Canad. Math. Bull. Vol. **48** (2), 2005 pp. 260–266

A Restriction Theorem for a k-Surface in $\mathbb{R}^n$

Daniel M. Oberlin

Abstract. We establish a sharp Fourier restriction estimate for a measure on a k-surface in $\mathbb{R}^n$, where $n = k(k+3)/2$.

Fix a positive integer k and let $n = k(k+3)/2$. If $x \in \mathbb{R}^k$, write $x = (x_1, \dots, x_k)$ and define $\phi : \mathbb{R}^k \to \mathbb{R}^n$ by

$$\phi(x) = (x_1, \dots, x_k, x_1^2, \dots, x_k^2, x_1x_2, \dots, x_1x_k, x_2x_3, \dots, x_2x_k, \dots, x_{k-1}x_k).$$

Write S for the k-surface in $\mathbb{R}^n$ which is the range of ϕ and let σ be the measure induced on S by Lebesgue measure on $\mathbb{R}^k$. We are interested in the operator R^* taking functions $f \in C_c^\infty(S)$ to functions on $\mathbb{R}^n$ which is given by

$$R^*(f)(\xi) = \widehat{f d\sigma}(\xi).$$

The operator R^* is the adjoint of the Fourier restriction operator associated with the surface S and the measure σ. The natural problem is to determine the indices $p, q \in [1, \infty]$ such that there is an *a priori* estimate

$$\|R^* f\|_{L^q(\mathbb{R}^n)} \le C(p,q)\|f\|_{L^p(\sigma)}. \tag{1}$$

There is also the analogous problem for the localized operator R_0^* given by

$$R_0^*(f)(\xi) = \widehat{f\psi d\sigma}(\xi)$$

where ψ is fixed in $C_c^\infty(S)$. For $k = 1$ these operators are associated with a parabola in $\mathbb{R}^2$. Their mapping properties are well understood and are analogous to those of the corresponding operator associated with the circle. For $k \ge 2$ the first result is due to Christ [C2], who obtained estimates for R_0^* when $p = 2$. Mockenhaupt [M1, M2] extended Christ's results to the cases $k \ge 3$. De Carli and Iosevich [CI] obtained a sharp L^2 result. Bak and Lee [BL] adapted Mockenhaupt's method to obtain the following nearly sharp result. (Their paper also contains a more detailed history of these problems.)

Received by the editors January 21, 2003; revised June 10, 2004.
The author was partially supported by the NSF.
AMS subject classification: 42B10.
Keywords: Fourier restriction.

https://doi.org/10.4153/CMB-2005-024-9 Published online by Cambridge University Press

Theorem 1 ([BL]) *If $k = 2$ or 3 then R^* is bounded from $L^p(S)$ to $L^q(\mathbb{R}^n)$ if and only if $\frac{1}{p} + \frac{k+2}{q} = 1$, $q > 2(k+1)$. If $k \geq 4$, then R_0^* is bounded from $L^p(S)$ to $L^q(\mathbb{R}^n)$ if $\frac{1}{p} + \frac{k+2}{q} < 1$, $q > 2(k+1)$, and R_0^* is unbounded from $L^p(S)$ to $L^q(\mathbb{R}^n)$ if $q \leq 2(k+1)$ or $\frac{1}{p} + \frac{k+2}{q} > 1$.*

The purpose of this note is to present a slight improvement on Theorem 1:

Theorem 2 *For $k \geq 2$, the operator R^* is bounded from $L^p(S)$ to $L^q(\mathbb{R}^n)$ if and only if $\frac{1}{p} + \frac{k+2}{q} = 1$, $q > 2(k+1)$.*

Quoting Christ [C2, p. 224]: "The strategy of our proof is not new: following Prestini [P], we utilize an argument originating in Fefferman [F] and Carleson and Sjölin [CS], based on a change of variables and the Hausdorff-Young inequality, to reduce (1) to an easier problem concerning estimates for positive integral operators." The same strategy is utilized in [BL]. The proof of Theorem 2 is a bit simpler than that of Theorem 1, depending on a change of variables different from that in [BL].

Proof of Theorem 2 As the necessity of the condition $\frac{1}{p} + \frac{k+2}{q} = 1$, $q > 2(k+1)$ is already established in [BL], it is enough to show the other implication.

We adopt the convention that C denotes a positive constant which may depend only on the relevant dimensions and/or indices. Writing $\|\cdot\|_r$ for $\|\cdot\|_{L^r(\mathbb{R}^n)}$, the Hausdorff–Young inequality shows that it is enough to prove the inequality

$$\|(f d\sigma) * \cdots * (f d\sigma)\|_{(\frac{q}{k+1})'} \leq C\, \|f\|^{k+1}_{L^p(S)},$$

where the convolution is $(k+1)$-fold. This is equivalent to the inequality

$$\int_{(\mathbb{R}^k)^{k+1}} \prod_{l=1}^{k+1} f(x^l) h\big(\phi(x^1) + \cdots + \phi(x^{k+1})\big) \leq C\, \|f\|^{k+1}_{L^p(\mathbb{R}^k)} \|h\|_{\frac{q}{k+1}} \tag{2}$$

for functions f on $\mathbb{R}^k$. For $j = 1, \ldots, k$, write $v_j = (x_j^1, \ldots, x_j^{(k+1)})$ and let d be the $(k+1)$-vector $(1, \ldots, 1)$. For fixed $v_2, \ldots, v_k$, define

$$\Phi(x_1^1, \ldots, x_1^{(k+1)}) = (v_1 \cdot d, |v_1|^2, v_1 \cdot v_2, \ldots, v_1 \cdot v_k).$$

Then the Jacobian J of Φ is the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 2x_1^1 & 2x_1^2 & \ldots & 2x_1^{(k+1)} \\ x_2^1 & x_2^2 & \ldots & x_2^{(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^1 & x_k^2 & \ldots & x_k^{(k+1)} \end{bmatrix}.$$

For $1 \leq s < 2$ we will estimate (2) by the product of

$$\Big(\int_{(\mathbb{R}^k)^{k+1}} h^{s'}\big(\phi(x^1) + \cdots + \phi(x^{k+1})\big) \cdot J\, dx^1 \cdots x^{k+1}\Big)^{1/s'} \tag{3}$$

https://doi.org/10.4153/CMB-2005-024-9 Published online by Cambridge University Press

and

$$\Big(\int_{(\mathbb{R}^k)^{k+1}} \prod_{l=1}^{k+1} f^s(x^l)\cdot J^{-s/s'}\,dx^1\cdots x^{k+1}\Big)^{1/s}. \tag{4}$$

Beginning with (3), write

$$\begin{aligned} h\big(\phi(x^1)+\cdots+\phi(x^{k+1})\big) \\ &= h\Big(\Phi(x_1^1,\dots,x_1^{(k+1)}), v_2\cdot d,\dots,v_k\cdot d, \\ &\qquad |v_2|^2,\dots,|v_k|^2, v_2\cdot v_3, v_2\cdot v_4,\dots,v_{k-1}\cdot v_k\Big) \\ &\doteq h\big(\Phi(x_1^1,\dots,x_1^{(k+1)}),\Psi(v)\big)\end{aligned}$$

where $v=(v_2,\dots,v_k)$. Thus $(3)^{s'}$ is

$$\int_{(\mathbb{R}^{k+1})^{k-1}}\int_{\mathbb{R}^{k+1}} h^{s'}\big(\Phi(x_1^1,\dots,x_1^{(k+1)}),\Psi(v)\big)\cdot J\,dx_1^1\cdots x_1^{(k+1)}dv.$$

The map Φ has multiplicity at most 2 for almost all v. For such v

$$\int_{\mathbb{R}^{k+1}} h^{s'}\big(\Phi(x_1^1,\dots,x_1^{(k+1)}),\Psi(v)\big)\cdot J\,dx_1^1\cdots x_1^{(k+1)} \le 2\int_{\mathbb{R}^{k+1}} h^{s'}\big(y,\Psi(v)\big)\,dy$$

and so it follows that $(3)^{s'}$ is bounded by

$$2\int_{(\mathbb{R}^{k+1})^{k-1}}\int_{\mathbb{R}^{k+1}} h^{s'}\big(y,\Psi(v)\big)\,dydv. \tag{5}$$

To bound (5) (by $C\|h\|_{s'}^{s'}$), recall that

$$\Psi(v_2,\dots,v_k)=(v_2\cdot d,\dots,v_k\cdot d,|v_2|^2,\dots,|v_k|^2,v_2\cdot v_3,v_2\cdot v_4,\dots,v_{k-1}\cdot v_k)$$

where $d=(1,1,\dots,1)$. We write $d'=d/\sqrt{k+1}$ and $v_j=d_jd'+c_j$ with $c_j\perp d'$. Then if g is a function on $\mathbb{R}^{(k-1)(k+2)/2}$ we have

$$\begin{aligned} &\int_{(\mathbb{R}^{k+1})^{k-1}} g\big(\Psi(v_2,\dots,v_k)\big)\,dv_2\cdots dv_k \\ &= \int_{\mathbb{R}^{k-1}}\int_{(\mathbb{R}^k)^{k-1}} g\Big(\sqrt{k+1}d_2,\dots,\sqrt{k+1}d_k,(d_2)^2+|c_2|^2,\dots,(d_k)^2+|c_k|^2, \\ &\qquad d_2d_3+c_2\cdot c_3, d_2d_4+c_2\cdot c_4,\dots,d_{k-1}d_k+c_{k-1}\cdot c_k\Big)\,d\,c_2\cdots c_k\,d\,d_2\cdots d_k.\end{aligned} \tag{6}$$

We require a lemma (whose proof is postponed until after the main argument).

https://doi.org/10.4153/CMB-2005-024-9 Published online by Cambridge University Press

Lemma 3 *The inequality*

$$\int_{\mathbb{R}^{k(k-1)}} \alpha\big(|c_2|^2, \dots, |c_k|^2, c_2 \cdot c_3, c_2 \cdot c_4, \dots, c_{k-1} \cdot c_k\big)\, dc_2 \cdots c_k \le C \int_{\mathbb{R}^{k(k-1)/2}} \alpha(x)\, dx$$

holds for nonnegative Borel functions α *on* $\mathbb{R}^{k(k-1)/2}$.

An application of this lemma to (6) now shows that

$$\int_{(\mathbb{R}^{k+1})^{k-1}} g\big(\Psi(v_2, \dots, v_k)\big)\, dv_2 \cdots dv_k \le C \int_{\mathbb{R}^{(k-1)(k+2)/2}} g(z)\, dz$$

(if g is a function on $\mathbb{R}^{(k-1)(k+2)/2}$). Applying this to (5) shows that

$$\Big(\int_{(\mathbb{R}^k)^{k+1}} h^{s'}\big(\phi(x^1) + \cdots + \phi(x^{k+1})\big) \cdot J\, dx^1 \cdots x^{k+1}\Big)^{1/s'} \le C\, \|h\|_{s'}, \tag{7}$$

completing the process of bounding (3).

To bound (4) we will need another lemma. Recall that $x^l = (x^l_1, \dots, x^l_k)$ and write D for the absolute value of the determinant of

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1^1 & x_1^2 & \dots & x_1^{(k+1)} \\ x_2^1 & x_2^2 & \dots & x_2^{(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_k^1 & x_k^2 & \dots & x_k^{(k+1)} \end{bmatrix}.$$

Lemma 4 ([C1, Theorem B]) *Suppose* $0 < \gamma < 1$, $1 \le r < 2$, *and* $\frac{1}{r} = 1 - \frac{\gamma}{k+1}$. *Then the inequality*

$$\int_{(\mathbb{R}^k)^{k+1}} \prod_1^{k+1} f_l(x^l) D^{-\gamma} dx^1 \cdots x^{k+1} \le C \prod_1^{k+1} \|f_l\|_{L^r(\mathbb{R}^k)}$$

holds for nonnegative Borel functions f_l.

An application of Lemma 4 to the integral in (4) bounds (4) by $C\, \|f\|^{k+1}_{L^{rs}(\mathbb{R}^k)}$ so long as the r defined by $\frac{1}{r} = 1 - \frac{s}{s'}\frac{1}{k+1}$ satisfies $1 \le r < 2$ (which follows from $1 \le s < 2$). That is, (4) is bounded by $C\|f\|^{k+1}_{L^p(\mathbb{R}^k)}$ where $p = rs$ and so $\frac{1}{p} = (\frac{k+2}{s} - 1)\frac{1}{k+1}$ (since $\frac{1}{r} = \frac{k+2-s}{k+1}$). With $q = s'(k+1)$, it follows that $\frac{1}{p} + \frac{k+2}{q} = 1$. Thus the bound (7) for (3) now yields (2) whenever $\frac{1}{p} + \frac{k+2}{q} = 1$ and $q > 2(k+1)$.

Proof of Lemma 3 The proof depends on a particular parametrization of $(\mathbb{R}^k)^{k-1}$. We begin by introducing notation.

Write e_j for the jth standard unit vector in $\mathbb{R}^k$. Fix a Borel mapping

$$\omega_{k-1} \mapsto O(\omega_{k-1})$$

of the unit sphere Σ_{k-1} in $\mathbb{R}^k$ into the orthogonal group on $\mathbb{R}^k$ in such a way that $O(\omega_{k-1})e_1 = \omega_{k-1}$. For $\omega_{k-1} \in \Sigma_{k-1}$, set

$$\omega_{k-1}^{'} = O(\omega_{k-1})e_2 \in \{\omega_{k-1}\}^{\perp} \cap \Sigma_{k-1},$$

realize Σ_{k-2} as $\{\omega_{k-1}\}^{\perp} \cup \Sigma_{k-1}$, and, as above, for $\omega_{k-2} \in \Sigma_{k-2}$, let $O(\omega_{k-1}, \omega_{k-2})$ be an orthogonal map on $\mathbb{R}^k$ which fixes ω_{k-1} and takes $\omega_{k-1}^{'}$ to ω_{k-2}. Then

$$O(\omega_{k-1}, \omega_{k-2})O(\omega_{k-1})$$

takes e_1 to ω_{k-1} and e_2 to ω_{k-2}. For such $\omega_{k-1}, \omega_{k-2}$ set

$$\omega_{k-2}^{'} = O(\omega_{k-1}, \omega_{k-2})O(\omega_{k-1})e_3 \in \{\omega_{k-1}, \omega_{k-2}\}^{\perp} \cap \Sigma_{k-1},$$

realize Σ_{k-3} as $\{\omega_{k-1}, \omega_{k-2}\}^{\perp} \cap \Sigma_{k-1}$, and, for $\omega_{k-3} \in \Sigma_{k-3}$, let $O(\omega_{k-1}, \omega_{k-2}, \omega_{k-3})$ be an orthogonal map on $\mathbb{R}^k$ which fixes $\omega_{k-1}, \omega_{k-2}$ and takes $\omega_{k-2}^{'}$ to ω_{k-3}. Continue this way until $O(\omega_{k-1}, \dots, \omega_1)$ is defined. Write $\omega = (\omega_1, \dots, \omega_{k-1})$ and

$$O(\omega) = O(\omega_{k-1}, \dots, \omega_1)O(\omega_{k-1}, \dots, \omega_2) \cdots O(\omega_{k-1}).$$

The notation $d\omega_1 \cdots \omega_{k-1}$ will represent integration with respect to the product of the surface area measures on (the realizations) of the spheres $\Sigma_1, \dots, \Sigma_{k-1}$.

For $\theta_i \in [0, \pi]$ define

$$\sigma_1(\theta_1) = (\cos\theta_1, \sin\theta_1) \in \Sigma_1$$

and

$$\sigma_j(\theta_1, \dots, \theta_j) = \big(\cos\theta_1; \sin\theta_1 \sigma_{j-1}(\theta_2, \dots, \theta_j)\big) \in \Sigma_j.$$

For $j = 1, \dots, k-2$, the notation $\big(\sigma_j(\theta_1, \dots, \theta_j), 0\big)$ stands for the k-vector obtained by following $\sigma_j(\theta_1, \dots, \theta_j)$ with $(k - j - 1)$ 0's.

The parametrization of $(\mathbb{R}^k)^{k-1}$ is now

$$\big(c_0; c_1; \dots; c_{k-2}\big) = \\ \big(\rho_0 O(\omega)e_1; \rho_1 O(\omega)(\sigma_1(\theta_1^1), 0); \dots; \rho_{k-2} O(\omega)(\sigma_{k-2}(\theta_1^{k-2}, \dots, \theta_{k-2}^{k-2}), 0)\big),$$

where the ρ_j's are positive. The volume element which corresponds to Lebesgue measure $dc_0 \cdots c_{k-2}$ on $(\mathbb{R}^k)^{k-1}$ is

$$\prod_0^{k-2} \rho_j^{k-1} d\rho_0 \cdots \rho_{k-2}\, d\omega_1 \cdots \omega_{k-1} \prod_1^{k-2} (\sin\theta_1^j)^{k-2} \\ \times \prod_2^{k-2} (\sin\theta_2^j)^{k-3} \cdots \prod_{k-2}^{k-2} (\sin\theta_{k-2}^j) d\theta_1^1 \theta_1^2 \theta_2^2 \cdots \theta_1^{k-2} \cdots \theta_{k-2}^{k-2}.$$

https://doi.org/10.4153/CMB-2005-024-9 Published online by Cambridge University Press

The ranges of integration for the ρ and θ variables are $(0, \infty)$ and $[0, \pi]$, respectively. (The ranges for the ω's were described above.) For example, in the case $k = 4$ we have

$$c_0 = \rho_0\omega_3, \quad c_1 = \rho_1(\cos\theta_1^1\omega_3 + \sin\theta_1^1\omega_2),$$
$$c_2 = \rho_2(\cos\theta_1^2\omega_3 + \sin\theta_1^2\cos\theta_2^2\omega_2 + \sin\theta_1^2\sin\theta_2^2\omega_1).$$

The volume element can be written

$$\rho_2^3(\sin\theta_1^2)^2\sin\theta_2^2 d\rho_2 d\theta_1^2\theta_2^2 d\omega_1 \cdot \rho_1^3(\sin\theta_1^1)^2 d\rho_1 d\theta_1^1 d\omega_2 \cdot \rho_0^2 d\rho_0 d\omega_3.$$

For fixed $\omega_3, \rho_0, \omega_2, \theta_1^1$, and ρ_1, dc_2 is $\rho_2^3(\sin\theta_1^2)^2\sin\theta_2^2 d\rho_2 d\theta_1^2\theta_2^2 d\omega_1$ since $d\omega_1$ gives "surface area" on $\{\omega_3, \omega_2\}^\perp \cap \Sigma_3$. And for fixed ω_3 and ρ_0, dc_1 is $\rho_1^3(\sin\theta_1^1)^2 d\rho_1 d\theta_1^1 d\omega_2$ since $d\omega_2$ gives surface area on $\{\omega_3\}^\perp \cap \Sigma_3$. Finally, dc_0 is $\rho_0^3 d\rho_0 d\omega_3$.

Lemma 3 is the statement that

$$\int_{\mathbb{R}^{k(k-1)}} \alpha\big(|c_0|^2, \dots, |c_{k-2}|^2, c_0\cdot c_1, c_0\cdot c_2, \dots, c_{k-3}\cdot c_{k-2}\big)\, dc_0 \cdots c_{k-2} \leq C\int_{\mathbb{R}^{k(k-1)/2}} \alpha(x)\, dx.$$

Since the orthogonal mappings $O(\omega)$ have no effect on the inner products $c_i \cdot c_j$, we define

$$\big(c'_0; c'_1; \dots; c'_{k-2}\big) = \Big(\rho_0 e_1; \rho_1\big(\sigma_1(\theta_1^1), 0\big); \dots; \rho_{k-2}\big(\sigma_{k-2}(\theta_1^{k-2}, \dots, \theta_{k-2}^{k-2}), 0\big)\Big).$$

Lemma 3 then follows by observing that

(8)
$$\int \alpha\big(|c'_0|^2, \dots, |c'_{k-2}|^2, c'_0\cdot c'_1, c'_0\cdot c'_2, \dots, c'_{k-3}\cdot c'_{k-2}\big)\prod_0^{k-2}\rho_j^{k-1} d\rho_0\cdots\rho_{k-2}$$
$$\times\prod_1^{k-2}(\sin\theta_1^j)^{k-2}\prod_2^{k-2}(\sin\theta_2^j)^{k-3}\cdots\prod_{k-2}^{k-2}(\sin\theta_{k-2}^j) d\theta_1^1\theta_1^2\theta_2^2\cdots\theta_1^{k-2}\cdots\theta_{k-2}^{k-2}$$
$$\leq C\int_{\mathbb{R}^{k(k-1)/2}} \alpha(x) dx.$$

We will explain this in the case $k = 4$, the general case being completely analogous. If $k = 4$ then

$$\big(|c'_0|^2, |c'_1|^2, |c'_2|^2, c'_0\cdot c'_1, c'_0\cdot c'_2, c'_1\cdot c'_2\big)$$
$$= \big(\rho_0^2, \rho_1^2, \rho_2^2, \rho_0\rho_1\cos\theta_1^1, \rho_0\rho_2\cos\theta_1^2, \rho_1\rho_2(\cos\theta_1^1\cos\theta_1^2 + \sin\theta_1^1\sin\theta_1^2\cos\theta_2^2)\big),$$

while the volume element can be written

$$\rho_1\rho_2\sin\theta_1^1\sin\theta_1^2\sin\theta_2^2 d\theta_2^2\cdot\rho_0\rho_2\sin\theta_1^2 d\theta_1^2\cdot\rho_0\rho_1\sin\theta_1^1 d\theta_1^1\cdot\rho_2 d\rho_2\cdot\rho_1 d\rho_1\cdot\rho_0 d\rho_0.$$

Thus (8) is evident.

https://doi.org/10.4153/CMB-2005-024-9 Published online by Cambridge University Press

Acknowledgement The author is grateful to Jong-Guk Bak for interesting and helpful correspondence. In particular, the author was in possession of a proof of Theorem 2 for the case $k = 2$ when he received the preprint [BL], and he appreciates Bak's encouraging him to extend that argument to $k \geq 3$.

References

[BL] J.-G. Bak and S. Lee, *Restriction of the Fourier transform to a quadratic surface in* $\mathbb{R}^n$. Math. Z. **247**(2004), 409–422.

[C1] M. Christ, *Estimates for the k-plane transform.* Indiana Univ. Math. J. **33**(1984), 891–910.

[C2] ———, *On the restriction of the Fourier transform to curves: endpoint results and the degenerate case.* Trans. Amer. Math. Soc. **287**(1985), 223–238.

[CI] L. de Carli and A. Iosevich, *Some sharp restriction theorems for homogeneous manifolds.* J. Fourier Analysis and Applications **4**(1998), 105–128.

[CS] L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc.* Studia Math. **44**(1972), 287–299.

[F] C. Fefferman, *Inequalities for strongly singular convolution operators.* Acta Math. **124**(1970), 9–36.

[M1] G. Mockenhaupt, *Bounds in Lebesgue spaces of oscillatory integral operators.* Habilitation thesis, Universität Siegen, 1996.

[M2] ———, *Some remarks on oscillatory integrals.* In: Geometric Analysis and Applications. Proc. Centre Math. Appl. Austral. Nat. Univ. 39, Canberra, 2001.

[M2] E. Prestini, *Restriction theorems for the Fourier transform to some manifolds in* $\mathbb{R}^n$. In: Harmonic analysis in euclidean spaces. Proc. Sympos. Pure Math. 35, American Mathematics Society, Providence, RI, 1979, pp. 101–109.

Department of Mathematics
Florida State University
Tallahassee, FL 32306-4510
U.S.A.
e-mail: oberlin@math.fsu.edu

https://doi.org/10.4153/CMB-2005-024-9 Published online by Cambridge University Press