak{r}_-}\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}}.
\end{aligned}$$

Let $(\mathcal{L}, f)$ be a framed brane, and let τ, $\sigma=\sigma_+,\sigma_-$ be defined as in Section 3.3. Recall that

$$\begin{aligned}
&I'_\sigma=\{i\in\{1,\dots,r'\}:\rho_i\subset\sigma\}=\{1,2,3\},\quad I_\sigma=\{1,\dots,r\}\setminus I'_\sigma,\\
&I'_\tau=\{i\in\{1,\dots,r'\}:\rho_i\subset\tau\}=\{2,3\},\quad I_\tau=\{1,\dots,r\}\setminus I'_\tau,\\
&\mathbb{K}_{\mathrm{eff},\sigma}=\{\beta\in\mathbb{L}_{\mathbb{Q}}:\langle D_i,\beta\rangle\in\mathbb{Z}_{\geq 0}\text{ for } i\in I_\sigma\}.
\end{aligned}$$

In case that $\mathcal{L}$ is inner,

$$\begin{aligned}
&I'_{\sigma_-}=\{2,3,4\},\ I_{\sigma_-}=\{1,\dots,r\}\setminus I'_{\sigma_-}\\
&\mathbb{K}_{\mathrm{eff},\sigma-}=\{\beta\in\mathbb{L}_{\mathbb{Q}}:\langle D_i,\beta\rangle\in\mathbb{Z}_{\geq 0}\text{ for } i\in I_{\sigma_-}\}.
\end{aligned}$$

Let $\mathsf{b}_{\sigma,i}=\iota_\sigma^*\bar D_i^{\mathcal{T}}\in H^2_{\mathbb{T}}(\mathfrak{p}_\sigma;\mathbb{Q})=H^2(B\mathbb{T};\mathbb{Q})$ for $1\leq i\leq r$, and then $\mathsf{b}_i=0$ for $r'+1\leq i\leq r$. For $\beta\in\mathbb{K}_{\mathrm{eff},\sigma}$, define an $H^*(B\mathbb{T};\mathbb{Q})$-valued

$$I(\sigma,\beta):=\prod_{i=1}^r\frac{\prod_{m=\lceil\langle D_i,\beta\rangle\rceil}^\infty(\mathsf{b}_{\sigma,i}+(\langle D_i,\beta\rangle-m)\frac{\mathsf{u}_1}{d_0})}{\prod_{m=0}^\infty(\mathsf{b}_{\sigma,i}+(\langle D_i,\beta\rangle-m)\frac{\mathsf{u}_1}{d_0})}. \tag{27}$$

Recall that $\iota_\sigma^*\bar p_a^{\mathcal{T}}=0$, so

$$\iota_\sigma^* I(q,z)|_{z=\frac{\mathsf{u}_1}{d_0}}=\sum_{\beta\in\mathbb{K}_{\mathrm{eff},\sigma}} e^{\frac{d_0}{\mathsf{u}_1}\log q'_0} q^\beta I(\sigma,\beta)\mathbf{1}_{v(\beta)}.$$

With the above notation, if $\mathcal{L}$ is an outer brane, we can rewrite $F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(q,X)$ as

$$F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(q,X)=\frac{1}{|G_\sigma|}\sum_{(d_0,\lambda)\in H_1(\mathcal{L};\mathbb{Z})}\ \sum_{\beta\in\mathbb{K}_{\mathrm{eff},\sigma},v(\beta)=h(d_0,\lambda)} x^{d_0}q^\beta D'_{d_0,\lambda,f} I^f(\sigma,\beta)\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}},$$

where $I^f(\sigma,\beta)=I(\sigma,\beta)|_{\mathsf{u}_1=\mathsf{u},\,\mathsf{u}_2=f\,\mathsf{u},\,\mathsf{u}_3=0}$, and

$$x=X\exp\Big(\frac{\log q'_0-\tau_0(q)}{\mathsf{u}_1}\Big)$$

is the B-brane moduli parameter.

Following [51, 59], we define *extended charge vectors*

$$\{m_i^{(a)}\}_{i=1,\dots,r}^{a=0,\dots,k}=\begin{pmatrix} w_1 & w_2 & w_3 & 0 & \dots\\ & & \{m_i^{(a)}\}_{i=1,\dots,r}^{a=1,\dots,k} & & \end{pmatrix},$$

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such that $m_i^{(0)} = w_i$ for $i = 1, 2, 3$ and $m_i^{(0)} = 0$ for $i = 4, \ldots, r$. Recall that

$$\tau_0 + \sum_{a=1}^{k'} \tau_a \bar{p}_a^{\mathcal{T}} + \sum_{a=k'+1}^{k} \tau_a \mathbf{1}_{b_{a+3}} = \log q_0' + \sum_{a=1}^{k'} \log q_a \bar{p}_a^{\mathcal{T}} + \sum_{i=1}^{r'} A_i(q) \bar{\mathcal{D}}_i^{\mathcal{T}} + \sum_{i=r'+1}^{r} A_i(q) \mathbf{1}_{b_i}.$$

We pull back the above identity under ι_σ^*. Since

$$\iota_\sigma^* \bar{p}_a^{\mathcal{T}} = 0, \quad \iota_\sigma^* \mathcal{D}_i^{\mathcal{T}}\Big|_{\mathsf{u}_1=\mathsf{u},\, \mathsf{u}_2=f\,\mathsf{u},\, \mathsf{u}_3=0} = m_i^{(0)} \mathsf{u},$$

we get

$$\tau_0(q_0', q) = \log q_0' + \sum_{i=1}^{r'} m_i^{(0)} A_i(q) \mathsf{v}.$$

So the open mirror map is given by

$$\log X = \log x + \sum_{i=1}^{r'} m_i^{(0)} A_i(q). \tag{28}$$

If $\mathcal{L}$ is inner, we further set $Q = q$. Denote the pullback of the disk potential $W^{\mathcal{X},(\mathcal{L},f)}(q,x)$ to be the pullback of $F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(q,X)$ under this open mirror map. Then by Lemma 4.2,

$$|G_\sigma| W^{\mathcal{X},(\mathcal{L},f)}(q,x) = \begin{cases} \displaystyle\sum_{\substack{d_0>0, \beta\in\mathbb{K}_{\mathrm{eff},\sigma} \\ v(\beta)=h(d_0,\lambda)}} x^{d_0} q^\beta D'_{d_0,\lambda,f} I^f(\sigma,\beta) \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}}, & \mathcal{L} \text{ is outer,} \\ \displaystyle\sum_{\substack{d_0>0, \beta\in\mathbb{K}_{\mathrm{eff},\sigma_+} \\ v(\beta)=h^+(d_0,\lambda)}} x^{d_0} q^\beta D'_{d_0,\lambda,f} I^f(\sigma^+,\beta) \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}} & \\ \displaystyle\qquad + \frac{\mathfrak{r}_+}{\mathfrak{r}_-} \sum_{\substack{d_0<0, \beta\in\mathbb{K}_{\mathrm{eff},\sigma_-} \\ v(\beta)=h^-(d_0,\lambda)}} x^{d_0} q^{\beta - d_0\alpha} D'_{d_0,\lambda,f} I^f(\sigma^-,\beta) \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}}, & \mathcal{L} \text{ is inner.} \end{cases} \tag{29}$$

Given $\widetilde{\beta} = (d_0, \beta) \in \mathbb{Z} \times \mathbb{K}_\sigma$, define the extended or open sector pairing to be

$$\langle D_i, \widetilde{\beta} \rangle = m_i^{(0)} d_0 + \langle D_i, \beta \rangle.$$

Recall that $\{D_i : i \in I_\sigma\}$ is a $\mathbb{Q}$-basis of $\mathbb{L}_{\mathbb{Q}}^\vee \cong \mathbb{Q}^k$ and a $\mathbb{Z}$-basis of $\mathbb{K}_\sigma^\vee \cong \mathbb{Z}^k$. Let $v_a = D_{a+3}$ for $a = 1, \ldots, k.$, and let $\{h_a\}_{a=1,\ldots,k}$ be the dual $\mathbb{Q}$-basis of $\mathbb{L}_{\mathbb{Q}}$. Then $\{h_a\}_{a=1,\ldots,k}$ is a $\mathbb{Z}$-basis of $\mathbb{K}_\sigma \cong \mathbb{Z}^k$, and

$$\mathbb{K}_{\mathrm{eff},\sigma} = \sum_{a=1}^{k} \mathbb{Z}_{\geq 0} h_a.$$

Given any $(d_0, \beta) \in \mathbb{Z} \times \mathbb{K}_\sigma$, define

$$q^{\widetilde{\beta}} = x^{d_0} q^\beta = x^{d_0} \prod_{a=1}^{k} q_a^{\langle p_a, \beta \rangle}.$$

Define

$$\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L}) = \{\widetilde{\beta} = (d_0,\beta) \in \mathbb{Z}\times\mathbb{K}_{\mathrm{eff},\sigma} : \langle D_1,\widetilde{\beta}\rangle \in \mathbb{Z}_{\geq 0}, d_0 \neq 0\}.$$

Theorem 4.3. *Assuming the Aganagic-Vafa brane $(\mathcal{L}, f)$ is either inner or outer,*

$$W^{\mathcal{X},(\mathcal{L},f)}(q,x) = \sum_{\substack{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})\\ v(\beta)=h(d_0,\lambda)}} q^{\widetilde{\beta}} A_{\widetilde{\beta}}^{\mathcal{X},(\mathcal{L},f)} \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}}, \tag{30}$$

where

$$A_{\widetilde{\beta}=(d_0,\beta)}^{\mathcal{X},(\mathcal{L},f)} = \frac{-(-1)^{\lfloor\langle D_3,\widetilde{\beta}\rangle\rfloor}}{\mathfrak{m}d_0\prod_{i\in I_\tau}\langle D_i,\widetilde{\beta}\rangle!}\cdot\frac{\Gamma(-\langle D_3,\widetilde{\beta}\rangle)}{\Gamma(\langle D_2,\widetilde{\beta}\rangle+1)}.$$

Proof. Assume $\mathcal{L}$ is an outer brane. Let $\widetilde{\beta} = (d_0,\beta)$, and let $\epsilon_j = c_j(v)$ for $j = 1, 2, 3$. By equation (29) (and recall that $|G_\sigma| = \mathfrak{r}\mathfrak{m}$),

$$W^{\mathcal{X},(\mathcal{L},f)}(q,x) = \frac{1}{\mathfrak{r}\mathfrak{m}} \sum_{\substack{d_0>0\\ \beta\in\mathbb{K}_{\mathrm{eff},\sigma}\\ v(\beta)=h(d_0,\lambda)}} x^{d_0} q^{\beta} D'_{d_0,\lambda,f} I^f(\sigma,\beta)\xi_0^{\bar{\lambda}}\mathbf{1}_{\lambda^{-1}}.$$

Given any $\beta\in\mathbb{K}_{\mathrm{eff},\sigma}$, we have $\lceil\langle D_j,\beta\rangle\rceil - \epsilon_j = \langle D_j,\beta\rangle$ for $j = 1, 2, 3$, and $\lceil\langle D_i,\beta\rangle\rceil = \langle D_i,\beta\rangle$ for $i\in I_\sigma$. The disk factors

$$\begin{aligned} D'_{d_0,\lambda,f} &= -(-1)^{\lfloor d_0 w_3 + \frac{\bar{\lambda}}{m}\rfloor}\frac{\mathfrak{r}}{d_0}\left(\frac{\mathsf{u}}{d_0}\right)^{\mathrm{age}(h(d_0,\lambda))}\frac{\prod_{a=1}^{\lfloor d_0 w_1\rfloor+\mathrm{age}(h(d_0,\lambda))-1}(d_0 w_2 + a - \epsilon_2)}{\lfloor d_0 w_1\rfloor!}\\ &= -(-1)^{\lfloor d_0 w_3 + \frac{\bar{\lambda}}{m}\rfloor}\frac{\mathfrak{r}}{d_0}\left(\frac{\mathsf{u}}{d_0}\right)^{\mathrm{age}(h(d_0,\lambda))}\frac{1}{\Gamma(w_1 d_0 - \epsilon_1 + 1)}\cdot\frac{\Gamma(-w_3 d_0 + \epsilon_3)}{\Gamma(w_2 d_0 - \epsilon_2 + 1)}. \end{aligned}$$

The pullback of the coefficients of the I-function is

$$\begin{aligned} I^f(\sigma,\beta) = \left(\frac{\mathsf{u}}{d_0}\right)^{-\mathrm{age}(h(d_0,\lambda))} &\frac{1}{\prod_{i\in I_\sigma}\Gamma(\langle D_i,\beta\rangle+1)}\\ &\cdot\frac{\Gamma(w_1 d_0 - \epsilon_1 + 1)}{\Gamma(\langle D_1,\widetilde{\beta}\rangle+1)}\cdot\frac{\Gamma(w_2 d_0 - \epsilon_2 + 1)}{\Gamma(\langle D_2,\widetilde{\beta}\rangle+1)}\cdot\frac{\Gamma(w_3 d_0 - \epsilon_3 + 1)}{\Gamma(\langle D_3,\widetilde{\beta}\rangle+1)}. \end{aligned}$$

Hence

$$W^{\mathcal{X},(\mathcal{L},f)}(q,x) = \sum_{\substack{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})\\ v(\beta)=h(d_0,\lambda)}} x^{d_0} q^{\beta} A_{(d_0,\beta)}^{\mathcal{X},(\mathcal{L},f)} \xi_0^{\bar{\lambda}} \mathbf{1}_{\lambda^{-1}},$$

where

$$A_{\widetilde{\beta}}^{\mathcal{X},(\mathcal{L},f)} = (-1)^{\lfloor\langle D_3,\widetilde{\beta}\rangle\rfloor}\frac{-1}{\mathfrak{m}d_0}\frac{1}{\prod_{i\in I_\tau}\Gamma(\langle D_i,\widetilde{\beta}\rangle+1)}\cdot\frac{\Gamma(-\langle D_3,\widetilde{\beta}\rangle)}{\Gamma(\langle D_2,\widetilde{\beta}\rangle+1)}.$$

Note that $\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L}) \subset \mathbb{Z}_{>0}\times\mathbb{K}_{\mathrm{eff},\sigma}$, and for any $(d_0,\beta)\in(\mathbb{Z}_{>0}\times\mathbb{K}_{\mathrm{eff},\sigma})\backslash\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})$, $A_{(d_0,\beta)}^{\mathcal{X},(\mathcal{L},f)} = 0$.

In case $\mathcal{L}$ is inner, by equation (29),

$$W^{\mathcal{X},(\mathcal{L},f)}(q,x) = I_+ + I_-,$$

where

$$I_+ = \sum_{\substack{d_0>0,\beta\in\mathbb{K}_{\mathrm{eff},\sigma_+}\\ v(\beta)=h^+(d_0,\lambda)}} x^{d_0}q^{\beta}\frac{-1}{\mathfrak{m}d_0}\frac{(-1)^{\lfloor w_3^+d_0+\langle D_3,\beta\rangle\rfloor}}{\prod_{i\in I_{\sigma^+}}\Gamma(\langle D_i,\beta\rangle+1)}$$
$$\cdot\frac{\Gamma(-w_3^+d_0-\langle D_3,\beta\rangle)}{\Gamma(w_2^+d_0+\langle D_2,\beta\rangle+1)\Gamma(w_1^+d_0+\langle D_1,\beta\rangle+1)}\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}},$$
$$I_- = \sum_{\substack{d_0<0,\beta\in\mathbb{K}_{\mathrm{eff},\sigma_-}\\ v(\beta)=h^-(d_0,\lambda)}} x^{d_0}q^{\beta-d_0\alpha}\frac{-1}{\mathfrak{m}d_0}\frac{(-1)^{\lfloor w_2^-d_0+\langle D_2,\beta\rangle\rfloor}}{\prod_{i\in I_{\sigma^-}}\Gamma(\langle D_i,\beta\rangle+1)}$$
$$\cdot\frac{\Gamma(-w_2^-d_0-\langle D_2,\beta\rangle)}{\Gamma(w_3^-d_0+\langle D_3,\beta\rangle+1)\Gamma(w_1^-d_0+\langle D_4,\beta\rangle+1)}\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}}$$
$$= \sum_{\substack{\beta\in\mathbb{K}_{\mathrm{eff},\sigma_-}\\ \langle\beta,D_4\rangle+w_1^-d_0\in\mathbb{Z}_{\geq 0}\\ d_0<0,v(\beta)=h^-(d_0,\lambda)}} x^{d_0}q^{\beta-d_0\alpha}\frac{-1}{\mathfrak{m}d_0}\frac{(-1)^{\lfloor w_2^-d_0+\langle D_2,\beta\rangle\rfloor}}{\prod_{i\in I_{\sigma^-}}\Gamma(\langle D_i,\beta\rangle+1)}$$
$$\cdot\frac{\Gamma(-w_2^-d_0-\langle D_2,\beta\rangle)}{\Gamma(w_3^-d_0+\langle D_3,\beta\rangle+1)\Gamma(w_1^-d_0+\langle D_4,\beta\rangle+1)}\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}}.$$

We have

$$\langle D_1,\alpha\rangle = w_1^+,\quad \langle D_2,\alpha\rangle = w_2^+-w_2^-,\quad \langle D_3,\alpha\rangle = w_3^+-w_3^-,\quad \langle D_4,\alpha\rangle = -w_1^-,$$

and $\langle D_i,\alpha\rangle = 0$ for $i\in I\backslash\{1,2,3,4\}$. So for $\beta\in\mathbb{K}_{\mathrm{eff},\sigma_-}$,

$$\begin{aligned}\langle D_1,\beta\rangle &= \langle D_1,\beta-d_0\alpha\rangle + d_0w_1^+,\\ w_2^-d_0+\langle D_2,\beta\rangle &= w_2^+d_0+\langle D_2,\beta-d_0\alpha\rangle,\\ w_3^-d_0+\langle D_3,\beta\rangle &= w_3^+d_0+\langle D_3,\beta-d_0\alpha\rangle,\\ d_0w_1^-+\langle D_4,\beta\rangle &= \langle D_4,\beta-d_0\alpha\rangle.\end{aligned}$$

Since the conditions $\langle\beta,D_4\rangle+w_1^-d_0\in\mathbb{Z}_{\geq 0}$ and $\beta\in\mathbb{K}_{\mathrm{eff},\sigma_-}$ imply $(d_0,\beta-d_0\alpha)\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})$,

$$I_- = \sum_{\substack{(d_0,\beta-d_0\alpha)\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})\\ d_0<0}} x^{d_0}q^{\beta-d_0\alpha}\frac{-1}{\mathfrak{m}d_0}\frac{(-1)^{\lfloor w_2^+d_0+\langle D_2,\beta-d_0\alpha\rangle\rfloor}}{\prod_{i\in I_{\sigma^+}}\Gamma(\langle D_i,\beta-d_0\alpha\rangle+1)}$$
$$\cdot\frac{\Gamma(-w_2^+d_0-\langle D_2,\beta-d_0\alpha\rangle)}{\Gamma(w_3^+d_0+\langle D_3,\beta-d_0\alpha\rangle+1)\Gamma(w_1^+d_0+\langle D_1,\beta-d_0\alpha\rangle+1)}\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}}.$$

So

$$I_++I_- = \sum_{\substack{\widetilde\beta\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})\\ v(\beta)=h^+(d_0,\lambda)}} x^{d_0}q^{\beta}\frac{-1}{\mathfrak{m}d_0}\frac{(-1)^{\lfloor\langle D_3,\widetilde\beta\rangle\rfloor}}{\prod_{i\in I_{\sigma^+}}\Gamma(\langle D_i,\widetilde\beta\rangle+1)}\cdot\frac{\Gamma(-\langle D_3,\widetilde\beta\rangle)}{\Gamma(\langle D_2,\widetilde\beta\rangle+1)\Gamma(\langle D_1,\widetilde\beta\rangle+1)}\xi_0^{\bar\lambda}\mathbf{1}_{\lambda^{-1}}.$$

□

Remark 4.4. When $\mathcal{L}$ is an outer brane, the condition $\widetilde{\beta} = (d_0, \beta) \in \mathbb{K}_{\mathrm{eff}}(\mathcal{X}, \mathcal{L})$ implies $d_0 > 0$. One has

$$\sum_{a=1}^{k} \langle D_{a+3}, \beta \rangle b_{a+3} = -\sum_{i=1}^{3} \langle D_i, \beta \rangle b_i.$$

Since $\langle D_{a+3}, \beta \rangle \geq 0$, the fan Σ is convex, and this is an outer brane, every 1-cone b_i is on the same side of the plane spanned by b_2, b_3. Therefore, $\langle \beta, D_1 \rangle \leq 0$. From $w_1 d_0 + \langle D_1, \beta \rangle \in \mathbb{Z}_{\geq 0}$, we see that $d_0 > 0$.

4.3. The B-model and the framed mirror curve

The mirror B-model to the toric Calabi-Yau threefold $\mathcal{X}$ is another noncompact Calabi-Yau hypersurface $Y \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2$, constructed as the Hori-Vafa mirror [45]. It is equivalent to an affine mirror curve $C_q \subset (\mathbb{C}^*)^2$. We state the relevant mirror prediction for disk amplitudes from [6, 5].

4.3.1. Toric degeneration

The main reference of this subsection is [61, Section 3].

The set

$$\Theta_0 = \bigcap_{I \in \mathcal{A}} \sum_{i \in I} \mathbb{Q}_{\geq 0} D_i \subset \mathbb{L}^\vee_{\mathbb{Q}}$$

is a top-dimensional convex cone in $\mathbb{L}^\vee_{\mathbb{Q}} \cong \mathbb{Q}^k$. The cone Θ_0 together with its faces is a fan in $\mathbb{L}^\vee_{\mathbb{R}}$ denoted by Θ. This fan determines a k-dimensional affine toric variety X_Θ.

Consider the exact sequence induced from equation (4) (notice $N_{\mathrm{tor}} = 0$)

$$0 \longrightarrow M' \xrightarrow{\phi'^\vee} \widetilde{M}' \xrightarrow{\psi'^\vee} \mathbb{L}^\vee \longrightarrow 0,$$

where $M' = M/\langle \mathsf{v}_3 \rangle$ and $\widetilde{M}' = \widetilde{M}/\langle \phi^\vee(\mathsf{v}_3) \rangle$. Let $D_i'^{\mathcal{T}}$ be the image of $D_i^{\mathcal{T}}$ when passing to $\widetilde{M}'$. For any proper subset $I \subset \{1, \dots, r\}$ and a cone $\upsilon \in \Theta$, define

$$\widetilde{\Xi}_I = \sum_{i \in I} \mathbb{Q}_{\geq 0} D_i'^{\mathcal{T}}, \quad \widetilde{\Theta}_{I,\upsilon} = (\psi'^\vee)^{-1}(\upsilon) \cap \widetilde{\Xi}_I.$$

Define a fan

$$\widetilde{\Theta} = \{\widetilde{\Theta}_{I,\upsilon} | I \subsetneq \{1, \dots, r\}, \upsilon \in \Theta\} \sqcup \{0\}.$$

This fan determines a toric variety $X_{\widetilde{\Theta}}$. There is a fan morphism $\rho' : \widetilde{\Theta} \to \Theta$, which induces a flat family of toric surfaces $\rho : X_{\widetilde{\Theta}} \to X_\Theta$.

Let $\Theta_0' \subset \mathbb{L}^\vee_{\mathbb{Q}}$ be the cone spanned by $p_1, \dots, p_k$. Let $\mathbb{L}'^\vee := \bigoplus_{a=1}^k \mathbb{Z} p_a$ and let $\mathbb{L}'$ be the dual lattice. Then $\mathbb{L}'^\vee$ is a sublattice of $\mathbb{L}^\vee$ of finite index, and $\mathbb{L}$ is a sublattice of $\mathbb{L}'$ of finite index. Let $\Theta_0^\vee$ and $\Theta_0'^\vee$ be the dual cones of Θ_0 and Θ_0', respectively. We have inclusions

$$\Theta_0' \subset \Theta_0, \quad \Theta_0^\vee \subset \Theta_0'^\vee.$$

Since $\Theta_0^\vee \cap \mathbb{L}$ is a subset of $\Theta_0'^\vee \cap \mathbb{L}'$, we have an injective ring homomorphism

$$\mathbb{C}[\Theta_0^\vee \cap \mathbb{L}] \to \mathbb{C}[\Theta_0'^\vee \cap \mathbb{L}'] = \mathbb{C}[q_1, \dots, q_k],$$

where $q_1, \dots, q_k$ are the variables in Section 4.1. Taking the spectrum, we obtain a morphism

$$\mathbb{A}^k = \mathrm{Spec}(\mathbb{C}[q_1, \dots, q_k]) \longrightarrow X_\Theta = \mathrm{Spec}\big(\mathbb{C}[\Theta_0^\vee \cap \mathbb{L}]\big)$$

and a cartesian diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\widetilde{\nu}"] \arrow[d, "\widetilde{\rho}"'] & X_{\widetilde{\Theta}} \arrow[d, "\rho"'] \\ \mathbb{A}^k \arrow[r, "\nu"] & X_\Theta \end{tikzcd} \tag{31}$$

where $\widetilde{\rho} : \mathfrak{X} \to \mathbb{A}^k$ is a flat family of toric surfaces.

We choose a Kähler class $[\omega(\eta)] \in H^2(X_\Sigma; \mathbb{Z})$ associated to a lattice point $\eta \in \mathbb{L}^\vee$; $[\omega(\eta)]$ is the first Chern class of some ample line bundle over X_Σ. Then it determines a toric graph $\Gamma \in M'_\mathbb{R} \cong \mathbb{R}^2$ up to translation by an element in $M' \cong \mathbb{Z}^2$ (see Section 3.2). The toric graph gives a *polyhedral decomposition* of $M_\mathbb{Q}$ in the sense of [61, Section 3]. It is a covering $\mathcal{P}$ of $M_\mathbb{Q}$ by strongly convex lattice polyhedra. The asymptotic fan of $\mathcal{P}$ is defined to be

$$\Sigma_\mathcal{P} := \{\lim_{a \to 0} a\Xi \subset M'_\mathbb{Q} : \Xi \in \mathcal{P}\}.$$

The fan $\Sigma_\mathcal{P} = \widetilde{\Theta} \cap \rho'^{-1}(0)$ defines a toric surface $\mathbb{S}$, which is the same as the toric surface given by the defining polytope Δ. For each $\Pi \in \mathcal{P}$, let $C(\Pi) \subset M'_\mathbb{Q} \times \mathbb{Q}_{\geq 0}$ be the closure of the cone over $\Xi \times \{1\}$ in $M'_\mathbb{Q} \times \mathbb{Q}$. Then

$$\widetilde{\Sigma}_\mathcal{P} := \{\sigma \text{ is a face of } C(\Pi) : \Pi \in \mathcal{P}\} = \widetilde{\Theta} \cap \rho'^{-1}(\mathbb{Q}_{\geq 0}\eta)$$

is a fan in $M'_\mathbb{Q} \times \mathbb{Q}$ with support $|\widetilde{\Sigma}_\mathcal{P}| = M'_\mathbb{Q} \times \mathbb{Q}_{\geq 0}$. The projection $\pi' : M'_\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ to the second factor defines a map from the fan $\widetilde{\Sigma}_\mathcal{P}$ to the fan $\{0, \mathbb{Q}_{\geq 0}\}$. This map of fans determines a flat toric morphism $\pi : X_{\widetilde{\Sigma}_\mathcal{P}} \to \mathbb{A}^1$, where $X_{\widetilde{\Sigma}_\mathcal{P}}$ is the toric 3-fold defined by the fan $\widetilde{\Sigma}_\mathcal{P}$. Let t be a closed point in $\mathbb{A}^1$, and let X_t denote the fibre of π over t. Then $X_t \cong \mathbb{S}$ for $t \neq 0$. As shown in [61], when $t = 0$, we have a union of irreducible components, where each $\mathbb{S}_\upsilon$ is the toric surface defined by the polytope Δ_υ (recall that each 3-cone is a cone over a triangle $\Delta_\upsilon \subset \Delta$ in $N_\mathbb{R}$)

$$X_0 = \bigcup_{\upsilon \in \Sigma(3)} \mathbb{S}_\upsilon.$$

If $\upsilon' \in \Sigma(2)$, $\upsilon \in \Sigma(3)$ and $\upsilon' \subset \upsilon$, υ' corresponds to a torus invariant divisor $\mathbb{D}_{\upsilon'} \subset \mathbb{S}_\upsilon$. We have the following commutative diagram:

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X_{\widetilde{\Sigma}_\mathcal{P}} \arrow[r, hook] \arrow[d, "\pi"] & \mathfrak{X} \arrow[r, "\widetilde{\nu}"] \arrow[d, "\widetilde{\rho}"] & X_{\widetilde{\Theta}} \arrow[d, "\rho"] \\ \{0\} \arrow[r, hook] & \mathbb{A}^1 \arrow[r, hook] & \mathbb{A}^k \arrow[r, "\nu"'] & X_\Theta. \end{tikzcd} \tag{32}$$

The polytope $\mathrm{Hull}(\tilde{b}_1, \dots, \tilde{b}_r) \subset \widetilde{N}$ lies on the hyperplane $\langle \phi^\vee(\mathrm{u}_3), \bullet \rangle = 1$. It determines a polytope on $\widetilde{N}' = \{\langle \phi^\vee(\mathrm{u}_3), \bullet \rangle = 0\}$ up to a translation. The associated line bundle $\mathfrak{L}$ on $X_{\widetilde{\Theta}}$ has sections u_i, $i = 1, \dots, r$ associated to each integer point in this polytope. Define

$$u = \sum_{i=1}^r u_i, \quad \widetilde{\mathfrak{C}} = u^{-1}(0).$$

The divisor $\widetilde{\mathfrak{C}} \subset X_{\widetilde{\Theta}}$. Let $\mathfrak{C} := \widetilde{\nu}^{-1}(\widetilde{\mathfrak{C}}) \subset \mathfrak{X}$ be the pullback divisor under the morphism $\widetilde{\nu} : \mathfrak{X} \to X_{\widetilde{\Theta}}$. Then $\mathfrak{C} \to \mathbb{A}^k$ is a flat family of curves over $\mathbb{A}^k$.

For $q \neq 0$, $\mathfrak{C}_q = \widetilde{\rho}^{-1}(q) \cap \mathfrak{C}$ can be identified with the zero locus of

$$H_q(x,y) = x^{\mathfrak{r}} y^{-\mathfrak{s}-\mathfrak{r}f} + y^{\mathfrak{m}} + 1 + \sum_{a=1}^{k} s_a(q) x^{m_a} y^{n_a - f m_a}, \tag{33}$$

where $x^{\mathfrak{r}} y^{-\mathfrak{s}-\mathfrak{r}f} = u_1 u_3^{-1}$, $y^{\mathfrak{m}} = u_2 u_3^{-1}$ while $s_a(q) x^{m_a} y^{n_a - f m_a} = u_{3+a} u_3^{-1}$ for $a = 1, \ldots, k$. Here x, y are affine coordinates of the toric surface $\mathbb{S}$.

For any $\beta \in \mathbb{K}_{\mathrm{eff}}$, let $s^\beta = \prod_{a=1}^k s_a^{\langle D_{a+3}, \beta \rangle}$. If we write $p_a = \sum_{b=1}^k p_a^b D_{b+3}$, we have

$$s_a = \prod_{b=1}^{k} q_b^{p_b^a}, \qquad s^\beta = q^\beta. \tag{34}$$

Denote $s^{\widetilde{\beta}} = x^{d_0} s^\beta$ for $\widetilde{\beta} = (d_0, \beta)$.

When $q = 0$, we have a union of irreducible components

$$\mathfrak{C}_0 = \bigcup_{\upsilon \in \Sigma(3)} \bar{C}_{0,\upsilon}.$$

Each irreducible component $\mathbb{S}_\upsilon$ of the central fibre X_0 is given by the equation $\{u_i = 0,\ b_i \notin \upsilon\}$. On $\mathbb{S}_{\sigma_+}$, the coordinates in the affine chart $u_3 \neq 0$ are

$$x^{\mathfrak{r}} y^{-\mathfrak{s}-\mathfrak{r}f} = u_1 u_3^{-1}, \ y^m = u_2 u_3^{-1},$$

while on $\mathbb{S}_{\sigma_-}$, the coordinates are

$$u_4 u_3^{-1} = (q^\alpha x^{-1})^{\mathfrak{s}^-} y^{n_1 + s^- f}, u_2 u_3^{-1} = y^{\mathfrak{m}}.$$

Here $b_4 = m_1 \mathsf{v}_1 + n_1 \mathsf{v}_2 + \mathsf{v}_3$, $m_1 = -\mathfrak{s}^-$ and $\alpha = [\mathfrak{l}_\tau]$. Define

$$\begin{aligned} U = & \{(q_1, \ldots, q_k) \in (\mathbb{C}^*)^k \times \mathbb{C}^{k-k'} : \mathfrak{C}_q \text{ is smooth} \\ & \text{and intersects } \partial\mathbb{S} \text{ transversally at distinct points}\}. \end{aligned}$$

Then U is a dense open subset of $\mathbb{A}^k$.

4.3.2. Mirror curve and the mirror conjecture for disk amplitudes

When $q \neq 0$, we denote $C_q = \mathfrak{C}_q \setminus (\partial\mathbb{S})$. Thus the mirror curve $C_q \subset (\mathbb{C}^*)^2$ is given by equation (33). On $\bar{C}_{0,\upsilon}$, when $x = 0$, there are m points, called large radius limit (LRL) points. They are given by

$$x = 0, \qquad y^{\mathfrak{m}} = -1.$$

If $\mathcal{L}$ is outer, these points are smooth points in $\bar{C}_{0,\sigma} \subset \mathfrak{C}_0$; if $\mathcal{L}$ is inner, they are the nodal points $\bar{C}_{0,\sigma_+} \cap \bar{C}_{0,\sigma_-}$.

The group $G_\sigma^* = \{(t_1, t_2) \in (\mathbb{C}^*)^2 | t_1^{\mathfrak{r}} = t_2^{\mathfrak{s}}, t_2^{\mathfrak{m}} = 1\}$ fits into the short exact sequence

$$1 \to \boldsymbol{\mu}_{\mathfrak{r}}^* \to G_\sigma^* \to \boldsymbol{\mu}_m^* \to 1,$$

where $G_\sigma^* \to \boldsymbol{\mu}_{\mathfrak{m}}^*$ is given by $(t_1, t_2) \mapsto t_2$. Let

$$\chi_1 = (e^{2\pi\sqrt{-1}\frac{1}{\mathfrak{r}}}, 1), \qquad \chi_2 = (e^{2\pi\sqrt{-1}\frac{\mathfrak{s}}{\mathfrak{r}\mathfrak{m}}}, e^{2\pi\sqrt{-1}\frac{1}{\mathfrak{m}}}).$$

Then $G_\sigma^* = \{\chi_1^j\chi_2^l | j\in\{0,\dots,\mathfrak{r}-1\}, l\in\{0,\dots,\mathfrak{m}-1\}\}$. It pairs with G_σ by

$$\chi_1(h) = e^{2\pi\sqrt{-1}c_1(h)},\quad \chi_2(h) = e^{2\pi\sqrt{-1}c_2(h)},\quad h\in G$$

and acts on the family of compactified mirror curves $\bar{\mathcal{C}}$ by

$$\chi_1\cdot(x,y,s_a) = (e^{2\pi\sqrt{-1}\frac{1}{\mathfrak{r}}}x, y, e^{-2\pi\sqrt{-1}c_1^\sigma(b_{a+3})}s_a),$$
$$\chi_2\cdot(x,y,s_a) = (e^{2\pi\sqrt{-1}\frac{s+\mathfrak{r}f}{\mathfrak{r}\mathfrak{m}}}x, e^{2\pi\sqrt{-1}\frac{1}{\mathfrak{m}}}y, e^{-2\pi\sqrt{-1}c_2^\sigma(b_{a+3})}s_a).$$

Here $c_i^\sigma(b_{a+3})$ is defined as $h_{a+3} = \sum_{i=1}^3 c_i^\sigma(b_{a+3})b_i$. The group $\boldsymbol{\mu}_{\mathfrak{m}}^*$ acts freely and transitively on the set of LRL points ($x=0$ on $\bar{C}_{0,\sigma_+}$).

Given $\bar\eta\in\{0,1,\dots,\mathfrak{m}-1\}$, let $\eta\in G_\tau^*$ be the element associated to the character

$$\chi_\eta : G_\tau \to \mathbb{C}^*,\quad \chi_\eta(\lambda) = \exp\left(\frac{2\pi\sqrt{-1}}{\mathfrak{m}}\bar\eta\bar\lambda\right).$$

Then $\bar\eta\mapsto\eta$ is a bijection from $\{0,1,\dots,\mathfrak{m}-1\}$ to G_τ^*. Given $\eta\in G_\tau^*$, define $\mathfrak{u}_\eta\in\mathfrak{C}_0$ by

$$\mathfrak{u}_\eta = (0, e^{\pi\sqrt{-1}(-1+2\bar\eta)/\mathfrak{m}}).$$

For a small ϵ, one can always find small $\epsilon'(\epsilon)<\epsilon$ such that when $\|q\|<\epsilon'(\epsilon)$ the following set

$$U^{\epsilon,\epsilon'} = \begin{cases}\{(x,q), |x|<\epsilon, U\}, & \mathcal{L} \text{ is outer;}\\ \{(x,q), |x|<\epsilon, |q^\alpha x^{-1}|<\epsilon \text{ whenever } \|q\|<\epsilon'\}, & \mathcal{L}\text{ is inner;}\end{cases}$$

is not empty. Let $U^\epsilon = U^{\epsilon,\epsilon'(\epsilon)}\times\{\|q\|<\epsilon'(\epsilon), q\in U\}\subset\mathfrak{C}$. When ϵ is sufficiently small, $U^{\epsilon,\epsilon'(\epsilon)}$ is a disjoint union of $\mathfrak{m}$ small contractible regions when $\mathcal{L}$ is outer, or is a disjoint union of m annuli when $\mathcal{L}$ is inner. Let U_η^ϵ be the unique connected component of U^ϵ containing $\mathfrak{u}_\eta$. So $\log y$ is well defined on U_η^ϵ up to an integral multiple of $2\pi\sqrt{-1}$, and it could be written as a power series in x when $\mathcal{L}$ is outer, and a Laurent series in x when $\mathcal{L}$ is inner.

Define

$$\phi_\eta := \frac{1}{\mathfrak{m}}\sum_{\lambda\in G_\tau}\chi_\eta(\lambda^{-1})\mathbf{1}_\lambda.$$

Then $\{\phi_\eta : \eta\in G_1^*\}$ is the canonical basis of $H^*_{\mathrm{CR}}(\mathcal{B}\mu_{\mathfrak{m}})$, and

$$\mathbf{1}_\lambda = \sum_{\eta\in\mu_{\mathfrak{m}}^*}\chi_\eta(\lambda)\phi_\eta.$$

We prove the following mirror theorem for disk amplitudes. Note that the ambiguity in $\log y$ does not play any role in the statement.

Theorem 4.5.

$$x\frac{\partial}{\partial x}\sum_{\eta\in\mu_m^*}(\log y|_{U_\eta^\epsilon})\phi_\eta = \left(x\frac{\partial}{\partial x}\right)^2 W^{\mathcal{X},(\mathcal{L},f)}(q,x) = \left(x\frac{\partial}{\partial x}\right)^2 F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}_2, X).$$

Here s and q are related by equation (34). The A-model flat coordinates $\boldsymbol{\tau}_2, X$ and the B-model coordinates q, x are related by the mirror maps in equation (25) and equation (28).

Remark 4.6.

(i) Theorem 4.5 for smooth toric Calabi-Yau 3-folds was conjectured in [6, 5] and proved in [33].
(ii) Theorem 4.5 can also be written as

$$\int \sum_{\eta\in\mu_m^*}\left(\log y\frac{dx}{x}|_{U_\eta^\epsilon}\right)\phi_\eta \text{``}=\text{''} W^{\mathcal{X},(\mathcal{L},f)}(q,x),$$

where the integral is indefinite and "=" means their instanton parts are equal in the following sense. The left side is the sum of a power series with no constant term in x and an extra term in the form of $f(q)\log x + c$. The power series part is equal to the right side. Note that the constant ambiguity in the indefinite integral is irrelevant here. If $\mathcal{X}$ is a smooth variety, then $\boldsymbol{\mu}_m^*$ is trivial, and we revert to the original form of the conjecture in [6, 5]. We will prove this conjecture in the next subsection.

4.4. Open mirror theorem for disk amplitudes

Lemma 4.7. *The solution v to the exponential polynomial equation*

$$\sum_{a=0}^{k} t_a e^{r_a v} - e^v + 1 = 0 \tag{35}$$

around $t_0 = \cdots = t_k = 0, v = 0$ is in the following power series form:

$$v = \sum_{\substack{l_0,\ldots,l_k=0\\(l_0,\ldots,l_k)\neq 0}}^{\infty} \frac{(r_0l_0+\ldots r_kl_k-1)_{(l_0+\cdots+l_k-1)}}{l_0!\ldots l_k!}t_0^{l_0}\ldots t_k^{l_k}. \tag{36}$$

Here we adopt the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} = \begin{cases} a(a-1)\cdots(a-n+1), & n>0;\\ 1, & n=0;\\ \frac{1}{(a+1)\ldots(a-n)}, & n<0;\end{cases}$$

where $a\in\mathbb{C}$ and $n\in\mathbb{Z}$.

Proof. See Appendix A. □

Starting from the above observation, we prove Theorem 4.5 in this section. To find the expansion of $\log y$ on U_η^ϵ, we assume

$$\log y = \log\xi_0 + \frac{2\pi\sqrt{-1}}{\mathfrak{m}}\bar\eta + \frac{v(q,x)}{\mathfrak{m}} = \frac{\pi\sqrt{-1}}{\mathfrak{m}}(-1+2\bar\eta) + \frac{v(q,x)}{\mathfrak{m}},$$

where v is a power series in q and x. Setting

$$\xi_{\bar\eta} = e^{\frac{2\pi\sqrt{-1}}{\mathfrak{m}}\bar\eta},\ t_0 = x^{\mathfrak{r}}(\xi_0\xi_{\bar\eta})^{-s-\mathfrak{r}f},\ r_0=-w_2\mathfrak{r},$$
$$t_a = s_a x^{m_a}(\xi_0\xi_{\bar\eta})^{n_a-fm_a},\ r_a = \frac{n_a-fm_a}{\mathfrak{m}},$$

the mirror curve is

$$H(x,y)=\sum_{a=0}^{k} t_a e^{r_a v} - e^{v} + 1 = 0.$$

Let $\mathbb{D}_{\mathrm{eff}}(\mathcal{X},\mathcal{L}) = \{\widetilde{\beta} = (d_0,\beta) \in \mathbb{Z}\times\mathbb{L} | \langle \widetilde{\beta}, D_i\rangle \in \mathbb{Z}_{\geq 0}, i \neq 2,3\}$. So $\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L}) \subset \mathbb{D}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})$. By Lemma 4.7,

$$\begin{aligned}
v &= \sum_{\substack{l_0,\ldots,l_k=0\\ (l_0,\ldots,l_k)\neq 0}}^{\infty} \frac{(r_0 l_0 + \ldots r_k l_k - 1)_{(l_0+\cdots+l_k-1)}}{l_0!\ldots l_k!} \prod_{a=0}^{k} t_a^{l_a}.\\
&= -\sum_{\widetilde{\beta}\in\mathbb{D}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})} (\xi_0\xi_{\bar\eta})^{-m\langle D_2,\widetilde{\beta}\rangle} \frac{(-\langle D_2,\widetilde{\beta}\rangle - 1)_{(-\langle D_2,\widetilde{\beta}\rangle-\langle D_3,\widetilde{\beta}\rangle-1)}}{\prod_{i\in I_\tau}\langle D_i,\widetilde{\beta}\rangle!} x^{d_0} \prod_{a=1}^{k} s_a^{\langle D_{a+3},\widetilde{\beta}\rangle}\\
&= -\sum_{\widetilde{\beta}\in\mathbb{D}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})} (\xi_0)^{-m\langle D_3,\widetilde{\beta}\rangle} (\xi_{\bar\eta})^{-m\langle D_2,\widetilde{\beta}\rangle} \frac{(-\langle D_3,\widetilde{\beta}\rangle - 1)_{(-\langle D_2,\widetilde{\beta}\rangle-\langle D_3,\widetilde{\beta}\rangle-1)}}{\prod_{i\in I_\tau}\langle D_i,\widetilde{\beta}\rangle!} x^{d_0} q^{\beta}.
\end{aligned}$$

Suppose that $\mathcal{L}$ is an outer brane. For any $\widetilde{\beta} = (d_0,\beta) \in \mathbb{D}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})$, we have

$$d_0(\mathsf{v}_1 + f\mathsf{v}_2) - v(\beta) + N_\sigma \in G_\tau \subset G_\sigma = N/N_\sigma.$$

Let $\lambda = d_0(\mathsf{v}_1 + f\mathsf{v}_2) - v(\beta) + N_\sigma \in G_\tau$. Then $h(d_0,\lambda) = v(\beta) \in G_\sigma$. If $\mathcal{L}$ is an inner brane, we replace σ by σ_+ in the above discussion and define $\lambda \in G_1$ similarly. Then

$$\langle D_2,\widetilde{\beta}\rangle \in \frac{\bar\lambda}{\mathfrak{m}} + \mathbb{Z}, \quad \langle D_3,\widetilde{\beta}\rangle \in -\frac{\bar\lambda}{\mathfrak{m}} + \mathbb{Z}.$$

So

$$\xi_{\bar\eta}^{-\langle D_2,\widetilde{\beta}\rangle} = \exp\left(-\frac{2\pi\sqrt{-1}}{\mathfrak{m}}\bar\eta\bar\lambda\right) = \chi_\eta(\lambda^{-1}).$$

It follows that

$$\begin{aligned}
&x\frac{\partial}{\partial x}\sum_{\eta\in\mu_m^*} (\log y)_{U_\eta^\epsilon}\phi_\eta\\
&= -\sum_{\substack{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})\\ h(d_0,\lambda)=v(\beta)}} (\xi_0)^{-\mathfrak{m}\langle D_3,\widetilde{\beta}\rangle} \frac{d_0(-\langle D_3,\widetilde{\beta}\rangle - 1)_{(-\langle D_2,\widetilde{\beta}\rangle-\langle D_3,\widetilde{\beta}\rangle-1)}}{\mathfrak{m}\prod_{i\in I_\tau}\langle D_i,\widetilde{\beta}\rangle!} \sum_{\eta\in\mu_m^*} \chi_\eta(\lambda^{-1})\phi_\eta x^{d_0} q^{\beta}\\
&= -\sum_{\substack{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})\\ h(d_0,\lambda)=v(\beta)}} (\xi_0)^{-\mathfrak{m}\lfloor\langle D_3,\widetilde{\beta}\rangle\rfloor} \frac{d_0(-\langle D_3,\widetilde{\beta}\rangle - 1)_{(-\langle D_2,\widetilde{\beta}\rangle-\langle D_3,\widetilde{\beta}\rangle-1)}}{\mathfrak{m}\prod_{i\in I_\tau}\langle D_i,\widetilde{\beta}\rangle!} \sum_{\eta\in\mu_{\mathfrak{m}}^*} (\xi_0)^{\bar\lambda}\chi_\eta(\lambda^{-1})\phi_\eta x^{d_0} q^{\beta}\\
&= \sum_{\substack{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})\\ h(d_0,\lambda)=v(\beta)}} -(-1)^{\lfloor\langle D_3,\widetilde{\beta}\rangle\rfloor} \frac{d_0\Gamma(-\langle D_3,\widetilde{\beta}\rangle)}{\mathfrak{m}\Gamma(\langle D_2,\widetilde{\beta}\rangle + 1)\prod_{i\in I_\tau}\Gamma(\langle D_i,\widetilde{\beta}\rangle + 1)} x^{d_0} q^{\beta} \xi_0^{\bar\lambda} \mathbf{1}_{\lambda^{-1}}.
\end{aligned}$$

On the other hand,

$$\left(x\frac{\partial}{\partial x}\right)^2 W^{\mathcal{X},(\mathcal{L},f)}(x,q)$$
$$= \sum_{\substack{\widetilde{\beta}\in\mathbb{K}_{\mathrm{eff}}(\mathcal{X},\mathcal{L})\\ h(d_0,\lambda)=v(\beta)}} -(-1)^{\lfloor\langle D_3,\widetilde{\beta}\rangle\rfloor}\frac{d_0\Gamma(-\langle D_3,\widetilde{\beta}\rangle)}{\mathfrak{m}\Gamma(\langle D_2,\widetilde{\beta}\rangle+1)\prod_{i\in I_\tau}\Gamma(\langle D_i,\widetilde{\beta}\rangle+1)}x^{d_0}q^{\beta}\xi_0^{\bar{\lambda}}\mathbf{1}_{\lambda^{-1}}.$$

Thus Theorem 4.5 follows.

A. Proof of Lemma 4.7

In this appendix, we obtain a power series solution to the following exponential polynomial where $r_a \in \mathbb{R}$

$$1 - e^{v} + \sum_{a=1}^{k} t_a e^{r_a v} = 0 \tag{37}$$

around $t_1 = \cdots = t_k = 0$ by oscillatory integral and inverse Laplace transform. Note that the notation here is slightly different from that in Lemma 4.7: the sum in the above equation (37) starts from $a = 1$, whereas the sum in equation (35) in Lemma 4.7 starts from $a = 0$.

We also consider the following equation where $f, r_a \in \mathbb{Z}_{>0}$:

$$L(X,Y) = 1 + XY^{-f} + Y + \sum_{a=1}^{k} s_a Y^{r_a} = 0. \tag{38}$$

Let $X = e^{-x}$ and $Y = e^{-y}$. This equation identifies with equation (37) after setting $X = 0$ and a change of variables $v = \sqrt{-1}\pi - y, t_a = (-1)^{r_a} s_a$.

Lemma A.1. *For equation (37), one can expand v as a power series in t_a, where each coefficient is a rational function of r_a. For equation (38), the variable Y can be expanded as a power series of $(-1)^{r_a} s_a$ and $(-1)^f X$ around $Y = -1$ with each coefficient rational in r_a and f. One can also expand Y as a power series of s_a and $(-X)^{\frac{1}{f}}$ around $Y = 0$ with each coefficient rational in r_a and f.*

Proof. This is done by elementary recursive calculation. We illustrate the expansion at $Y = 0$ for equation (38). The equation can be written as

$$Y^f + Y^{f+1} + \sum_{a=1}^{k} s_a Y^{r_a+f} = ((-X)^{\frac{1}{f}})^f.$$

Implicit function theorem (applying to $Y(Y + 1 + \sum_{a=1}^{k} s_a Y^{r_a})^{\frac{1}{f}} = (-X)^{\frac{1}{f}}$) says Y is analytic in $(-X)^{\frac{1}{f}}$ and s_a around $(X, Y, s) = (0, 0, 0)$, and recursive calculation shows each coefficient is a rational function of lower degree coefficients. $\square$

We consider an affine curve

$$C := \{(X,Y) \in (\mathbb{C}^*)^2 \mid L(X,Y) = 0\}$$

and its partial compactification $\bar{C} \subset \mathbb{C}^2$ with two points $(X,Y) = (0,0)$ and $(X,Y) = (0,-1+O(s))$ added. Let e^{-x_0} be the branch point of the map $(X,Y) \mapsto X$ such that $e^{-x_0} = -\frac{1}{4} + O(s)$. Let γ_s be the oriented Lefschetz thimble that passes through the ramification point $(e^{-x_0}, e^{-y_0}) = (-\frac{1}{4} + O(s), -\frac{1}{2} +$

$O(s)$) and goes from $(X, Y) = (0, -1 + O(s))$ to $(X, Y) = (0, 0)$. So the coordinate on γ_s is z such that $x - x_0 = z^2$. We choose the sign of z such that $(X, Y) = (0, 0)$ is at $z = +\infty$.

Lemma A.2.

$$\int_{\gamma_s} e^{-ux} y dx = \sum_{l_1,\ldots,l_k \geq 0} e^{\sqrt{-1}\pi(-(f+1)u + \sum_{a=1}^k r_a l_a)} \frac{\Gamma(u)\Gamma(fu + \sum_{a=1}^k r_a l_a)}{\Gamma((f+1)u + \sum_{a=1}^k (r_a - 1)l_a + 1)} \frac{\prod_{a=1}^k s_a^{l_a}}{l_1! \ldots l_k!}.$$

Proof. Consider a Landau-Ginzburg model $W_s : (\mathbb{C}^*)^3 \to \mathbb{C}$, where

$$W_s = X_1 X_2^{-f} X_3 + X_2 X_3 + X_3 + \sum_{a=1}^k s_a X_2^{r_a} X_3 - u \log X_1.$$

Define $\check{t}_1 = X_1 X_2^{-f} X_3, \check{t}_2 = X_2 X_3, \hat{t}_1 = X_1 X_2^{-f}, \hat{t}_2 = X_2$.

Let $X_3 \in \Gamma_3$ be a cycle that counterclockwise encircles the positive real axis, starting and ending on the positive real infinity. We require that the argument of each $X_3 \in \Gamma_3$ takes every value in $(0, 2\pi)$ once. Define the relative connected cycle Γ_s to be

$$\Gamma_s = \{(X_1, X_2, X_3) \in (\mathbb{C}^*)^3 \mid \check{t}_1 > 0, \check{t}_2 > 0, X_3 \in \Gamma_3, \text{when } X_3 < 0 \text{ and } s = 0, X_2 \in \mathbb{R}^-\}.$$

When $|s| < \epsilon$ for small ϵ, the superpotential $\mathrm{Re}(W) \to \infty$ in the noncompact direction of Γ_s. On Γ_s the logarithm is taken in the following way: when $X_3 < 0$ and $s = 0$,

$$\arg(X_1) = -(f+1)\pi, \quad \arg(X_2) = -\pi, \quad \arg(X_3) = \pi.$$

Since the cycle Γ_s is simply connected and deforms continuously with respect to s, this choice is fixed. Evaluate the following oscillatory integral of W_s:

$$\begin{aligned}
I(u) &= \int_{\Gamma_s} e^{-W_s} \frac{dX_1}{X_1} \frac{dX_2}{X_2} \frac{dX_3}{X_3} \\
&= \int_{\Gamma_s} \exp(-\sum_{a=1}^k s_a \check{t}_2^{r_a} X_3^{1-r_a} - \check{t}_1 - \check{t}_2 - X_3 + u \log \check{t}_1 + fu \log \check{t}_2 - (f+1)u \log X_3) \frac{d\check{t}_1}{\check{t}_1} \frac{d\check{t}_2}{\check{t}_2} \frac{dX_3}{X_3} \\
&= \int_{\Gamma_s} e^{-\sum_{a=1}^k s_a \check{t}_2^{r_a} X_3^{1-r_a} - \check{t}_1 - \check{t}_2 - X_3} \check{t}_1^u \check{t}_2^{fu} e^{-(f+1)u(\log X_3 - \sqrt{-1}\pi)} e^{-(f+1)\sqrt{-1}\pi u} \frac{d\check{t}_1}{\check{t}_1} \frac{d\check{t}_2}{\check{t}_2} \frac{dX_3}{X_3} \\
&= -e^{-(f+1)\sqrt{-1}\pi u} \sum_{l_1,\ldots,l_k \geq 0} (-1)^{\sum_{a=1}^k (r_a - 1)l_a} \prod_{a=1}^k \frac{(-s_a)^{l_a}}{l_a!} \Big(\int_{\check{t}_1 > 0} e^{-\check{t}_1} \check{t}_1^{u-1} d\check{t}_1 \Big) \\
&\quad \cdot \Big(\int_{\check{t}_2 > 0} e^{-\check{t}_2} \check{t}_2^{\sum_{a=0}^k r_a l_a + fu - 1} d\check{t}_2 \Big) \cdot \Big(\int_{X_3 \in \Gamma_3} e^{-X_3} e^{(\log X_3 - \sqrt{-1}\pi)(\sum_{a=1}^k (1-r_a)l_a - (f+1)u - 1)} dX_3 \Big) \\
&= 2\pi\sqrt{-1} e^{-(f+1)\sqrt{-1}\pi u} \sum_{l_1,\ldots,l_k \geq 0} (-1)^{\sum_{a=1}^k r_a l_a} \prod_{a=1}^k \frac{s_a^{l_a}}{l_a!} \frac{\Gamma(u)\Gamma(fu + \sum_{a=1}^k r_a l_a)}{\Gamma((f+1)u + \sum_{a=1}^k (r_a - 1)l_a + 1)}.
\end{aligned}$$

Here we use the Hankel's formula:

$$\frac{\sqrt{-1}}{2\pi} \int_{\Gamma_3} e^{-z(\log(t) - \sqrt{-1}\pi)} e^{-t} dt = \frac{1}{\Gamma(z)}.$$

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By Hori-Iqbal-Vafa [44], this oscillatory integral could be reduced to a Laplace transform on the curve C. Introduce two variables $v^+, v^- \in \mathbb{C}$, and the extended cycle

$$\tilde{\Gamma}_s = \Gamma_s \times \{v^+ = -\overline{v^-}\}.$$

Define $H = \frac{W_s}{X_3}$. Define the holomorphic volume form

$$\Omega = \frac{dX_1}{X_1}\frac{dX_2}{X_2}\frac{dv^-}{v^-} = dxdy\frac{dv^-}{v^-}.$$

We reduce the oscillatory integral to the curve C as follows. Let $\tilde{\Gamma}_{\mathrm{red}} = \{(\hat{t}_1, \hat{t}_2) | \arg \hat{t}_1 = \arg \hat{t}_2\} \times \{v^+ = -\overline{v^-}\}$.

$$\begin{aligned} I(u) =& \frac{1}{2\sqrt{-1}\pi}\int_{\tilde{\Gamma}_s} e^{-X_3(H-v^+v^-)}e^{-ux}\frac{dX_1}{X_1}\frac{dX_2}{X_2}dX_3dv^+dv^- \\ =& \frac{1}{2\sqrt{-1}\pi}\int_{\tilde{\Gamma}_{\mathrm{red}}} \delta(H-v^+v^-)e^{-ux}\frac{dX_1}{X_1}\frac{dX_2}{X_2}dv^+dv^- \\ =& -\int_{\tilde{\Gamma}_{\mathrm{red}}\cap\{H-v^+v^-=0\}} e^{-ux}\frac{dX_1}{X_1}\frac{dX_2}{X_2}\frac{dv^-}{v^-}. \end{aligned}$$

This integration is further reduced to the curve $C = \{H(e^{-x}, e^{-y}) = 0\}$ as follows:

$$I(u) = -\int_{\tilde{\Gamma}_{\mathrm{red}}\cap\{H-v^+v^-=0\}} e^{-ux}dxdy\frac{dv^-}{v^-} = 2\sqrt{-1}\pi\int_{\gamma_s} e^{-ux}ydx.$$

Notice that we use the fact that $d(e^{-ux}ydx\frac{dv^-}{v^-}) = -e^{-ux}\Omega$ near $\tilde{\Gamma}_{\mathrm{red}} \cap \{H - v^+v^- = 0\}$. □

The function $\frac{dy}{dx}$ is a meromorphic function on the partially compactified curve $\bar{C}$ with the only pole at x_0. Its expansion at $(X,Y) = (0, -1+O(s))$ is a power series in X, while its expansion at $(X,Y) = (0,0)$ is a series in $X^{\frac{1}{f}}$. Denote $g^\pm(x) = \frac{dy}{dx}|_{z=\pm\sqrt{x-x_0}}$. Then g^- is a power series in X and s_a, while g^+ is a power series in $X^{\frac{1}{f}}$ and s_a. Since they are expansions of $\frac{dy}{dx} = \frac{XdY}{YdX}$ regarding the curve equation (38), as series of $(-1)^f X$, $(-1)^{r_a}s_a$ and $(-X)^{\frac{1}{f}}$, s_a respectively, their coefficients are rational in r_a and f by Lemma A.1.

By Lemma A.2, the 'classical Laplace transform' is

$$\begin{aligned} \mathfrak{G}(u) &= \int_{x-x_0\in\mathbb{R}^+} e^{-u(x-x_0)}(g^+(x)-g^-(x))d(x-x_0) \\ &= \int_{x-x_0\in\mathbb{R}^+} e^{-u(x-x_0)}\left(\frac{dy}{dx}\right)d(x-x_0) = ue^{ux_0}\int_{\gamma_s} e^{-ux}ydx \\ &= \sum_{l_1,\dots,l_k\geq 0} ue^{ux_0}e^{\sqrt{-1}\pi(-(f+1)u+\sum_{a=1}^k r_al_a)}\frac{\Gamma(u)\Gamma(fu+\sum_{a=1}^k r_al_a)}{\Gamma((f+1)u+\sum_{a=1}^k(r_a-1)l_a+1)}\frac{\prod_{a=1}^k s_a^{l_a}}{l_1!\dots l_k!}. \end{aligned}$$

By the inverse Laplace transform formula,

$$(g^+ - g^-) = \int_{u=-\infty\sqrt{-1}+T}^{u=+\infty\sqrt{-1}+T} \mathfrak{G}(u)e^{u(x-x_0)}du,$$

where T is large enough that all poles of $\mathfrak{G}(u)$ are on the left of the integration contour. Here the inverse Laplace transform takes residues around poles of $\Gamma(u)$ and $\Gamma(fu + \sum_{a=1}^k r_al_a)$. Taking the residues around all poles (other than the possible pole at $u = 0$) of $\Gamma(fu + \sum_{a=1}^{k} r_a l_a)$ gives a series of $(-X)^{\frac{1}{f}}$ with coefficients rational in r_a and f, denoted by h^+

$$
\begin{aligned}
h^+ &= \sum_{l>0, l_1,\dots,l_k \geq 0} h^+_{l,l_1,\dots,l_k} \prod_{a=1}^{k} s_a^{l_a} ((-X)^{\frac{1}{f}})^l \\
&= \sum_{l>0, l_1,\dots,l_k \geq 0} (-l/f) e^{\sqrt{-1}\pi(-\frac{f+1}{f}l + \sum_{a=1}^{k} r_a l_a)} (X^{\frac{1}{f}})^l \frac{\Gamma(-\frac{l}{f}) \mathrm{Res}_{u=-\frac{l}{f}} \Gamma(fu + \sum_{a=1}^{k} r_a l_a)}{\Gamma(-\frac{l(f+1)}{f} + \sum_{a=1}^{k} (r_a - 1) l_a + 1)} \frac{\prod_{a=1}^{k} s_a^{l_a}}{l_1! \dots l_k!} \\
&= \sum_{l>0, l_1,\dots,l_k \geq 0} (-l/f) (-X)^{\frac{l}{f}} \frac{\Gamma(-\frac{l}{f})}{\Gamma(-\frac{l(f+1)}{f} + \sum_{a=1}^{k} (r_a - 1) l_a + 1) f} \frac{\prod_{a=1}^{k} s_a^{l_a}}{l! l_1! \dots l_k!};
\end{aligned}
$$

while taking residues around the poles of $\Gamma(u)$ (other than the possible pole at $u = 0$), we get a power series in X

$$
\begin{aligned}
h^- &= \sum_{l>0, l_1,\dots,l_k \geq 0} (-l) X^l e^{\sqrt{-1}\pi(-(f+1)l + \sum_{a=1}^{k} r_a l_a)} \frac{\mathrm{Res}_{u=-l}(\Gamma(u)) \Gamma(-fl + \sum_{a=1}^{k} r_a l_a)}{\Gamma(-(f+1)l + \sum_{a=1}^{k} (r_a - 1) l_a + 1)} \frac{\prod_{a=1}^{k} s_a^{l_a}}{l_1! \dots l_k!} \\
&= \sum_{l>0, l_1,\dots,l_k \geq 0} (-l) ((-1)^f X)^l \frac{\Gamma(-fl + \sum_{a=1}^{k} r_a l_a)}{\Gamma(-(f+1)l + \sum_{a=1}^{k} (r_a - 1) l_a + 1) l!} \frac{\prod_{a=1}^{k} ((-1)^{r_a} s_a)^{l_a}}{l_1! \dots l_k!}.
\end{aligned}
$$

So $g^+ - g^- = h^+ + h^- + \text{const.}$, where the constant difference (in X) arises since we don't consider the residue around $u = 0$. For any degree $l \geq 1$, choose $f > l$ such that the term $((-X)^{\frac{1}{f}})^l \prod_{a=1}^{k} s_a^{l_a}$ in g^+ is not a monomial in X and thus can only come from h^+. Since the coefficient of the term is rational in f, it has to be equal to the corresponding term $h^+_{l,l_1,\dots,l_k}$ for all $f > 0$. Therefore $g^+ = h^+ + \text{const.}$ and $-g^- = h^-$. Note here that g^- is the expansion of $\frac{dy}{dx}$, and since y is analytic in X at $(X, Y) = (0, -1 + O(s))$, $-g^-$ has no degree 0 term and does not differ from h^- by a degree 0 term in X.

Suppose the expansion of y at $(X, Y) = (0, -1 + O(s))$ is $y = A_0 + \sum_{l>0} A_l X^l$, then the expansion of $\frac{dy}{dx}$ at this point is

$$
g^- = \frac{dy}{dx} = -\sum_{l>0} l A_l X^l.
$$

Therefore, for $l \geq 1$,

$$
A_l = -\sum_{l_1,\dots,l_k \geq 0} e^{\sqrt{-1}\pi(-fl + \sum_{a=1}^{k} r_a l_a)} \frac{\Gamma(-fl + \sum_{a=1}^{k} r_a l_a)}{\Gamma(-(f+1)l + \sum_{a=1}^{k} (r_a - 1) l_a + 1)} \frac{\prod_{a=1}^{k} s_a^{l_a}}{l_1! \dots l_k!}. \tag{39}
$$

We prove Lemma 4.7 by induction. The statement is true for $k = 0$ trivially. Assume it is true for $k = \mathfrak{m} - 1$ ($\mathfrak{m} \geq 1$). For $k = \mathfrak{m}$, we first assume $r_{\mathfrak{m}} \in \mathbb{Z}_{<0}$ and all other r_a ($a = 1, \dots, \mathfrak{m} - 1$) are positive integers. By the induction assumption, we know the expansion is given as in Lemma 4.7 for terms of degree 0 in $t_{\mathfrak{m}}$. Let $f = -r_{\mathfrak{m}}$. After a change of variables $v = \sqrt{-1}\pi - y$, $(-1)^{r_a} s_a = t_a$ for $a = 1, \dots, \mathfrak{m} - 1$ and $X = (-1)^f t_{\mathfrak{m}}$, we obtain Equation (38). Then from Equation (39), we know the expansion of y for positive degree terms in X and thus conclude that for positive degree terms in $t_{\mathfrak{m}}$, the lemma also holds. Then for all degrees, the lemma holds

$$
v = \sum_{\substack{l_1,\dots,l_k=1 \\ (l_1,\dots,l_k) \neq 0}}^{\infty} \frac{(r_1 l_1 + \dots r_k l_k - 1)_{(l_1 + \dots + l_k - 1)}}{l_1! \dots l_k!} t_1^{l_1} \dots t_k^{l_k}.
$$

Each coefficient is a rational function of $r_1, \ldots, r_{\mathfrak{m}}$. The above equation holds for $r_1, \ldots, r_{\mathfrak{m}-1} \in \mathbb{Z}_{>0}$ and $r_{\mathfrak{m}} \in \mathbb{Z}_{<0}$, so it is true for all $r_a \in \mathbb{R}$.

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