Part I

CHAOS

https://doi.org/10.1017/S0252921100065866 Published online by Cambridge University Press

Instability, Chaos and Predictability
in Celestial Mechanics and
Stellar Dynamics

ISBN 1-56072-054-9

CHAOS

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ABSTRACT

This is an introductory article on Chaos giving the definition of Chaos, source and tools of Chaos, routes to chaos, measurement of chaos, chaos through resonance. The Duffing equation has been mentioned as a problem of double resonance leading to chaos and finally some problems relating to chaos in the Solar system have been described.

INTRODUCTION

The discovery of new types of dynamic behaviour in physical systems in the last decade has brought about new analytic and experimental techniques in dynamics. Principal amongst these new discoveries is the existence of chaotic, unpredictable behaviour in many non-linear deterministic systems. Observations of chaotic and prechaotic behaviour have been observed in all areas of classical physics including solid and fluid mechanics, thermo-fluid phenomena, electro-magnetic systems and in the area of acoustics.

To mention a few, the following scientists have contributed to the development of classical and modern techniques studying chaotic behaviour of a dynamical system.

a) Classical : Newton, Lagrange, Hamilton, Poin-care, Birkhoff

b) Modern : Kolomogrov, Arnold, Moser (KAM); Henon, Feignenbaum, Wisdom, Henrard, Chirikov, Cohen, Froschle, Deprit, Szebehely, Contropolus, Brumberg, Bhatnagar, Buty, Moon, Lie and many others.

Deterministic and chaotic motion

In the Newtonian deterministic system, initial conditions determine an orbit.

A deterministic system is one in which the values of x_i for $i \geq (n+1)$ can be determined from the values of x_i, $i \leq n$. In the simplest case this is written in the form

$$x_{n+1} = f(x_n)$$

This can be recognized as a difference equation. The idea can be generalised for more than one variable.

CHAOS

Def. 1: If with very slight changes in the initial conditions of a deterministic system, the orbit is random, then the motion is said to be chaotic.

Def. 2: Almost all bounded motions with atleast one positive Liapunov characteristic exponents (L.C.E) are chaotic.

SOURCE AND TOOLS OF CHAOS

The modern advances in non-linear dynamics in both mathematical theory and analytical techniques have been matched by the development of new experimental and numerical tools for studying the dynamics of non-linear systems. The list of experimental and numerical tools to study systems with chaotic, dynamics includes the following:

- Phase plane methods
- Pseudo phase plane methods
- Bifurcation diagrams
- Fast Fourier transforms
- Auto-correlation functions
- Poincare maps
- Double poincare maps
- Reduction to one-dimensional maps
- Liapunov exponents
- Fractal dimensions
- Invariant distributions
- Chaos diagrams
- Basic boundary diagrams

These techniques expand the scientist ability to analyse the dynamics of complex systems. The specific technique depends in part on the particular nature of the system. However

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in the case of simple one degree-of-freedom non-linear systems, a simple procedure has been developed to look for chaotic behaviour.

The ability to classify the nature of oscillations can provide a clue as to how to control them. For example, if the system is thought to be linear, large periodic oscillations may be traced to a resonance effect. However, if the system is non-linear, a limit cycle may be the source of periodic vibration, which in turn may be traced to some dynamic instability in the system.

In order to identify non-periodic or chaotic motions the following check list is provided:

i) Identify nonlinear elements in the system
ii) Check for sources of random input in the system
iii) Observe time history of measured signal
iv) Look at phase plane history
v) Observing limit cycle
vi) Examine Fourier spectrum of signal
vii) Take poincaré map of signal
viii) Vary system parameters (bifurcation diagram).

Now we shall discuss briefly the above mentioned elements through which we identify non-periodic or chaotic motions:

1. Nonlinear Elements

A chaotic system must have nonlinear elements or properties. A linear system cannot exhibit chaotic behaviour. In a linear system periodic in-puts produce periodic out-puts of the same period.

In a nonlinear system periodic inputs can produce periodic or sub-harmonic or chaotic motion.

2. Random Inputs

By definition, chaotic motion arise from deterministic physical systems or deterministic differential or difference equations. A very low input disturbance is required if one is to attribute non-periodic (or chaotic) response to a deterministic system behaviour.

3. Time History

Usually the first clue to chaotic motion, in general, is obtained, through the study of amplitude with time. Non-periodicity and instability leads to chaos.

The motion is observed to exhibit no visible pattern or periodicity.

4. Limit Cycle

Def: An isolated periodic orbit is a limit cycle .Following are examples of 1-lt. cycle, 2-lt. cycle, 3-lt cycle.

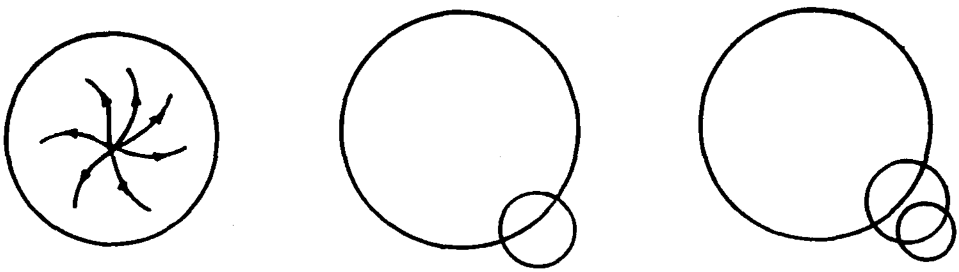

Fig. 1 (a) 1-Lt. Cycle

Fig. 1 (b) 2-Lt. Cycle

Fig. 1 (c) 3-Lt. Cycle

In general 3-lt cycle leads to chaos.

5. Phase Space (or phase plane in the case of one-degree of freedom)

Def: The phase-space is defined as the set of points $(x,\dot{x})$ or (x,p) or $(q,\dot{q})$.

When the motion is periodic (Fig. 2), the phase-plane orbit traces out a closed curve. Whereas chaotic motions on the other hand have orbit which never close or repeat.

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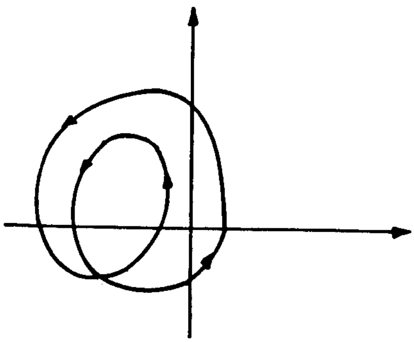

Fig.2 (a)

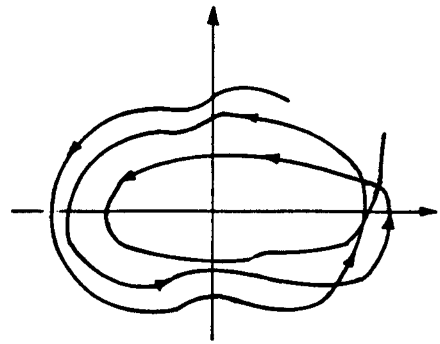

Fig. 2 (b)

6. <u>Poincare Map</u>

Def: The $[x(t_n), \dot{x}(t_n)]$ is called a Poincare map, where t_n is selected according to some rule (Fig. 3)

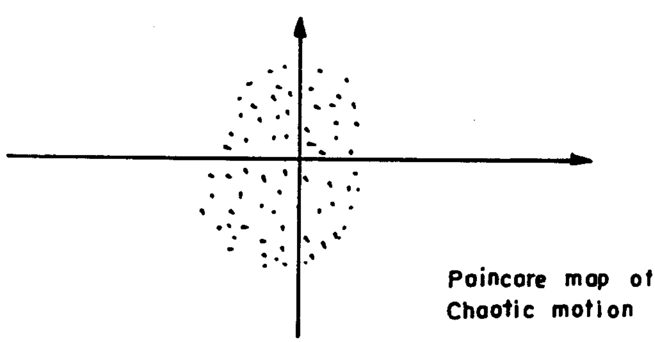

Fig. 3

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a) When t_n=1-period, the orbit consists of one point.

b) When t_n = 2-period, the orbit consists of two points.

c) 3-period leads to chaos.

Example: $x_1 = c_1 \sin(w_1 t + d_1) + c_2 \sin(w_2 t + d_2)$.

This represents

a) a periodic motion, when $\frac{w_1}{w_2}$ is rational.

b) quasi-periodic, when $\frac{w_1}{w_2}$ is irrational.

c) chaos.

If the Poincare map does not consist of either a finite set of points or closed orbit then the motion may be chaotic.

7. Fourier Spectrum

Whenever there is the appearance of a broad spectrum of frequencies in the output when the input is a single frequency harmonic motion there is a possibility of a chaotic motion. Suppose initially there is a dominant frequency a precursor to chaos is the appearance of sub-harmonics in the frequencies spectrum w_o/n, mw_o/n; $m,n \in I$.

8. Resonance

Consider the system

$\ddot{x} + n^2 x = f(x)$; f(x) is of period m. If m and n are commensurable, resonance occurs.

Def: If the basic frequencies of a dynamical system are commensurable, the phenomenon of resonance occurs.

Example: (i) In the solar system the satellite hyperion is probably trumbling chaotically.

(ii) Double resonance leads to chaos. It is shown by the study of the Duffing equation.

9. Routes to Chaos

i) Variation of parameter

In varying the controlling parameter, the appearance of subharmonic periodic terms leads to chaos. Several models of prechaotic behaviour have been observed in both numerical and physical experiments.

ii) Period Doubling

In the period doubling phenomenon, one starts with a

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fundamental period motion. Then as some experimental parameter is varied, say λ the motion undergoes a bifurcation or change to a periodic motion with twice the period of original oscillation. As λ is changed, further, the system undergoes bifurcation to periodic motions with twice the period of the previous oscillation.

Feigenbaum has discovered a very interesting result for the critical values of λ:

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}, \quad \delta = 4.6692016\ldots, \quad n \to \infty$$

In practice, this limit approaches by the third or 4th bifurcation.

This process accumulates at a critical value of the parameter after which the motion becomes chaotic.

iii) Quasi-periodic

Def: An orbit on a torus is quasi-periodic if it is closed.

In case the orbit on a torus is not closed, it leads to chaos.

iv) Intermittances

In this case, one observes long periods of periodic motion with bursts of chaos. As one varies the controlling parameter the chaotic bursts become more frequent and longer.

v) Fixed Points

Def: If $f(x) = x$ then x is called a fixed point.

There are stable fixed points and unstable fixed points. Unstable fixed points lead to chaos.

Strange Attractors: When in the neighbourhood of a fixed point we get similar structure within a structure - we call it a strange attractor. We came across strange attractors in plasma physics.

vi) Measurement of chaos

(a) Fractal dimension:

Consider N discrete points of the orbit

$$\bar{x}_i, \quad i = 1,2,\ldots,N.$$

We define distance between two points

$$S_{ij} = |\bar{x}_i - \bar{x}_j|$$

and correlation function

$$c(r) = \lim_{n\to\infty} \frac{1}{N^2} \quad \text{(Number of pairs of } i,j \text{ for which } S_{ij} < r\text{)}$$

<u>Def:</u> Fractional dimension

$$d = \lim_{r\to 0} \frac{\log c(r)}{\log (r)}$$

1) Orbit is periodic, when $d = 0$
2) Orbit is quasi-periodic, when $d = 1$
3) Orbit is chaotic if $1 < d < 2$

b) Lyapunov Exponents:

Lyapunov exponents enable us to decide whether or not a system is chaotic. Chaos in deterministic systems implies a sensitive dependence on initial conditions. This means that if two trajectories start close to one another in phase-space, they will more exponentially away from each other for small times on the average.

Thus if d_o is a measure of the initial distance between the two starting points, at a latter time the distance is

$$d(t) = d_o\, 2^{\lambda t} \quad \text{(for continuous orbit)}$$

$$d_n = d_o\, 2^{\Lambda n} \quad \text{(for discrete orbit)}$$

The choice of base 2 is convenient but arbitrary are called Lyapunov exponents. We have

$$\lambda = \frac{1}{t_N - t_o} \sum_{k=1}^{n} \log \frac{d(t_k)}{d_o(t_k - 1)}$$

$$\Lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{n} \log_2 \left| \frac{df(x)}{dx} \right|$$

where discrete mapping is $x_{n+1} = f(x_n)$. Then the criterion for chaos becomes

$\lambda, \Lambda > 0$ chaotic,

$\lambda\ \Lambda \leq 0$ regular motion.

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vi) Chaos Through Double Resonance

KAM - Result:

If we are given an integrable system with N degree of freedom, then its trajectory in the 2N-dimensional phase-space are constrained to lie on N-dimensional surfaces in the phase-space. These N-dimensional surfaces are called KAM surfaces.

If we perturb such a system by a weak perturbation which makes the system non-integrable, most KAM surfaces remain intact. However, the perturbation induces resonance zones locally in the phase-space which make the system chaotic in the region of the chaotic zones.

As the perturbation grows, these resonance zones grow and destroy the KAM surfaces around them. Overlap of two resonance zones destroys KAM surfaces. When all KAM surfaces are destroyed, chaos starts.

Example 1: Duffing Oscillator: Reichl and Zheng (1987) has discussed somewhat in detail about the transition from regular to chaotic behaviour in the conservative Duffing Oscillator.

They have analysed the phenomenon of resonance overlap using renormalization group methods and have derived and discussed mappings which contain the essential features of the passage from regular to chaotic behaviour in local regions of the phase-space.

Example 2: Chaos in the Solar System: The chaotic behaviour in the Solar system has been reviewed by Wisdom (1987). We may state here, briefly some aspects of chaotic behaviour in the Solar system.

1. Saturn satellite hyperion is currently trumbling chaotically, its rotation rate and spin axis orientation undergoes significant chaotic variations on a time scale of only a couple of orbit periods (Wisdom, Peale, Mignard, 1984).

2. Chaotic orbital evolution seems to be an essential ingredient in the explanation of the Kirkwood gaps in the distribution of asteroids. The predicted boundary of the Kirkwood gap is in close agreement with the observed population of asteroids (Wisdom, 1985).

Henrard (1988) has also stated that if Wisdom's effect can explain the 3/1 gap just as well as resonance sweeping does, can it do the same for 2/1 gap? This is not evident according to Henrard.

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