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A NOTE ON THE STABILITY OF SWIRLING FLOWS WITH RADIUS-DEPENDENT DENSITY WITH RESPECT TO INFINITESIMAL AZIMUTHAL DISTURBANCES

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Abstract

We study the stability of inviscid, incompressible swirling flows of variable density with respect to azimuthal, normal mode disturbances. We prove that the wave velocity of neutral modes is bounded. A further refinement of Fung's semi-elliptical instability region is given. This new instability region depends not only on the minimum Richardson number, and the lower and upper bounds for the angular velocity like Fung's semi-ellipse, but also on the azimuthal wave number and the radii of the inner and outer cylinders. An estimation for the growth rate of unstable disturbances is obtained and it is compared to some of the recent asymptotic results.

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1. Introduction

Recent research on the stability of vortices is motivated by the aircraft trailing vortex problem [10]. It has been pointed out in the review article of Spalart [14] that the vorticity profiles appropriate for these problems should be continuous. However, the analytical study of the instability of constant density swirling flows of inviscid incompressible fluids with respect to infinitesimal normal mode disturbances was initiated long ago by Howard and Gupta [7]. When the disturbances are only axisymmetric, a number of general analytical results, such as the Richardson number criterion for instability, a semicircle theorem for the instability region and an estimate for the growth rate of unstable modes, can be found in that article. Since obtaining general analytical results for the three-dimensional disturbances is very difficult, they have considered the problem of stability of swirling flows with respect to a special class

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of nonaxisymmetric disturbances, namely, the azimuthal (also called two-dimensional) disturbances. Let (r, θ, z) be a cylindrical polar coordinate system. Consider the motion of an inviscid, incompressible homogeneous fluid in the annular region between two infinite cylinders at $r = R_1, R_2$, where $0 < R_1 < R_2 < \infty$. Then the basic flow has the velocity $(0, V(r), 0)$, and the pressure is calculated from the Euler equations. The azimuthal disturbances are of the form $f(r)e^{im(\theta - ct)}$, where m is a nonzero integer, called the *azimuthal wave number*, $c = c_r + ic_i$ is the complex wave velocity and $\Omega = V/r$ is the angular velocity. The only general result for this problem known at that time was the Rayleigh theorem on instability, namely, a necessary condition for instability was that DZ should change sign at least once, with $D = d/dr$ as the differential operator and Z the basic flow vorticity [3]. However, substantial progress has been made on this problem subsequently. Shukhman [13] has constructed the angular velocity profile

$$\Omega(r) = \pm\left[\frac{\Omega_1 + \Omega_2}{2} + \left(\frac{\Omega_2 - \Omega_1}{2}\right)\tanh\left(\frac{1}{d}\log\frac{r}{R}\right)\right]$$

as an analogue of the mixing layer profile in the parallel flow theory, where Ω_1, Ω_2, d and R are the parameters of the model. Note that d is a shear width parameter, $R > 0$ is a scale parameter and $\pm\Omega_1$ and $\pm\Omega_2$ are limiting values of the angular velocity in the limits $r \to 0$ and $r \to +\infty$, respectively. In the absence of the cylinders, that is, when the fluid is in an open environment, Shukhman [13] has shown that this flow is stable when $m = 1$ for all values of d, but, when $m = 2$, it is stable for $d = d_{\text{critical}} = 1/2$ and unstable for $d < d_{\text{critical}}$. Subbiah [15] has found an estimate for the growth rate of unstable modes and also upper and lower bounds on the phase velocity of neutral modes. Maslowe and Nigam [10] have developed a nonlinear critical layer analysis for general profiles, while Shukhman [13] has constructed an analysis for a special profile.

Now we will discuss the problem under consideration in this paper, namely, the two-dimensional stability of swirling flows of inviscid, incompressible and variable density fluids. This problem has also been studied in many articles (see e.g. the articles by Pierro and Abid [2, 11]) previously. The stability of a heterogeneous swirling flow with velocity $(0, V(r), 0)$ and density $\rho_0(r)$ confined within an annular region $R_1 \le r \le R_2$ between two concentric cylinders was considered by Fung and Kurzweg [5]. Following the usual normal mode analysis, they have considered infinitesimal azimuthal disturbances, that is, disturbances of the form $f(r)e^{im(\theta - ct)}$, and derived the linear stability equation and the associated boundary conditions. Subsequently, Fung [4] has obtained some of the important general analytical results. In particular, he has defined a local Richardson number, $J = \Omega^2(D\rho_0)/\rho_0 r(D\Omega)^2$, with D as the differential operator, and derived a semicircle theorem and a semi-ellipse theorem for instability regions. If $a \le \Omega(r) \le b$, then for basic flows satisfying the condition $abD\rho_0 \ge 0$, a semicircle theorem giving the instability region within the complex wave velocity of unstable modes was derived. In addition, he has derived a semi-ellipse theorem for the instability region, when the condition $J < 1/4$ is satisfied by the Richardson number. The semicircle theorem was proved by adapting the method of Howard [6], while the semi-ellipse theorem was proved by that of Kochar and Jain [8], both developed in the context of parallel shear flows. However, unlike the semi-ellipse of Kochar and Jain [8], Fung's semi-ellipse depended not only on the minimum Richardson number J_m (which is the minimum of $J(r)$ over the flow domain $[R_1, R_2]$), but also on the upper and lower bounds of the azimuthal velocity. Instability of the constant angular velocity basic flow was also demonstrated by Fung [4] when $D\rho_0 < 0$, and it was interpreted as an instability due to a density variation, only because there was no shear in this flow. Davalos-Orozco and Vazquez-Luis [1] have studied the instability of an interface between two inviscid fluids inside a rotating annulus between two cylinders with infinite dimensions, and pointed out the reason for not excluding the disturbance with the azimuthal wave number $m = 1$, when the inner cylinder was present.

In a recent work on this problem, Dixit and Govindarajan [2] have studied the instability of the Rankine vortex. Though the stability equation studied is the equation of Fung and Kurzweg [5], they have considered the stability problem in the absence of the cylinders with infinite dimensions and, consequently, they have considered only the modes $m \neq 1$. As there was no shear in the Rankine vortex, they studied the instability by computing the growth rate of unstable modes as a function of Atwood number [2] rather than the Richardson number. In the case of swirling flows without cylinders, the instability of slowly varying velocity profiles was studied asymptotically in the large azimuthal wave number limit by Pierro and Abid [11]. The basic flows considered in that work include general velocity profiles of the form $(0, V(r), W(r))$ and the disturbances considered are also general, that is, they are of the form $f(r)e^{i(kz+m\theta-\omega t)}$. Of course, the density of the basic flow is a nonconstant function of r. They have also pointed out that swirling flows are important for many application devices and as a fundamental problem considering their relevance to aircraft trailing vortices, vortical transport of momentum and energy in meteorology and vortex breakdown [12]. As such, these problems are widely studied, but the physical mechanisms of their instabilities, when density variations are present, are not generally discussed. In a variable density swirling flow, there are ingredients for the development of two fundamental instabilities, namely, instabilities due to density variations and instabilities due to differential rotation. Also, it is pointed out that the instabilities due to density variations is characterized by the formation of *bubbles* and *spikes* for positive and negative buoyant fluids, respectively.

The instability of swirling flows considered by Pierro and Abid [11] was restricted to slowly varying basic velocity profiles, and the instability analysis was an asymptotic analysis valid for large axial or azimuthal wave numbers. For the case of azimuthal disturbances, they have obtained an expression for the growth rate, and observed that the growth rate was unbounded for $|m| \gg 1$ (see equation (22) of their article [11]). They have introduced the equivalent of the Brunt–Väisäla frequency as $N^2(r) = -(V^2/r)(D\rho_0/\rho_0)$ (in our notation). Though Pierro and Abid [11] did not have any restriction on the sign of $D\rho_0$ in their asymptotic analysis, $N(r)$ should be real only for negatively buoyant fluids. They have also numerically studied the instability of Batchelor-like vortices [12] with density of the basic flow $\rho_0(r) = \rho_\infty[1 + (s-1)\exp(-r^2)]$, where ρ_∞ is the density at $r = \infty$ and s the axial to ambient density ratio. Note that $(s-1)/(s+1)$ is the Atwood number. Their numerical results demonstrate the instability of the Batchelor-like vortices if $s \gtrsim 1$, and this corresponds to negatively buoyant fluids again. Pierro and Abid [12] have shown numerically that the linear asymptotical results in their earlier article [11] that are valid for large azimuthal wave numbers also work for lower values of m, and they are valid in the nonlinear regime.

Now we state the results obtained in the present paper. Upper and lower bounds for the wave velocity of neutral modes are determined here. We have obtained a generalized semi-elliptical region that depends not only on the minimum Richardson number, and upper and lower bounds on the angular velocity in common with Fung's semi-elliptical region, but also on the azimuthal wave number and the radii of the cylinders. For an infinite R_2 our generalized semi-ellipse reduces to Fung's semi-ellipse, but for finite R_2 it lies inside Fung's semi-ellipse. Since $J_m \geq 1/4$ implies stability of the swirling flow [4], the instability region should reduce to the line $c_i = 0$ as $J_m \to (1/4)-$. This is true for the instability region obtained in this paper and this may be compared with that of Fung [4], which remains a semi-ellipse even in the limit $J_m \to (1/4)-$. A further consequence of the derivation of our generalized semi-elliptical region is that we are able to obtain an estimate for the growth rate of unstable disturbances. This estimate for growth rate is derived independently, and a further improvement of this is also given. This can be compared with the estimation of Pierro and Abid [11]. Our estimation is valid for all values of the azimuthal wave number m, while theirs is valid for only large values of m. Another difference is that our estimation is valid for positively buoyant fluids and for all smooth velocity profiles, whereas theirs is valid for slowly varying velocity profiles.

2. Formulation of the problem

We consider the motion of an inviscid, incompressible and density stratified fluid between concentric cylinders of radii R_1 and R_2 with $0 < R_1 < R_2 < \infty$. Also, we consider a basic flow with velocity $(0, V(r), 0)$, pressure $P_0(r)$ and density $\rho_0(r)$, where the pressure is related to the velocity and density by $dP_0/dr = \rho_0(V^2/r)$. The disturbed flows are given by the velocity $(u, V+v, w)$, pressure $P_0(r) + p$ and density $\rho_0(r) + \rho$ with equations

$$(\rho+\rho_0)\Big(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{(V+v)}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{(V+v)^2}{r}\Big) = -\frac{\partial (P_0+p)}{\partial r},$$

$$(\rho+\rho_0)\Big(\frac{\partial (V+v)}{\partial t} + u\frac{\partial (V+v)}{\partial r} + \frac{(V+v)}{r}\frac{\partial (V+v)}{\partial \theta} + w\frac{\partial (V+v)}{\partial z} + \frac{u}{r}(V+v)\Big)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta},$$

$$(\rho+\rho_0)\Big(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{(V+v)}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}\Big) = -\frac{\partial p}{\partial z},$$

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$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial (V+v)}{\partial \theta} + \frac{\partial w}{\partial z} = 0,$$
$$\frac{\partial \rho}{\partial t} + u\frac{\partial (\rho+\rho_0)}{\partial r} + \frac{(V+v)}{r}\frac{\partial \rho}{\partial \theta} + w\frac{\partial \rho}{\partial z} = 0.$$

For infinitesimal disturbances, the nonlinear terms can be neglected to get the following linear system of partial differential equations:

$$\rho_0\Big(\frac{\partial u}{\partial t} + \frac{V}{r}\frac{\partial u}{\partial \theta} - \frac{2V}{r}v\Big) - \frac{V^2}{r}\rho = -\frac{\partial p}{\partial r}, \tag{2.1}$$
$$\rho_0\Big(\frac{\partial v}{\partial t} + u\frac{dV}{dr} + \frac{V}{r}\frac{\partial v}{\partial \theta} + \frac{V}{r}u\Big) = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{2.2}$$
$$\rho_0\Big(\frac{\partial w}{\partial t} + \frac{V}{r}\frac{\partial w}{\partial \theta}\Big) = -\frac{\partial p}{\partial z}, \tag{2.3}$$
$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \tag{2.4}$$
$$\frac{\partial \rho}{\partial t} + u\frac{d\rho_0}{dr} + \frac{V}{r}\frac{\partial \rho}{\partial \theta} = 0. \tag{2.5}$$

We consider the two-dimensional (that is, disturbances with $w = 0$ and $\partial/\partial z(u, v) = 0$) normal mode disturbances which are of the form

$$(u, v, p, \rho) = (\hat{u}(r), \hat{v}(r), \hat{p}(r), \hat{\rho}(r))e^{im(\theta - ct)},$$

where m is a positive integer, called the *azimuthal wave number*, θ is the *azimuthal angle* and $c = c_r + ic_i$ is the (complex) phase velocity. When $mc_i > 0$, it is called the growth rate of an unstable mode. Substituting these variables into the equations (2.1)–(2.5), we get the following system of ordinary differential equations:

$$\begin{aligned}
\rho_0(im(\Omega - c)\hat{u} - 2\Omega\hat{v}) - \frac{V^2}{r}\hat{\rho} &= -\frac{d\hat{p}}{dr},\\
\rho_0\Big(im(\Omega - c)\hat{v} + \Big(\frac{dV}{dr} + \frac{V}{r}\Big)\hat{u}\Big) &= -\frac{im\hat{p}}{r},\\
\frac{d\hat{u}}{dr} + \frac{\hat{u}}{r} + \frac{im\hat{v}}{r} &= 0,\\
im(\Omega - c)\hat{\rho} + \frac{d\rho_0}{dr}\hat{u} &= 0.
\end{aligned} \tag{2.6}$$

Evaluating all variables in terms of $\hat{u}(r)$, and dropping $\widehat{\ }$,

$$\rho = -\frac{D\rho_0}{im(\Omega - c)}u, \tag{2.7}$$
$$v = -\frac{r}{im}D^*u, \tag{2.8}$$
$$p = \frac{-r\rho_0}{im}(-r(\Omega - c)D^*u + D^*Vu), \tag{2.9}$$

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with the differential operators $D = d/dr$ and $D^* = D + 1/r$. Now, substituting (2.7)–(2.9) in (2.6), we get the second order ordinary differential equation

$$D(\rho_0 r^2 D^* u) + \left\{-\rho_0 m^2 - \frac{rD[\rho_0 D^*(r\Omega)]}{\Omega - c} + \frac{r\Omega^2 D\rho_0}{(\Omega - c)^2}\right\} u = 0 \tag{2.10}$$

with boundary conditions

$$u = 0 \quad \text{at } r = R_1, R_2. \tag{2.11}$$

The eigenvalue problem consisting of equations (2.10) and (2.11) has been derived by Fung and Kurzweg [5]. If we take $u = (\Omega - c)F$, then

$$D(\rho_0 r^2 D^*[(\Omega - c)F]) + \left\{-m^2\rho_0(\Omega - c) - rD(\rho_0 D^*(r\Omega)) + \frac{r\Omega^2 D\rho_0}{\Omega - c}\right\} F = 0 \tag{2.12}$$

with boundary conditions

$$F(R_1) = 0 = F(R_2). \tag{2.13}$$

For unstable modes $c_i > 0$, $G = (\Omega - c)^{1/2} F$ is well defined and

$$\begin{aligned}(\Omega - c)D(\rho_0 r^2 D^* G) &+ \frac{\rho_0 r^2 D\Omega D^* G}{2} + \frac{D(\rho_0 r^2 D\Omega G)}{2} - \frac{\rho_0 r^2 (D\Omega)^2 G}{4(\Omega - c)} \\ &+ \left\{-m^2\rho_0 - rD(\rho_0 D^*(r\Omega)) + \frac{r\Omega^2 D\rho_0}{\Omega - c}\right\} G = 0 \end{aligned} \tag{2.14}$$

with boundary conditions

$$G(R_1) = 0 = G(R_2). \tag{2.15}$$

If the perturbation stream function is $\phi(r)e^{im(\theta - ct)}$, then $u = -im\phi/r$ and, substituting this in (2.10),

$$\rho_0\left(D^* D - \frac{m^2}{r^2}\right)\phi + D\rho_0 D\phi + \left\{\frac{\Omega^2 D\rho_0}{r(\Omega - c)^2} - \frac{D(\rho_0 Z)}{r(\Omega - c)}\right\}\phi = 0 \tag{2.16}$$

with $Z = rD\Omega + 2\Omega$ as the basic vorticity and the boundary conditions

$$\phi = 0 \quad \text{at } r = R_1, R_2. \tag{2.17}$$

When the fluid is homogeneous, that is, when ρ_0 is constant, equation (2.16) reduces to equation (2.2) of Shukhman [13].

3. Boundedness of the wave velocity of neutral modes

The wave velocity c is real for neutral modes. For singular neutral modes, it is bounded since $a \le c \le b$. So, it is enough to prove that c is bounded for nonsingular neutral modes. For nonsingular neutral modes, either $c < a$ or $c > b$, and so $\Omega - c \neq 0$ in $[R_1, R_2]$. For such modes, we have the following general result.

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THEOREM 3.1. *The wave velocity of any nonsingular neutral mode is bounded, and the lower and upper bounds are given by*

$$a + \frac{D(\rho_0 Z)_{\min}(R_2 - R_1)^2}{2\pi^2(\rho_0)_{\min} R_1} - \frac{(R_2 - R_1)^2 \sqrt{\Delta_{\max}}}{2\pi^2(\rho_0)_{\min} R_1}$$
$$\leq c \leq b + \frac{D(\rho_0 Z)_{\max}(R_2 - R_1)^2}{2\pi^2(\rho_0)_{\min} R_1} + \frac{(R_2 - R_1)^2 \sqrt{\Delta_{\max}}}{2\pi^2(\rho_0)_{\min} R_1},$$

where

$$\Delta_{\max} = \max_{[R_1, R_2]} [(D(\rho_0 Z))^2 + (4(\rho_0)_{\min} R_1 \pi^2 \Omega^2 D\rho_0/(R_2 - R_1)^2)].$$

PROOF. Multiplying equation (2.16) by $r\phi$ (which is real) and integrating over the interval (R_1, R_2),

$$\int_{R_1}^{R_2} \rho_0 \Big(D^* D - \frac{m^2}{r^2}\Big)\phi(r\phi)\,dr + \int_{R_1}^{R_2} D\rho_0 D\phi(r\phi)\,dr + \int_{R_1}^{R_2} \Big\{\frac{\Omega^2 D\rho_0}{(\Omega - c)^2} - \frac{D(\rho_0 Z)}{(\Omega - c)}\Big\}|\phi|^2\,dr = 0. \tag{3.1}$$

Now

$$\begin{aligned}\int_{R_1}^{R_2} \rho_0 D^* D\phi \cdot (r\phi)\,dr &= \int_{R_1}^{R_2} \rho_0 \frac{D(rD\phi)}{r}(r\phi)\,dr \\ &= \int_{R_1}^{R_2} D(rD\phi)\rho_0\phi\,dr \\ &= [rD\phi]_{R_1}^{R_2} - \int_{R_1}^{R_2} rD\phi(D\rho_0\phi + \rho_0 D\phi)\,dr.\end{aligned}$$

Consequently,

$$\int_{R_1}^{R_2} \rho_0 D^* D\phi \cdot (r\phi)\,dr = -\int_{R_1}^{R_2} D\phi D\rho_0 r\phi\,dr - \int_{R_1}^{R_2} \rho_0 r|D\phi|^2\,dr \tag{3.2}$$

(by applying the boundary conditions (2.17)).

Using (3.2) and (3.1),

$$\int_{R_1}^{R_2} \rho_0 r|D\phi|^2\,dr + \int_{R_1}^{R_2} \frac{m^2}{r}|\phi|^2\,dr + \int_{R_1}^{R_2} \Big\{\frac{D(\rho_0 Z)(\Omega - c) - \Omega^2 D\rho_0}{(\Omega - c)^2}\Big\}|\phi|^2\,dr = 0. \tag{3.3}$$

Note that in (3.3), if c is sufficiently large, the last integral is dominated by the first integral (by Sobolev embedding) and, therefore, the sum of the left-hand side must be strictly positive, which is a contradiction. Consequently, the wave velocity c is bounded. Now we use the well-known Raleigh–Ritz inequality [3], which yields

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$$\int_{R_1}^{R_2} \rho_0(|D\phi|^2 r\,dr \geq (\rho_0)_{\min} R_1 \int_{R_1}^{R_2} |D\phi|^2\,dr$$
$$\geq \frac{(\rho_0)_{\min} R_1 \pi^2}{(R_2-R_1)^2} \int_{R_1}^{R_2} |\phi|^2\,dr.$$

Using this estimate and dropping the second term in (3.3),

$$\int_{R_1}^{R_2} \Big[\frac{(\rho_0)_{\min} R_1 \pi^2}{(R_2-R_1)^2}(\Omega-c)^2 + D(\rho_0 Z)(\Omega-c) - \Omega^2 D\rho_0\Big]\frac{|\phi|^2}{(\Omega-c)^2}\,dr \leq 0.$$

This means that there is a $r_0 \in (R_1, R_2)$ such that

$$\frac{(\rho_0)_{\min} R_1 \pi^2}{(R_2-R_1)^2}(\Omega(r_0)-c)^2 + D(\rho_0 Z)(r_0)(\Omega(r_0)-c) - \Omega^2(r_0) D\rho_0(r_0) \leq 0.$$

We can rewrite this inequality as

$$\frac{(\rho_0)_{\min} R_1 \pi^2}{(R_2-R_1)^2}c^2 - \Big[\frac{2\Omega(\rho_0)_{\min} R_1 \pi^2}{(R_2-R_1)^2} + D(\rho_0 Z)(r_0)\Big]c + \frac{(\rho_0)_{\min} R_1 \pi^2 \Omega^2(r_0)}{(R_2-R_1)^2}$$
$$+ D(\rho_0 Z)(r_0)\Omega(r_0) - \Omega^2(r_0)\rho_0(r_0) \leq 0, \tag{3.4}$$

which is a quadratic inequality in c. If we only consider the equality sign in (3.4), we have a quadratic equation in c with discriminant

$$\Delta = [D(\rho_0 Z)(r_0)]^2 + \frac{4(\rho_0)_{\min} R_1 \pi^2 \Omega^2(r_0) D\rho_0(r_0)}{(R_2-R_1)^2}.$$

A negative quadratic discriminant means the nonexistence of the nonsingular neutral modes. A nonnegative quadratic discriminant and the fact that the coefficient of c^2 is positive imply that c should be bounded from below and above, and

$$\Omega(r_0) + \frac{D(\rho_0 Z)(r_0)(R_2-R_1)^2}{2\pi^2(\rho_0)_{\min} R_1} - \frac{(R_2-R_1)^2\sqrt{\Delta(r_0)}}{2\pi^2(\rho_0)_{\min} R_1}$$
$$\leq c \leq \Omega(r_0) + \frac{D(\rho_0 Z)(r_0)(R_2-R_1)^2}{2\pi^2(\rho_0)_{\min} R_1} + \frac{(R_2-R_1)^2\sqrt{\Delta(r_0)}}{2\pi^2(\rho_0)_{\min} R_1}.$$

This implies that

$$a + \frac{D(\rho_0 Z)_{\min}(R_2-R_1)^2}{2\pi^2(\rho_0)_{\min} R_1} - \frac{(R_2-R_1)^2\sqrt{\Delta_{\max}}}{2\pi^2(\rho_0)_{\min} R_1}$$
$$\leq c \leq b + \frac{D(\rho_0 Z)_{\max}(R_2-R_1)^2}{2\pi^2(\rho_0)_{\min} R_1} + \frac{(R_2-R_1)^2\sqrt{\Delta_{\max}}}{2\pi^2(\rho_0)_{\min} R_1},$$

which completes the proof. □

When ρ_0 is constant, our result reduces to that of Subbiah [15]. Pierro and Abid [11] have pointed out an important difference between swirling flows with $D\rho_0 \geq 0$ and $D\rho_0 \leq 0$. The flows are characterized by *bubbles* for positively buoyant fluids (that is,

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when $D\rho_0 \geq 0$) and *spikes* for negatively buoyant fluids (that is, when $D\rho_0 \leq 0$). For the positively buoyant fluids, the discriminant Δ is always nonnegative, whereas for negatively buoyant fluids the discriminant Δ is nonnegative if the density $\rho_0(r)$ and the velocity $V(r)$ satisfy the condition

$$|D\rho_0|_{\max} \leq \frac{[D(\rho_0 Z)]^2{}_{\min}(R_2 - R_1)^2}{4(\rho_0)_{\min} R_1 \pi^2 \Omega^2{}_{\max}}.$$

The following is an example of a basic flow satisfying this condition:

$$\Omega(r) = \pm\left[\frac{\Omega_1 + \Omega_2}{2} + \left(\frac{\Omega_2 - \Omega_1}{2}\right)\tanh\left(\frac{1}{d}\log\frac{r}{R}\right)\right]$$

with $\Omega_1 = 0.5, \Omega_2 = 1, d = 1/4, R = 2$ and

$$\rho_0(r) = \frac{1}{2}\left[(\rho_1 + \rho_2) - (\rho_1 - \rho_2)\tanh\left(\frac{r - (R_1 + R_2)/2}{0.1}\right)\right]$$

with $\rho_1 = 6, \rho_2 = 7, R_1 = 1$ and $R_2 = 11$.

4. Instability regions for arbitrary angular velocity and density profiles

We obtain instability regions within which the complex phase velocities, c, corresponding to unstable modes lie in the (c_r, c_i)-plane using the classical integral method. Since our results are improvements on the results of Fung [4], our analysis is a continuation of his analysis. The details of the derivation of his results are given in the Appendix.

The semi-elliptical instability region of Fung [4] is valid for flows satisfying the condition $abJ > 0$. As pointed out in [4], this semi-ellipse is different from that of Kochar and Jain [8] for stratified parallel shear flows, since the minor axis of the semi-ellipse of Fung depends on the upper and lower bounds of the angular velocity. Note that Fung's semi-elliptical region has an important limitation, namely, it remains a semi-ellipse even when $J_m = 1/4$. Since a sufficient condition for stability is $J \geq 1/4$ (see Appendix), it is desirable that the instability region should reduce to the line $c_i = 0$ when $J_{\min} \to (1/4)-$. Since this does not happen similarly to the instability regions obtained in [4], we try to improve the semi-elliptical instability region of Fung [4]. Here our analysis is based on the adaptation of the method of Makov and Stepanyants [9] to the present context of rotating flows. In an analogous inequality used in the proof of Fung's semi-ellipse theorem, we combine the first term and the fourth term of the inequality (A.11). Consequently, the dependence on the wave number m vanishes in the final result. However, we do not combine the terms at this stage, and the wave number m appears in the inequality.

Next we rewrite the inequality (A.11) as

$$E^2 + B^2 + m^2 D^2 - \int_{R_1}^{R_2} \rho_0 r^3 |D\Omega|\,|F|\,|D^* F|\,dr \leq (1 - 4J_m)B^2, \tag{4.1}$$

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where

$$E^2 = \int_{R_1}^{R_2} \rho_0 |\Omega - c| r^3 |D^* F|^2 \, dr, \quad B^2 = \int_{R_1}^{R_2} \frac{\rho_0 r^3 |D\Omega|^2}{4|\Omega - c|} |F|^2 \, dr$$

$$\text{and} \quad D^2 = \int_{R_1}^{R_2} \rho_0 |\Omega - c| \, |F|^2 r \, dr.$$

Since

$$\begin{aligned}
&\int_{R_1}^{R_2} \rho_0 r^3 |D\Omega| \, |F| \, |D^* F| \, dr \\
&\quad \le \Big(4 \int_{R_1}^{R_2} \rho_0 r^3 |\Omega - c| \, |D^* F|^2 \, dr\Big)^{1/2} \Big(\int_{R_1}^{R_2} \frac{\rho_0 r^3 |D\Omega|^2}{4|\Omega - c|} |F|^2 \, dr\Big)^{1/2} \\
&\quad = 2EB,
\end{aligned}$$

using the estimate in (4.1),

$$E^2 + B^2 + m^2 D^2 - 2EB \le (1 - 4J_m) B^2,$$
$$\text{that is, } E^2 + m^2 D^2 - 2EB + 4J_m B^2 \le 0.$$

Solving this inequality with respect to E,

$$B - (B^2 - m^2 D^2 - 4J_m B^2)^{1/2} \le E \le B + (B^2 - m^2 D^2 - 4J_m B^2)^{1/2},$$
$$\text{which yields } E^2 + m^2 D^2 \le 2B^2 \Big(1 - 2J_m + \Big(1 - m^2 \frac{D^2}{B^2} - 4J_m\Big)^{1/2}\Big). \tag{4.2}$$

Now

$$\begin{aligned}
\frac{D^2}{B^2} &= \frac{\int_{R_1}^{R_2} \rho_0 |\Omega - c| \, |F|^2 r \, dr}{\int_{R_1}^{R_2} |F|^2 \rho_0 r^3 |D\Omega|^2 / 4|\Omega - c| \, dr} \\
&\ge \frac{4c_i^2 \int_{R_1}^{R_2} \rho_0 |F|^2 r \, dr}{|D\Omega|_{\max}^2 R_2^2 \int_{R_1}^{R_2} \rho_0 |F|^2 r \, dr},
\end{aligned}$$

$$\text{that is, } \frac{D^2}{B^2} \ge \frac{4c_i^2}{|D\Omega|_{\max}^2 R_2^2}. \tag{4.3}$$

Since $|\Omega - c| \ge c_i$,

$$\begin{aligned}
E^2 + m^2 D^2 &= \int_{R_1}^{R_2} \rho_0 |\Omega - c| r^3 |D^* F|^2 \, dr + m^2 \int_{R_1}^{R_2} \rho_0 |\Omega - c| \, |F|^2 r \, dr \\
&\ge c_i \int_{R_1}^{R_2} \rho_0 [r^2 |D^* F|^2 + m^2 |F|^2] r \, dr,
\end{aligned}$$

$$\text{that is,} \quad E^2 + m^2 D^2 \ge c_i \int_{R_1}^{R_2} Q \, dr, \tag{4.4}$$

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and

$$B^2 = \int_{R_1}^{R_2} \frac{\rho_0 r^3 |D\Omega|^2}{4|\Omega - c|} |F|^2 \, dr \le \frac{1}{4c_i} \int_{R_1}^{R_2} \rho_0 r^3 |F|^2 |D\Omega|^2 \, dr. \tag{4.5}$$

Using the estimates (4.3)–(4.5) in (4.2),

$$\int_{R_1}^{R_2} \rho_0 r^3 |F|^2 |D\Omega|^2 \, dr \ge \frac{2c_i^2}{1 - 2J_m + \sqrt{1 - 4J_m - 4m^2 c_i^2 / R_2^2 (D\Omega)^2_{\max}}} \int_{R_1}^{R_2} Q \, dr. \tag{4.6}$$

Now

$$\begin{aligned} ab \int_{R_1}^{R_2} D\rho_0 r^2 |F|^2 \, dr &= ab \int_{R_1}^{R_2} \frac{r\Omega^2 D\rho_0}{\rho_0 r^2 (D\Omega)^2)} \cdot \frac{\rho_0 r^3 (D\Omega)^2}{\Omega^2} |F|^2 \, dr \\ &= ab \int_{R_1}^{R_2} J \frac{\rho_0 r^3 (D\Omega)^2}{\Omega^2} |F|^2 \, dr \\ &\ge abJ_m \int_{R_1}^{R_2} \frac{\rho_0 r^3 (D\Omega)^2}{\Omega^2} |F|^2 \, dr \\ &\ge \frac{aJ_m}{b} \int_{R_1}^{R_2} \rho_0 r^3 (D\Omega)^2 |F|^2 \, dr \\ &\ge \frac{aJ_m}{b} \frac{2c_i^2}{1 - 2J_m + \sqrt{1 - 4J_m - 4m^2 c_i^2 / R_2^2 (D\Omega)^2_{\max}}} \int_{R_1}^{R_2} Q \, dr, \end{aligned}$$

which yields

$$\begin{aligned} &ab \int_{R_1}^{R_2} D\rho_0 r^2 |F|^2 \, dr \\ &\quad \ge \frac{aJ_m}{b} \frac{2c_i^2}{1 - 2J_m + \sqrt{1 - 4J_m - 4m^2 c_i^2 / R_2^2 (D\Omega)^2_{\max}}} \int_{R_1}^{R_2} Q \, dr. \end{aligned} \tag{4.7}$$

Substituting (4.7) in (A.8) and using the fact that $\int_{R_1}^{R_2} Q \, dr > 0$, we have the following generalized semi-ellipse theorem.

THEOREM 4.1. *The complex phase velocity c of any unstable azimuthal mode must lie inside the generalized semi-ellipse*

$$\left(c_r - \frac{a+b}{2}\right)^2 + \left(1 + \frac{2aJ_m/b}{1 - 2J_m + \sqrt{1 - 4J_m - 4m^2 c_i^2 / R_2^2 (D\Omega)^2_{\max}}}\right) c_i^2 \le \left(\frac{b-a}{2}\right)^2. \tag{4.8}$$

Note that in the absence of an outer cylinder when R_2 is infinite, the generalized semi-ellipse in Theorem 4.1 reduces to the semi-ellipse of Fung [4]. However, when the outer cylinder is present at $r = R_2 < \infty$, the generalized semi-elliptical region of Theorem 4.1 definitely lies inside the semi-elliptical region of Fung [4]. Another consequence of this result is given in the next section.

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5. An estimation of the growth rate in the presence of the cylinders

One more important relation follows explicitly from (4.8): stipulation of a nonnegative radicand involved in the condition

$$m^2 c_i^2 \le (\tfrac{1}{4} - J_m) R_2^2 \max_{[R_1,R_2]} \{(D\Omega)^2\}, \tag{5.1}$$

which, in fact, sets an additional restriction on c_i. The inequality (5.1) is derived from (A.10), adapting the method of Howard [6]. Dropping the term $\int_{R_1}^{R_2} \rho_0 r^3 |D^* G|^2 \, dr$ from the left-hand side and using $1/|\Omega - c|^2 \le 1/c_i^2$,

$$m^2 \int_{R_1}^{R_2} \rho_0 |G|^2 r \, dr \le \frac{(1/4 - J_m)(D\Omega)^2_{\max} R_2^2}{c_i^2} \int_{R_1}^{R_2} \rho_0 |G|^2 r \, dr,$$

that is, $m^2 c_i^2 \le (1/4 - J_m)(D\Omega)^2_{\max} R_2^2$. Since the integration is over a finite interval, $[R_1, R_2]$, the dropped term may be estimated by using the well-known Rayleigh–Ritz inequality [3] by adapting the method of Makov and Stepanyants [9]:

$$\begin{aligned}
\int_{R_1}^{R_2} \rho_0 r^3 |D^* G|^2 \, dr &= \int_{R_1}^{R_2} \rho_0 r^3 \Big(\frac{|D(rG)|}{r} \Big)^2 dr \\
&\ge (\rho_0)_{\min} R_1 \int_{R_1}^{R_2} |D(rG)|^2 \, dr \\
&\ge \frac{(\rho_0)_{\min} R_1 \pi^2}{(R_2 - R_1)^2} \int_{R_1}^{R_2} |rG|^2 \, dr \\
&\ge \frac{(\rho_0)_{\min} R_1^3 \pi^2}{(R_2 - R_1)^2} \int_{R_1}^{R_2} |G|^2 \, dr.
\end{aligned}$$

Therefore, from (A.10),

$$\begin{aligned}
\Big(\frac{(\rho_0)_{\min} R_1^3 \pi^2}{m^2 (R_2 - R_1)^2} + (\rho_0)_{\min} R_1 \Big) m^2 &\le \Big(\frac{1}{4} - J_m \Big) \int_{R_1}^{R_2} \frac{\rho_0 r^3 (D\Omega)^2}{4|\Omega - c|^2} |G|^2 \, dr \\
&\le \frac{1}{c_i^2} \Big(\frac{1}{4} - J_m \Big) R_2^3 (D\Omega)^2_{\max} (\rho_0)_{\max},
\end{aligned}$$

which yields the estimate

$$m^2 c_i^2 \le \frac{(\rho_0)_{\max}}{(\rho_0)_{\min}} \frac{(1/4 - J_m) R_2^3 (D\Omega)^2_{\max}}{(R_1^3 \pi^2 / m^2 (R_2 - R_1)^2 + R_1)}. \tag{5.2}$$

This estimate for growth rate of an unstable mode depends on the radii of the cylinders R_1 and R_2 and their difference $R_2 - R_1$. If the distance between the two cylinders tends to zero, then it follows that the growth rate $mc_i \to 0$. The boundedness of the growth rate of unstable modes given in (5.1) and the improvement given in (5.2) are only valid in the presence of an outer cylinder at a finite distance $r = R_2$. Note that the growth rate

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of unstable modes has been found by Pierro and Abid [11] by the large azimuthal wave number asymptotic analysis, that is, for $m \gg 1$. Their result shows that the growth rate (see equation (22) of Pierro and Abid [11]) is an unbounded function of m for $m \gg 1$. The reason for the difference between our result and theirs is as follows.

(i) Their result is an asymptotic result valid for large $|m|$ but without any restriction on the sign of $(D\rho_0)$, whereas our result is valid for any value of $|m|$ but restricted to the case of $(D\rho_0)$ being nonnegative.
(ii) The presence of the cylinders at $r = R_1$ and R_2 gives our bound, while in their analysis the cylinders are absent.

The instability of Batchelor-like vortices with density profile $\rho_0(r) = \rho_\infty[1 + (s-1)e^{-r^2}]$ has been numerically studied by Pierro and Abid [11], where s is the axial to ambient density ratio. In the case of two-dimensional disturbances, they have shown instability for $s \gtrsim 1$. This corresponds to $D\rho_0 \le 0$, that is, density is a decreasing function of r.

6. Further refinement of instability regions for specific angular velocity profiles

The general results obtained in Section 4 apply to arbitrary density and velocity profiles for a stratified flow. Now we shall show that the generalized semi-elliptical region of Section 4 can be improved for a special class of angular velocity profiles. Recall that the apparent inequality

$$\begin{aligned} 0 &\ge \int_{R_1}^{R_2} (\Omega - a)(\Omega - b) Q_1\, dr \\ &= \int_{R_1}^{R_2} \Omega^2 Q_1\, dr - (a+b)\int_{R_1}^{R_2} \Omega Q_1\, dr + ab \int_{R_1}^{R_2} Q_1\, dr \end{aligned} \tag{6.1}$$

plays a central role in proving the semicircle, semi-ellipse and generalized semi-ellipse theorems. Substituting the integrals [4]

$$\int_{R_1}^{R_2} \Omega Q_1\, dr = c_r \int_{R_1}^{R_2} Q\, dr, \quad \int_{R_1}^{R_2} \Omega^2 Q_1\, dr = (c_r^2 + c_i^2)\int_{R_1}^{R_2} Q\, dr$$
$$\text{and} \quad \int Q_1\, dr = \int Q\, dr + \int (D\rho_0) r^2 |F|^2\, dr \tag{6.2}$$

in (6.1), we obtain (A.8), which yields Fung's semicircle after dropping the last term on the left-hand side. The closer the right-hand side of (6.1) is to its upper limit of zero, the more exact is the estimate of the (c_r, c_i) parameter region for growing perturbations. For this purpose, we decrease the absolute value of the first integral in (6.1), substituting Q_1 by a smaller nonnegative function

$$r\rho_0[r|DF| - \sqrt{m^2-1}|F|]^2 = Q_1 - 2r^2\rho_0\sqrt{m^2-1}|DF|\,|F|,$$

which yields

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$$
\begin{aligned}
0 &\geq \int_{R_1}^{R_2} (\Omega - a)(\Omega - b) r\rho_0 [r|DF| - \sqrt{m^2-1}|F|]^2 \, dr \\
&= \int_{R_1}^{R_2} (\Omega - a)(\Omega - b) Q_1 \, dr - 2\sqrt{m^2-1} \int_{R_1}^{R_2} r^2 \rho_0 (\Omega - a)(\Omega - b)|DF|\,|F| \, dr \\
&= \int_{R_1}^{R_2} \Omega^2 Q_1 \, dr - (a+b) \int_{R_1}^{R_2} \Omega Q_1 \, dr + ab \int_{R_1}^{R_2} Q_1 \, dr \\
&\quad + 2\sqrt{m^2-1} \int_{R_1}^{R_2} r^2 \rho_0 \Big[\Big(\frac{b-a}{2}\Big)^2 - \Big(\Omega - \frac{a+b}{2}\Big)^2\Big] |DF|\,|F| \, dr. \qquad (6.3)
\end{aligned}
$$

Using (6.2) in the above inequality (6.3),

$$
\begin{aligned}
0 \geq \{c_r^2 + c_i^2 - (a+b)c_r + ab\} &\int_{R_1}^{R_2} Q \, dr + ab \int_{R_1}^{R_2} (D\rho_0) r^2 |F|^2 \, dr \\
&+ 2\sqrt{m^2-1} \int_{R_1}^{R_2} r^2 \rho_0 \Big[\Big(\frac{b-a}{2}\Big)^2 - \Big(\Omega - \frac{a+b}{2}\Big)^2\Big] |DF|\,|F| \, dr. \qquad (6.4)
\end{aligned}
$$

This inequality generalizes (A.8) and contains an additional final term greater than zero. Neglecting the second and third terms in (6.4) yields Fung's semicircle theorem [4], while neglecting the third term with a suitable estimate of the second term gives his semi-ellipse. An estimate of the second term of (6.4) different from the one obtained by Fung gives our generalized semi-ellipse Theorem 4.1 in Section 4. Now, keeping the third term in (6.4), we will find an estimate for the same, which will give an improved instability region.

Hence, we now estimate the third term on the right-hand side of (6.4), substituting it either by an equivalent expression or by one with a smaller absolute value. For this purpose, we see that when the condition

$$
\int_{R_1}^{R_2} r^2 \rho_0 \Big[\Big(\frac{b-a}{2}\Big) - \frac{\{\Omega - (a+b)/2\}^2}{(b-a)/2}\Big] |DF|\,|F| \, dr \geq \int_{R_1}^{R_2} d r^3 \rho_0 |DF|\,|F|\,|D\Omega| \, dr, \qquad (6.5)
$$

where d is a constant with dimension of distance, is satisfied, the inequality (6.4) can be strengthened, expressing the last component in terms of the integral $\int Q\, dr$. Unfortunately, validity of (6.5) in the general case has not been proved. Now we will identify a class of basic angular velocity profiles that will satisfy the equation (6.5). In particular, equating the integrand on the left- and right-hand sides of (6.5), we obtain the differential equation

$$
D\Omega = \pm \frac{2}{(b-a)} \Big[\Big(\frac{b-a}{2}\Big)^2 - \Big(\Omega - \frac{a+b}{2}\Big)^2\Big], \qquad (6.6)
$$

which has a solution

$$
\Omega(r) = \pm\Big[\frac{a+b}{2} + \Big(\frac{b-a}{2}\Big) \tanh\Big(\frac{\log r}{d} + e\Big(\frac{b-a}{2}\Big)\Big)\Big], \qquad (6.7)
$$

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where e is an arbitrary constant. If we choose e such that $e\{(b-a)/2\} = -(\log R)/d$, where R is a constant, then

$$\Omega(r) = \pm\left[\frac{a+b}{2} + \left(\frac{b-a}{2}\right)\tanh\left(\frac{1}{d}\log\frac{r}{R}\right)\right], \tag{6.8}$$

which is the profile given by Shukhman [13]. Note that Shukhman [13] has constructed this angular velocity profile as a model for a mixing layer of rotating fluid which is an analogue of the mixing layer for a plane flow widely used in the literature. He has discussed the linear instability of this model flow with homogeneous density and the nonlinear instability as well. It is interesting that the Shukhman profile [13] given in (6.8) is obtained as a solution of an ordinary differential equation satisfied by the angular velocity that is necessary for the applicability of the improved instability region.

The inequality (6.5) is satisfied for the profiles (6.7) or smoother ones, that is, profiles for which

$$|D\Omega| \le \frac{2}{b-a}\left[\left(\frac{b-a}{2}\right)^2 - \left(\Omega - \frac{a+b}{2}\right)^2\right].$$

For these profiles, the inequality (6.4) takes the form

$$\begin{aligned} 0 \ge &\left[\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2\right]\int_{R_1}^{R_2} Q\,dr + ab\int_{R_1}^{R_2} (D\rho_0)r^2|F|^2\,dr \\ &+ 2d\sqrt{m^2-1}\left(\frac{b-a}{2}\right)\int_{R_1}^{R_2} \rho_0 r^3|D\Omega|\,|DF|\,|F|\,dr. \end{aligned} \tag{6.9}$$

Now we estimate the last term by using the relation between F and G, that is, from $G = (\Omega - c)^{1/2}F$,

$$|DG|^2 \ge |\Omega - c|\,|DF|^2 + \frac{|D\Omega|^2}{4|\Omega - c|}|F|^2 - |D\Omega|\,|F|\,|DF|.$$

This yields

$$\begin{aligned} &\int_{R_1}^{R_2} \rho_0(r^2|DG|^2 + m^2|G|^2)r\,dr \\ &\quad\ge \int_{R_1}^{R_2} \rho_0 r|\Omega - c|(r^2|DF|^2 + (m^2-1)|F|^2)\,dr \\ &\qquad+ \int_{R_1}^{R_2} \frac{r^3\rho_0|D\Omega|^2|F|^2}{4|\Omega - c|}\,dr - \int_{R_1}^{R_2} r^3\rho_0|D\Omega|\,|F|\,|DF|\,dr, \\ &\qquad\text{that is,} \int_{R_1}^{R_2} \rho_0(r^2|DG|^2 + m^2|G|^2)r\,dr \\ &\quad\ge \int_{R_1}^{R_2} |\Omega - c|Q_1\,dr + \int_{R_1}^{R_2} \frac{r^3\rho_0|D\Omega|^2}{4|\Omega - c|}|F|^2\,dr \\ &\qquad- \int_{R_1}^{R_2} r^3\rho_0|D\Omega|\,|F|\,|DF|\,dr. \end{aligned} \tag{6.10}$$

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Now, using integration by parts and the boundary conditions (2.15),

$$
\begin{aligned}
&\int_{R_1}^{R_2} \rho_0[r^2|D^*G|^2 + m^2|G|^2]r\,dr \\
&\geq (\rho_0)_{\min}R_1 \int_{R_1}^{R_2} [r^2|D^*G|^2 + m^2|G|^2]\,dr \\
&= (\rho_0)_{\min}R_1\Big[\int_{R_1}^{R_2} (r^2|DG|^2 + m^2|G|^2)\,dr\Big] \\
&\quad + (\rho_0)_{\min}R_1\Big[\int_{R_1}^{R_2} |G|^2\,dr + (r|G|^2)_{R_1}^{R_2} - \int_{R_1}^{R_2} |G|^2\,dr\Big] \\
&= (\rho_0)_{\min}R_1\Big[\int_{R_1}^{R_2} (r^2|DG|^2 + m^2|G|^2)\,dr\Big] \\
&\geq \frac{(\rho_0)_{\min}R_1}{(\rho_0)_{\max}R_2}\Big[\int_{R_1}^{R_2} \rho_0(r^2|DG|^2 + m^2|G|^2)r\,dr\Big], \\
&\quad \text{that is,}\ \int_{R_1}^{R_2} \rho_0(r^2|DG|^2 + m^2|G|^2)r\,dr \leq \lambda \int_{R_1}^{R_2} \rho_0[r^2|D^*G|^2 + m^2|G|^2]r\,dr,
\end{aligned}
$$

where $\lambda = [(\rho_0)_{\max}R_2/(\rho_0)_{\min}R_1] > 1$. Then, from (A.10),

$$
\int_{R_1}^{R_2} \rho_0(r^2|DG|^2 + m^2|G|^2)r\,dr \leq \lambda\Big(\frac{1}{4} - J_m\Big) \int_{R_1}^{R_2} \frac{\rho_0 r^3(D\Omega)^2}{|\Omega - c|^2}|G|^2\,dr. \tag{6.11}
$$

Substituting (6.11) in (6.10) and since $|\Omega - c| \geq c_i$ and $1/|\Omega - c| \geq 1/(b-a)$,

$$
\int_{R_1}^{R_2} \rho_0 r^3|D\Omega|\,|DF|\,|F|\,dr \geq c_i \int_{R_1}^{R_2} Q_1\,dr + \frac{\lambda(J_m - 1/4) + 1/4}{b-a} \int_{R_1}^{R_2} \rho_0 r^3(D\Omega)^2|F|^2\,dr,
$$

provided that $\lambda(J_m - 1/4) + 1/4 > 0$, that is, $J_m \geq (1/4)(1 - 1/\lambda)$.

Using the inequality (4.6) and since $\int_{R_1}^{R_2} Q_1\,dr \geq (a/b)\int_{R_1}^{R_2} Q\,dr$, (6.11) reduces to

$$
\begin{aligned}
\int_{R_1}^{R_2} \rho_0 r^3|D\Omega|\,|DF|\,|F|\,dr &\geq \frac{c_i a}{b} \int_{R_1}^{R_2} Q\,dr + \frac{\lambda(J_m - 1/4) + 1/4}{b-a} \int_{R_1}^{R_2} \rho_0 r^3(D\Omega)^2|F|^2\,dr \\
&\geq \frac{c_i a}{b} \int_{R_1}^{R_2} Q\,dr + \frac{\lambda(J_m - 1/4) + 1/4}{b-a} \\
&\quad \times \frac{2c_i^2}{1 - 2J_m + \sqrt{1 - 4J_m - 4m^2c_i^2/R_2^2(D\Omega)^2_{\max}}} \int_{R_1}^{R_2} Q\,dr.
\end{aligned}
$$

Substituting this estimate in (6.9), dropping the term $ab\int_{R_1}^{R_2} D\rho_0 r^2|F|^2\,dr$ (since we have taken $D\rho_0 \geq 0$) and since $\int_{R_1}^{R_2} Q\,dr > 0$,

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$$\left(c_r - \frac{a+b}{2}\right)^2 + \left\{c_i + \frac{ad}{b}\sqrt{m^2-1}\left(\frac{b-a}{2}\right)\right\}^2 + \frac{2d\sqrt{m^2-1}\{\lambda(J_m - 1/4) + 1/4\}c_i^2}{1 - 2J_m + \sqrt{1 - 4J_m - 4m^2c_i^2/R_2^2(D\Omega)^2_{\max}}} \le \left(\frac{b-a}{2}\right)^2\left[1 + \frac{a^2d^2}{b^2}(m^2-1)\right].$$

And, if we do not drop the term $ab\int_{R_1}^{R_2} D\rho_0 r^2|F|^2\,dr$ and use the estimate given in (4.7),

$$\left(c_r - \frac{a+b}{2}\right)^2 + \left\{c_i + \frac{ad}{b}\sqrt{m^2-1}\left(\frac{b-a}{2}\right)\right\}^2 + 2\frac{d\sqrt{m^2-1}\{\lambda(J_m - 1/4) + 1/4\} + aJ_m/b}{1 - 2J_m + \sqrt{1 - 4J_m - 4m^2c_i^2/R_2^2(D\Omega)^2_{\max}}}c_i^2 \le \left(\frac{b-a}{2}\right)^2\left[1 + \frac{a^2d^2}{b^2}(m^2-1)\right]. \tag{6.12}$$

Thus, we have proved the following result.

Theorem 6.1. *For angular velocity profiles satisfying the conditions* (6.6) *and* (6.4)*, the instability region for arbitrary unstable modes is given by* (6.12)*.*

If we put $m = 1$, then (6.12) reduces to (4.8), but for other values of m the instability region given by (6.12) lies not only inside the semicircle and semi-ellipse of Fung [4] but also inside the generalized semi-ellipse given by (4.8) of the present paper. As noted earlier in the context of an improvement of Fung's semi-ellipse, the presence of an outer cylinder at a finite distance $r = R_2$ is also essential for the above result.

7. Discussion of the result

For the stability problem of inviscid swirling flows with variable density with respect to two-dimensional disturbances, we have obtained a number of new results in the present paper. The boundedness of the wave velocity of neutral modes has been proved for the first time here.

Dixit and Govindarajan [2] and Pierro and Abid [11] studied the stability problem of swirling flows in the absence of cylinders. This means that $R_1 = 0$ and $R_2 = \infty$ and, in this situation, the condition of Theorem 3.1 becomes $[D(\rho_0 Z)]^2 \ge 0$, which is satisfied without any restriction on the basic velocity or density.

Our generalized semi-elliptical region of instability is an improvement on the semicircular and semi-elliptical regions of Fung [4] in the following way: the instability region given by (4.8):

(i) lies inside the semicircular region of Fung and it reduces to his semicircle when $J_m \to 0$;

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(ii) it depends on the minimum Richardson number J_m and the minimum and maximum of the angular velocity like Fung's semi-ellipse; in addition, it also depends on the azimuthal wave number m and the radius of the outer cylinder R_2. In fact, when $R_2 \to \infty$, it reduces to Fung's semi-elliptical region;

(iii) it reduces to the line $c_i = 0$ when $J_m \to (1/4)-$ in accordance with the sufficient condition for stability, derived in the Appendix;

(iv) an estimate for the growth rate mc_i follows from the nonnegativity of the term inside the square root in (4.8).

The estimate of growth rate (5.2) of an unstable mode depends on the radii of the cylinders R_1 and R_2 and their difference $R_2 - R_1$. If the distance between the two cylinders tends to zero, then it follows that the growth rate $mc_i \to 0$. The boundedness of the growth rate of unstable modes given in (5.1) and the improvement given in (5.2) are valid in the presence of the outer cylinder at a finite distance $r = R_2$ only. Note that the growth rate of unstable modes has been found by Pierro and Abid [11] by the large azimuthal wave number asymptotic analysis, that is, for $m \gg 1$. Their result shows that the growth rate (see equation (22) of their article [11]) is an unbounded function of m for $m \gg 1$. The reason for the difference between our result and theirs is as follows.

(i) Their result is an asymptotic result valid for large $|m|$, but without any restriction on the sign of $(D\rho_0)$; whereas our result is valid for any value of $|m|$, but restricted to the case of $(D\rho_0)$ being nonnegative.

(ii) The presence of the cylinders at $r = R_1$ and R_2 gives our bound, while in their analysis the cylinders are absent and, consequently, $R_1 = 0$ and $R_2 = \infty$.

Pierro and Abid [11] have studied the instability of Batchelor-like vortices with density profile $\rho_0(r) = \rho_\infty[1 + (s - 1)e^{-r^2}]$, where s is the axial to ambient density ratio. In the case of two-dimensional disturbances, they have shown instability for $s \gtrsim 1$. This corresponds to $D\rho_0 \leq 0$, that is, density is a decreasing function of r.

For further understanding of the results of the present paper, namely, the estimates for the growth rate and the instability regions, we plot the curves defining the ranges of c_r and c_i by choosing $\Omega(r) = (a + b)/2 + [(b - a)/2] \tanh((1/d) \log (r/R))$ with $a = 0.5, b = 1, d = 1/4, R = 2$ and for a particular value of $J_{\min}$, namely $J_m = 0.2$. In Figure 1, the breaks of the curves indicate that the radicand in (4.8) becomes negative for sufficiently large m. Therefore, the additional restriction stipulated by (5.1) occurs in the imaginary part of the wave velocity. In this figure, the bold line is Fung's semi-ellipse and the dashed line is Fung's semicircle. It is clear from this figure that the maximum of the imaginary part of the wave velocity depends on the azimuthal wave number as well as J_m. The inequalities (4.8) and (5.1) enable us to estimate the dependence of the maximum growth rate on J_m. The family of lines in Figure 2 represent the estimates of mc_i versus J_m at different values of m.

We now consider a velocity profile of the type (6.8), assuming that density is an arbitrary smooth function of r. The range of (c_r, c_i) defined by (6.12) is shown in Figure 3 for different values of m and $J_m = 0.2$. The value of λ is computed by taking

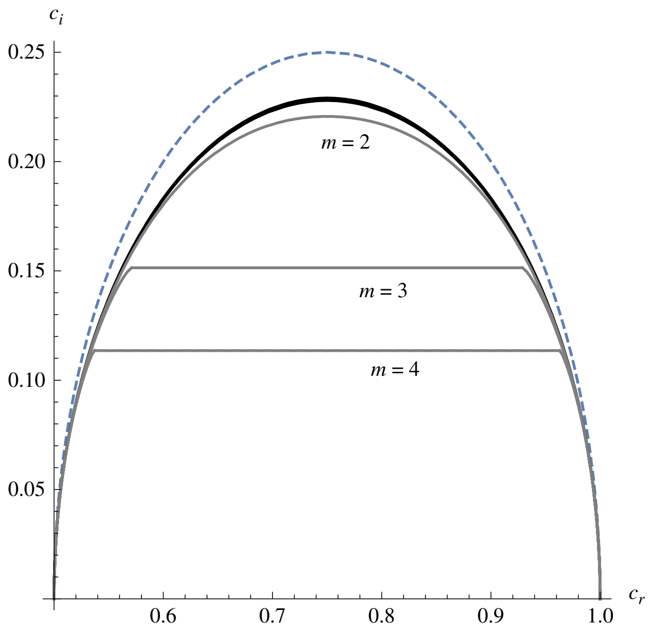

FIGURE 1. Dependence of the complex wave velocity range for growing perturbations on the wave number m, plotted using (4.8) at $J_m = 0.2$. The dashed line shows Fung's semicircle, the heavy curve is given for Fung's semi-ellipse and the generalized semi-ellipse is given for different values of m, namely, $m = 2, m = 3, m = 4$.

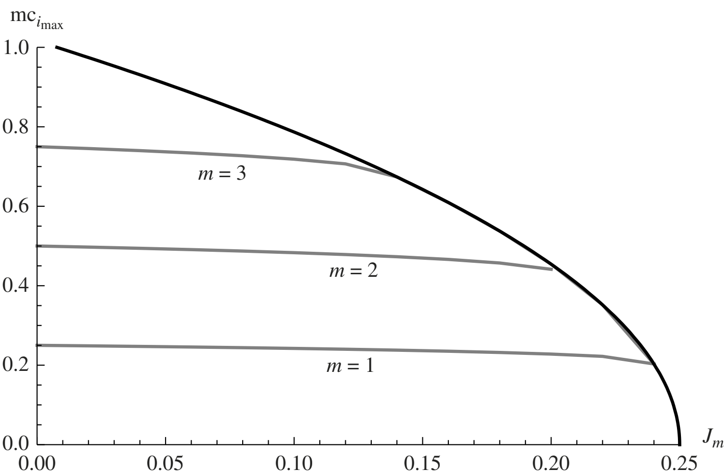

FIGURE 2. Estimates of the maximum growth rate mc_i versus the minimum Richardson number J_m at different m. The thick line gives the growth rate given by (5.1). The improved estimates following from the generalized semi-ellipse (4.8) are given for different values of m.

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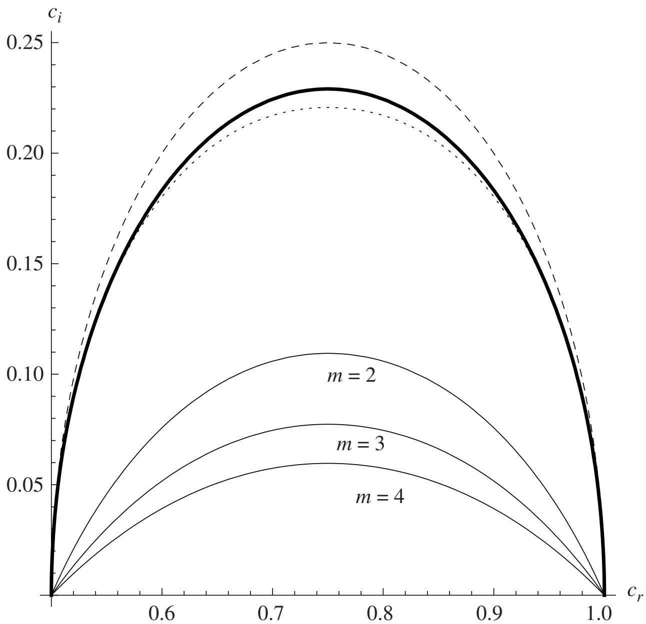

FIGURE 3. Dependence of the complex wave velocity range for growing perturbations on the dimensionless wave number m for a particular velocity profile of the type (6.8), plotted using (6.12) at $J_m = 0.2$. The dashed line shows Fung's semicircle and the heavy curve is given for Fung's semi-ellipse. The dotted line gives the generalized semi-ellipse for $m = 2$.

the density profile

$$\rho_0 = \frac{1}{2}\Big[(\rho_1 + \rho_2) - (\rho_1 - \rho_2)\tanh\Big(\frac{r - (R_1 + R_2)/2}{0.1}\Big)\Big],$$

which Dixit and Govindarajan [2] have used by choosing $\rho_1 = 6, \rho_2 = 7, R_1 = 1, R_2 = 4$, and this yields $\lambda = 14/3$. Note that the curves corresponding to equal values of m are located in Figure 3 lower than in Figure 1. From Figure 3, we see that the instability region for $m = 2$ as given by (6.12) lies inside that given by (4.8). The improved instability region is valid for disturbances with $m \neq 1$. It may be noted here that Fung [4] has discussed a basic flow which is stable for $m = 1$ but unstable for $m \neq 1$. Also, Shukhman [13] has shown that the angular velocity profile constructed by him is stable for $m = 1$ and all values of d but it becomes unstable for $m = 2$ when $d < 1/2$. In the context of parallel shear flows, Makov and Stepanyants [9] have plotted the growth rate versus the wave number curve for different values of the minimum Richardson number. Unlike that situation, in the case of the stability of swirling flows to azimuthal disturbances, the wave number m varies over discrete values, and we have plotted the growth rate mc_i versus the minimum Richardson number $J_{\min}$ curve for different values of the wave number.

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8. Concluding remarks

We will make some remarks on the stability problem of variable density swirling flows with particular reference to the problem that needs to be solved for a better understanding of the problem. It is necessary to compute the maximum growth rate curves as a function of the minimum Richardson number, J_m, for positively buoyant fluids so that it can be compared with the growth rate curves given in the present paper. For constant density swirling flows, a necessary condition for instability is that DZ should change sign in the flow domain. Now, we should find the role of DZ in the stability analysis of variable density swirling flows. In particular, we should find instability regions that depend on the term DZ. Finally, the wave velocity of neutral modes should be computed so that it can be compared with the bounds given in the present paper.

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Appendix

Multiplying equation (2.12) by rF^*, where F^* is the complex conjugate of F, and integrating the final equation over the flow region,

$$\int_{R_1}^{R_2} (\Omega - c)D(\rho_0 r^2 D^*[(\Omega - c)F])rF^*\,dr + \int_{R_1}^{R_2} \{-m^2\rho_0(\Omega - c)^2 - rD(\rho_0 D^*(r\Omega))(\Omega - c) + r\Omega^2 D\rho_0\}r|F|^2\,dr = 0. \quad \text{(A.1)}$$

Now, by using the integration by parts formula and making use of the boundary conditions (2.13),

$$\begin{aligned}\int_{R_1}^{R_2} &(\Omega - c)D(\rho_0 r^2 D^*[(\Omega - c)F])rF^*\,dr \\ &= \int_{R_1}^{R_2} \rho_0 r^2 D\Omega(\Omega - c)|F|^2\,dr + \int_{R_1}^{R_2} D\rho_0 r^3 D\Omega(\Omega - c)|F|^2\,dr \\ &\quad + \int_{R_1}^{R_2} \rho_0 r^3 D^2\Omega(\Omega - c)|F|^2\,dr - \int_{R_1}^{R_2} (\Omega - c)^2\rho_0 r^3 |D^*F|^2\,dr.\end{aligned}$$

Substituting the above equation in (A.1) yields

$$\int_{R_1}^{R_2} (\Omega - c)^2 Q\,dr + \int_{R_1}^{R_2} [2rD(\rho_0\Omega)(\Omega - c) - r\Omega^2(D\rho_0)]|F|^2 r\,dr = 0, \quad \text{(A.2)}$$

where $Q = \rho_0[r^2|D^*F|^2 + m^2|F|^2]r \geq 0$.

Further, let $Q_1 = \rho_0[r^2|DF|^2 + (m^2 - 1)|F|^2]r \geq 0$. Then equation (A.2) reduces to a relatively simple form

$$\int_{R_1}^{R_2} (\Omega^2 - 2c\Omega)Q_1\,dr + c^2 \int_{R_1}^{R_2} Q\,dr = 0. \tag{A.3}$$

The real and imaginary parts of (A.3) give

$$\int_{R_1}^{R_2} \Omega^2 Q_1\,dr - 2c_r \int_{R_1}^{R_2} \Omega Q_1\,dr + (c_r^2 - c_i^2) \int_{R_1}^{R_2} Q\,dr = 0, \tag{A.4}$$

$$\int_{R_1}^{R_2} \Omega Q_1\,dr = c_r \int_{R_1}^{R_2} Q\,dr. \tag{A.5}$$

Since $a = \Omega_{\min}$ and $b = \Omega_{\max}$ over $R_1 \leq r \leq R_2$, $a \leq \Omega \leq b$, and we have the apparent inequality

$$\int_{R_1}^{R_2} (\Omega - a)(\Omega - b)Q_1\,dr \leq 0, \tag{A.6}$$

which is the basis for the proof of Fung's semicircle theorem.

It follows that

$$\int_{R_1}^{R_2} Q_1\,dr = \int_{R_1}^{R_2} Q\,dr + \int_{R_1}^{R_2} D\rho_0 r^2 |F|^2\,dr. \tag{A.7}$$

Using (A.4), (A.5) and (A.7) in (A.6),

$$\{c_r^2 + c_i^2 - (a + b)c_r + ab\} \int_{R_1}^{R_2} Q\,dr + ab \int_{R_1}^{R_2} (D\rho_0)r^2|F|^2\,dr \leq 0. \tag{A.8}$$

If $abD\rho_0 \geq 0$, then the last term in (A.8) can be dropped and, since $\int_{R_1}^{R_2} Q\,dr > 0$, the semicircle theorem of Fung [4], namely,

$$\left(c_r - \frac{a + b}{2}\right)^2 + c_i^2 \leq \left(\frac{b - a}{2}\right)^2,$$

follows.

For nonaxisymmetric instabilities of heterogeneous swirling flows the semicircle theorem of Fung [4] holds under a restriction, namely, the product of the density gradient and the upper and lower bounds of the velocity is greater than or equal to zero. If this restriction is violated, then the phase velocity may no longer be bounded and an example supporting this has been given by Fung. However, when $abD\rho_0 > 0$, we have the semicircular instability region of Fung that has been derived just above. Adapting the method of Kochar and Jain [8] to the case of swirling flows, one gets from (2.14) the equality

$$\int_{R_1}^{R_2} \rho_0[r^2|D^*G|^2 + m^2|G|^2]r\,dr + \int_{R_1}^{R_2} \left(J - \frac{1}{4}\right) \frac{\rho_0 r^2 (D\Omega)^2}{|\Omega - c|^2} |G|^2 r\,dr = 0, \tag{A.9}$$

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where J is the local Richardson number defined earlier. It follows from (A.9) that a necessary condition for instability is $J < 1/4$ at least once in (R_1, R_2). Consequently, it is necessary that $J_{\min} < 1/4$ for instability. Alternatively, it means that a sufficient condition for stability is $J_{\min} \geq 1/4$.

From equation (A.9),

$$\int_{R_1}^{R_2} \rho_0[r^2|D^*G|^2 + m^2|G|^2]r\,dr \leq (1 - 4J_m)\int_{R_1}^{R_2} \frac{\rho_0 r^3(D\Omega)^2}{4|\Omega - c|^2}|G|^2\,dr. \tag{A.10}$$

Now, since $G = (\Omega - c)^{1/2}F$,

$$|D^*G|^2 \geq |\Omega - c|\,|D^*F|^2 + \frac{|D\Omega|^2}{4|\Omega - c|}|F|^2 - |D\Omega|\,|F|\,|D^*F|.$$

Substituting the above inequality in (A.10) yields

$$\begin{aligned}\int_{R_1}^{R_2} \rho_0 r^3|\Omega - c|\,|D^*F|^2\,dr + \int_{R_1}^{R_2} \frac{\rho_0 r^3|D\Omega|^2}{4|\Omega - c|}|F|^2\,dr - \int_{R_1}^{R_2} \rho_0 r^3|D\Omega|\,|F|\,|D^*F|\,dr \\ + m^2\int_{R_1}^{R_2} \rho_0 r|\Omega - c|\,|F|^2\,dr \leq (1 - 4J_m)\int_{R_1}^{R_2} \frac{\rho_0 r^3|D\Omega|^2}{4|\Omega - c|}|F|^2\,dr.\end{aligned} \tag{A.11}$$

Using the transformation $G = (\Omega - c)^{1/2}F$ and adapting the method of Kochar and Jain [8],

$$\int_{R_1}^{R_2} \rho_0 r^3|D\Omega|^2|F|^2\,dr \geq \frac{4c_i^2\int_{R_1}^{R_2} Q\,dr}{[1 + (1 - 4J_m)^{1/2}]^2}. \tag{A.12}$$

Now, using (A.12),

$$\begin{aligned} ab\int_{R_1}^{R_2} D\rho_0 r^2|F|^2\,dr &= ab\int_{R_1}^{R_2} J\frac{\rho_0 r^3(D\Omega)^2}{\Omega^2}|F|^2\,dr \\ &\geq \frac{aJ_m}{b}\int_{R_1}^{R_2} \rho_0 r^3(D\Omega)^2|F|^2\,dr \\ &\geq \frac{aJ_m}{b}\frac{4c_i^2\int_{R_1}^{R_2} Q\,dr}{[1 + (1 - 4J_m)^{1/2}]^2}.\end{aligned}$$

Hence,

$$ab\int_{R_1}^{R_2} D\rho_0 r^2|F|^2\,dr \geq \frac{aJ_m}{b}\frac{4c_i^2\int_{R_1}^{R_2} Q\,dr}{[1 + (1 - 4J_m)^{1/2}]^2}. \tag{A.13}$$

Substituting (A.13) in (A.8) and, since $\int_{R_1}^{R_2} Q\,dr > 0$, we have the following semi-ellipse of Fung [4]:

$$\left(c_r - \frac{a + b}{2}\right)^2 + \left(1 + \frac{4aJ_m}{b[1 + (1 - 4J_m)^{1/2}]^2}\right)c_i^2 \leq \left(\frac{b - a}{2}\right)^2.$$

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