11
Covariant analysis

Consider the scattering of a Dirac electron in an external field created by an electromagnetic transition current density in a hadronic target. We assume to start with that the target makes a transition from the ground state to some discrete excited state. The external vector potential $A_\mu^{\text{ext}}(\mathbf{x}, t)$ can be related to that current density through the use of Maxwell's equations[1]

$$\left(\frac{\partial}{\partial x_\nu}\right)^2 A_\mu^{\text{ext}}(\mathbf{x}, t) = -e_p \langle f|\hat{J}_\mu(\mathbf{x}, t)|i\rangle \tag{11.1}$$

Here $|i\rangle$ and $|f\rangle$ are exact Heisenberg eigenstates of the target with energies E, E' respectively. It follows that the time dependence of the target matrix element can be extracted as

$$\left(\frac{\partial}{\partial x_\nu}\right)^2 A_\mu^{\text{ext}}(\mathbf{x}, t) = -e_p \langle f|\hat{J}_\mu(\mathbf{x})|i\rangle e^{-i(E-E')t} \tag{11.2}$$

The states can similarly be taken as eigenstates of four-momentum $p_\mu = (\mathbf{p}, iE)$, so the entire space-time dependence can be extracted as

$$\left(\frac{\partial}{\partial x_\nu}\right)^2 A_\mu^{\text{ext}}(\mathbf{x}, t) = -e_p \langle f|\hat{J}_\mu(0)|i\rangle e^{i(p-p')\cdot x} \tag{11.3}$$

First-order time-dependent perturbation theory and the interaction of Eq. (10.24) lead to the scattering operator

$$\hat{S} \doteq -i \int \hat{\mathscr{H}}_I^{(1)}(\mathbf{x}, t)\, d^4x \tag{11.4}$$

[1] These are Maxwell's equations in the Lorentz gauge for the external field where, by current conservation, $\partial A_\mu^{\text{ext}}/\partial x_\mu = 0$.

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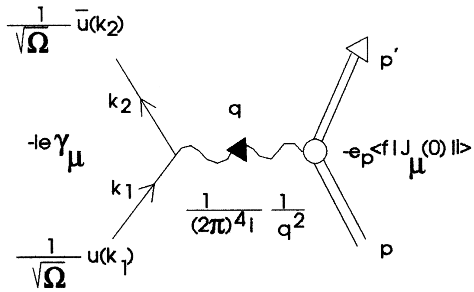

Fig. 11.1. Feynman diagram for electron scattering from a hadronic target.

Here the interaction representation for the electrons has been introduced and they carry the free field time dependence $\exp(\pm ik\cdot x)$. Now take matrix elements of the scattering operator between appropriate initial and final electron states

$$\begin{aligned}\langle \mathbf{k}_2, s_2|\hat{S}|\mathbf{k}_1, s_1\rangle &= -\frac{e}{\sqrt{\Omega^2}}\bar{u}(\mathbf{k}_2, s_2)\gamma_\mu u(\mathbf{k}_1, s_1)\int e^{-iq\cdot x}A_\mu^{\text{ext}}(x)\,d^4x \\ q &\equiv k_2 - k_1 \end{aligned} \tag{11.5}$$

If one proceeds directly to the continuum limit, what is required is the four-dimensional Fourier transform of the external field

$$\tilde{A}_\mu^{\text{ext}}(q) = \int e^{-iq\cdot x}A_\mu^{\text{ext}}(x)\,d^4x \tag{11.6}$$

The Fourier transform is inverted with the relation

$$A_\mu^{\text{ext}}(x) = \int e^{iq\cdot x}\tilde{A}_\mu^{\text{ext}}(q)\,\frac{d^4q}{(2\pi)^4} \tag{11.7}$$

Substitute Eq. (11.7) on the left hand side of Eq. (11.3). The right hand side is then reproduced if one chooses

$$-q^2\tilde{A}_\mu^{\text{ext}}(q) = -e_p\langle f|\hat{J}_\mu(0)|i\rangle(2\pi)^4\delta^{(4)}(p-p'-q) \tag{11.8}$$

The required S-matrix thus takes the form

$$\begin{aligned}\langle f|\hat{S}|i\rangle &= -\frac{ee_p}{\sqrt{\Omega^2}}\bar{u}(k_2)\gamma_\mu u(k_1)\frac{1}{q^2}\langle f|\hat{J}_\mu(0)|i\rangle(2\pi)^4\delta^{(4)}(p-p'-q) \\ q &= k_2 - k_1 = p - p'\end{aligned} \tag{11.9}$$

The spin quantum numbers for the electron have been suppressed.

The amplitude in Eq. (11.9) can be represented as a *Feynman diagram* as shown in Fig. 11.1. There is a corresponding set of *Feynman rules* for the S-matrix:

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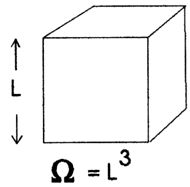

Fig. 11.2. Quantization volume.

1. Include a factor of $(-i)$ for each order of perturbation theory; here second order;

2. Include a factor of $(-eJ_\mu)$ for each vertex; here

   - $-ie\gamma_\mu$; for electron vertex
   - $-e_p\langle f|\hat{J}_\nu(0)|i\rangle$; for hadronic vertex (lowest order)

3. Include factors of $u(k_1)/\sqrt{\Omega}$ and $\bar{u}(k_2)/\sqrt{\Omega}$ for the initial and final electron legs;

4. Include the following factor for the virtual photon propagator

$$\frac{1}{(2\pi)^4 i}\frac{1}{q^2}\delta_{\mu\nu} \tag{11.10}$$

   Since both the electron and target currents are conserved, one could just as well use the following expression for the photon propagator

$$\frac{1}{(2\pi)^4 i}\frac{1}{q^2}\left[\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}(1-\bar{\alpha})\right] \tag{11.11}$$

   The extra term in $q_\mu q_\nu$ vanishes in the S-matrix element (see below). Here different choices of $\bar{\alpha}$ correspond to different gauges for the internal vector potential;

5. Include a factor $(2\pi)^4\delta^{(4)}(\sum_i p_i)$ at each vertex;

6. Integrate $\int d^4p$ over internal lines.

The factors in the above diagram can be checked according to $(-i)^2(-ie) \times(-e_p)(2\pi)^8/(2\pi)^4 i\sqrt{\Omega^2} = -ee_p(2\pi)^4/\sqrt{\Omega^2}$.

As indicated previously, we choose to quantize in a big box of volume Ω (Fig. 11.2) and in the end let $\Omega \to \infty$. This fictitious volume must

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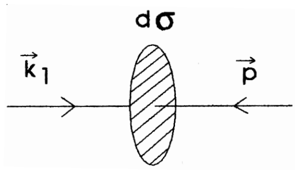

Fig. 11.3. Flux and cross section in any frame obtained by a Lorentz transformation along the incident electron direction.

disappear from any physical result. The end term in Eqn. (11.9) should really be written in the form

$$(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}'-\mathbf{q}) = \int_{\text{box}} e^{i(\mathbf{p}-\mathbf{p}'-\mathbf{q})\cdot\mathbf{x}}\, d^3x = \Omega\,\delta_{\mathbf{p}',\mathbf{p}-\mathbf{q}} \tag{11.12}$$

Thus the S-matrix for the problem at hand actually takes the form

$$\begin{aligned}\langle f|\hat{S}|i\rangle &\equiv -2\pi i\delta(W_f - W_i)T_{fi} \\ T_{fi} &= -iee_p\bar{u}(k_2)\gamma_\mu u(k_1)\frac{1}{q^2}\langle f|\hat{J}_\mu(0)|i\rangle\,\delta_{\mathbf{p}',\mathbf{p}-\mathbf{q}}\end{aligned} \tag{11.13}$$

Here W_f, W_i are the final and initial total energies. The cross section can now be evaluated immediately with the aid of the *Golden Rule*

$$d\sigma = \frac{R_{fi}}{\text{Flux}} = 2\pi|T_{fi}|^2\delta(W_f - W_i)d\rho_f\,\frac{1}{\text{Flux}} \tag{11.14}$$

It follows immediately that

$$[\delta_{\mathbf{p}',\mathbf{p}-\mathbf{q}}]^2 = \delta_{\mathbf{p}',\mathbf{p}-\mathbf{q}} \tag{11.15}$$

Only one of the momenta in the final state is independent. The number of states for the final electron in a big box with periodic boundary conditions is

$$d\rho_f = \frac{\Omega}{(2\pi)^3}d^3k_2 \tag{11.16}$$

Consider the incident flux in any frame obtained by a Lorentz transformation along the incident electron direction as shown in Fig. 11.3. The incident flux is defined by v_{rel}/Ω and is given by

$$\begin{aligned}\text{Flux} &= \frac{1}{\Omega}v_{\text{rel}} = \frac{1}{\Omega}\left(\frac{k_1}{\varepsilon_1}+\frac{p}{E}\right) \\ &= \frac{1}{\Omega\varepsilon_1 E}(\varepsilon_1 p + k_1 E) \\ &= \frac{1}{\Omega\varepsilon_1 E}\sqrt{(k_1\cdot p)^2}\end{aligned} \tag{11.17}$$

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The last relation follows for $\mathbf{k}_1$ antiparallel to $\mathbf{p}$ and massless electrons since then $k_1 \cdot p = -k_1 p - \varepsilon_1 E = -(k_1 E + \varepsilon_1 p)$. The reader can verify that the same result holds if $\mathbf{k_1} \| \mathbf{p}$, which we now use to generically identify this case. A combination of the above results then leads to

$$\begin{aligned} d\sigma &= 2\pi \frac{e^2 e_p^2}{q^4} |\bar{u}(k_2)\gamma_\mu u(k_1)\langle f|\hat{J}_\mu(0)|i\rangle|^2 \delta(W_f - W_i) \frac{\Omega\, d^3 k_2}{(2\pi)^3} \\ & \quad \times \delta_{\mathbf{p}',\mathbf{p}-\mathbf{q}} \frac{\Omega \varepsilon_1 E}{\sqrt{(k_1 \cdot p)^2}} \end{aligned} \tag{11.18}$$

If the electron beam is unpolarized and its final polarization unmeasured, one must average over initial electron spins and sum over final spins.[2] If the target is unpolarized and unobserved, one must average over initial target states and sum over final states. The cross section thus takes the form (recall $e^2/4\pi = \alpha$, the fine-structure constant)

$$\begin{aligned} d\sigma &= \frac{4\alpha^2}{q^4} \left(\frac{d^3 k_2}{2\varepsilon_2} \right) \frac{1}{\sqrt{(k_1 \cdot p)^2}} (2\varepsilon_1 \varepsilon_2) \frac{1}{2} \sum_{s_1} \sum_{s_2} \overline{\sum_{i}} \sum_{f} \\ & \quad \times |\bar{u}(k_2)\gamma_\mu u(k_1)\langle f|\hat{J}_\mu(0)|i\rangle|^2 (\Omega E)(2\pi)^3 \delta^{(4)}(p - p' - q) \end{aligned} \tag{11.19}$$

Here Eqn. (11.12) has again been employed. The product of the matrix element and its complex conjugate can now be written out. In taking the complex conjugate, one must use $\{\gamma_4, \gamma_i\} = 0$ to restore the gamma matrices to the proper order and also remember that $\hat{J}_\mu = (\hat{\mathbf{J}}, i\hat{\rho})$ has an imaginary fourth component. We therefore arrive at the final, important result

$$\begin{aligned} d\sigma &= \frac{4\alpha^2}{q^4} \left(\frac{d^3 k_2}{2\varepsilon_2} \right) \frac{1}{\sqrt{(k_1 \cdot p)^2}} \eta_{\mu\nu} W_{\mu\nu} \qquad (11.20) \\ \eta_{\mu\nu} &\equiv -2\varepsilon_1 \varepsilon_2 \frac{1}{2} \sum_{s_1} \sum_{s_2} \bar{u}(k_1)\gamma_\nu u(k_2) \bar{u}(k_2)\gamma_\mu u(k_1) \\ W_{\mu\nu} &= (2\pi)^3 \overline{\sum_{i}} \sum_{f} \langle i|\hat{J}_\nu(0)|f\rangle \langle f|\hat{J}_\mu(0)|i\rangle (\Omega E) \delta^{(4)}(p' - p + q) \end{aligned}$$

This expression represents the cross section in any frame where $\mathbf{k}_1 \| \mathbf{p}$. It is evident from Fig. 11.3 that $d\sigma$ is a small element of transverse area and, as such, is invariant under Lorentz transformations along the incident electron direction. The initial factors in this result are all Lorentz invariant. The quantity $\eta_{\mu\nu}$ transforms as a second rank tensor (see below). Hence

[2] We shall later relax these conditions.

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one concludes that $W_{\mu\nu}$ must also be a second rank tensor.[3] The right hand side of Eq. (11.20) is explicitly Lorentz invariant and can now be evaluated in any Lorentz frame; it represents the physical cross section in any frame where $\mathbf{k}_1 \| \mathbf{p}$.

If one were doing elastic scattering from a point Dirac particle, the matrix elements of the current would each be proportional to $1/\Omega$ and the final momentum would be determined by the use of Eq. (11.12). The quantity $W_{\mu\nu}$ would thus be independent of Ω and the quantization volume would then cancel from Eq. (11.20), as it must. This is in fact a general result, as we shall see in all our applications.

Although Eq. (11.20) has been derived under the assumption of a discrete final state of the target, the generalization to an arbitrary final state of the target, which might include the production of many particles, is now immediate. One simply calculates the appropriate inelastic matrix element of the current and then sums over the correct number of final states at given (p, q) in $W_{\mu\nu}$.

With the aid of the positive-energy projection operators for the Dirac equation in Eq. (10.20), the lepton response tensor can be evaluated for massless electrons as follows

$$\begin{aligned}
\eta_{\mu\nu} &\equiv -2\varepsilon_1\varepsilon_2\frac{1}{2}\sum_{s_1}\sum_{s_2}\bar{u}(k_1)\gamma_\nu u(k_2)\bar{u}(k_2)\gamma_\mu u(k_1) \\
&= -\varepsilon_1\varepsilon_2 \text{ trace}\left[\gamma_\nu\left(\frac{-i\gamma_\lambda k_{2\lambda}}{2\varepsilon_2}\right)\gamma_\mu\left(\frac{-i\gamma_\rho k_{1\rho}}{2\varepsilon_1}\right)\right] \\
&= \frac{1}{4}4\left[k_{2\nu}k_{1\mu} + k_{1\nu}k_{2\mu} - (k_1\cdot k_2)\delta_{\mu\nu}\right] \\
\eta_{\mu\nu} &= k_{2\nu}k_{1\mu} + k_{1\nu}k_{2\mu} - (k_1\cdot k_2)\delta_{\mu\nu} \qquad (11.21)
\end{aligned}$$

This is evidently a second rank Lorentz tensor, as advertised.

The target response tensor $W_{\mu\nu}$ is a second rank Lorentz tensor built out of the two remaining independent four-vectors p and q; everything else has been summed over. The electromagnetic current is conserved. With the aid of the Heisenberg equations of motion, one concludes that

$$\begin{aligned}
\frac{\partial}{\partial x_\mu}\langle f|\hat{J}_\mu(x)|i\rangle &= e^{i(p-p')\cdot x}\, i(p-p')_\mu\langle f|\hat{J}_\mu(0)|i\rangle = 0 \\
q_\mu\langle f|\hat{J}_\mu(0)|i\rangle &= 0 \qquad (11.22)
\end{aligned}$$

Hence current conservation for the target implies

$$q_\mu W_{\mu\nu} = W_{\mu\nu}q_\nu = 0 \qquad (11.23)$$

[3] This can be proven directly from the Lorentz transformation properties of the states, but the argument is more involved.

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The Dirac equation for the massless electrons implies that

$$\bar{u}(k_2)\gamma_\lambda q_\lambda u(k_1) = \bar{u}(k_2)(\gamma_\lambda k_{2\lambda} - \gamma_\lambda k_{1\lambda})u(k_1) = 0 \tag{11.24}$$

It follows that the lepton response tensor in Eq. (11.21) obeys the same conditions

$$q_\mu \eta_{\mu\nu} = \eta_{\mu\nu} q_\nu = 0 \tag{11.25}$$

The two independent Lorentz scalars that can be constructed from p and q are q^2 and $q \cdot p$. Recall $p^2 = -M_T^2$ is fixed by the target mass. In the laboratory frame, for massless electrons, $q^2 = (\mathbf{k}_2 - \mathbf{k}_1)^2 - (k_2 - k_1)^2 = 2k_1k_2(1 - \cos\theta)$ where θ is the scattering angle. Furthermore, the target is at rest in that frame so $p = (\mathbf{0}, iM_T)$. Hence one can identify these scalers in the laboratory frame according to

$$\begin{aligned} q^2 &= 4k_1k_2 \sin^2\frac{\theta}{2} \qquad ;\text{laboratory frame} \\ q \cdot p &= -q_0 M_T \end{aligned} \tag{11.26}$$

The conditions in Eq. (11.23) then imply that the target response tensor must have the following form

$$\begin{aligned} W_{\mu\nu} &= W_1(q^2, q\cdot p)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \\ &\quad + W_2(q^2, q\cdot p)\frac{1}{M_T^2}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right) \end{aligned} \tag{11.27}$$

This result is due to Bjorken [Bj60], Von Gehlen [Vo60], and Gourdin [Go61]. It forms the basis for the subsequent analysis. It makes use only of Lorentz covariance and current conservation, and it holds for *any* hadronic target, independent of its internal structure. Note that it is $\overline{\sum_i}\sum_f$ that yields the simplicity of the form in Eq. (11.27). Upon substitution of this expression, the cross section in Eq. (11.20) is then exact to lowest order in α. We proceed to the proof of this important result.

Write the most general tensor[4] one can make out of the four-vectors p and q

$$\begin{aligned} W_{\mu\nu} &= W_1\delta_{\mu\nu} + W_2\frac{p_\mu p_\nu}{M_T^2} + A\frac{q_\mu q_\nu}{M_T^2} + B\frac{1}{M_T^2}(p_\mu q_\nu + p_\nu q_\mu) \\ &\quad + C\frac{1}{M_T^2}(p_\mu q_\nu - p_\nu q_\mu) \end{aligned} \tag{11.28}$$

[4] Note that $\varepsilon_{\mu\nu\rho\sigma}q_\rho p_\sigma$ is a *pseudo*tensor. We shall return to this later.

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Use the current conservation relations in Eq. (11.23)

$$\begin{aligned}
W_1 q_\nu + W_2 \frac{p\cdot q\, p_\nu}{M_T^2} + A\frac{q^2 q_\nu}{M_T^2} + B\frac{1}{M_T^2}(p\cdot q\, q_\nu + q^2 p_\nu) & \\
+ C\frac{1}{M_T^2}(p\cdot q\, q_\nu - q^2 p_\nu) &= 0 \\
W_1 q_\mu + W_2 \frac{p\cdot q\, p_\mu}{M_T^2} + A\frac{q^2 q_\mu}{M_T^2} + B\frac{1}{M_T^2}(p\cdot q\, q_\mu + q^2 p_\mu) & \\
+ C\frac{1}{M_T^2}(q^2 p_\mu - p\cdot q\, q_\mu) &= 0
\end{aligned} \tag{11.29}$$

Since p and q are linearly independent four-vectors, their coefficients must individually vanish

$$\begin{aligned}
W_1 + \frac{q^2}{M_T^2}A + \frac{p\cdot q}{M_T^2}B + \frac{p\cdot q}{M_T^2}C &= 0 \\
\frac{p\cdot q}{M_T^2}W_2 + \frac{q^2}{M_T^2}B - \frac{q^2}{M_T^2}C &= 0 \\
W_1 + \frac{q^2}{M_T^2}A + \frac{p\cdot q}{M_T^2}B - \frac{p\cdot q}{M_T^2}C &= 0 \\
\frac{p\cdot q}{M_T^2}W_2 + \frac{q^2}{M_T^2}B + \frac{q^2}{M_T^2}C &= 0
\end{aligned} \tag{11.30}$$

The solution to these linear equations is

$$\begin{aligned}
C &= 0 \\
B &= -\frac{p\cdot q}{q^2}W_2 \\
A &= -\frac{M_T^2}{q^2}W_1 + \left(\frac{p\cdot q}{q^2}\right)^2 W_2
\end{aligned} \tag{11.31}$$

This is the desired result.

The next task is to combine the expressions in Eqs. (11.21, 11.27) to get the cross section in Eq. (11.20). With the aid of Eq. (11.25), the required expression reduces to

$$\begin{aligned}
\eta_{\mu\nu} W_{\mu\nu} &= (k_{1\mu}k_{2\nu} + k_{1\nu}k_{2\mu} - k_1\cdot k_2\,\delta_{\mu\nu})\left(W_1\delta_{\mu\nu} + W_2\frac{p_\mu p_\nu}{M_T^2}\right) \\
&= W_1(-2k_1\cdot k_2) + W_2\frac{1}{M_T^2}(2\,p\cdot k_1\,p\cdot k_2 - p^2\,k_1\cdot k_2)
\end{aligned} \tag{11.32}$$

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Now employ some kinematics in the laboratory frame. Since the electrons are massless here

$$
\begin{aligned}
q &= k_2 - k_1 \\
q^2 &= -2k_1 \cdot k_2 = -2\mathbf{k}_1 \cdot \mathbf{k_2} + 2\varepsilon_1\varepsilon_2 \\
&= 2\varepsilon_1\varepsilon_2(1-\cos\theta) = 4\varepsilon_1\varepsilon_2 \sin^2\frac{\theta}{2}
\end{aligned}
\tag{11.33}
$$

Also, since $p = (\mathbf{0}, iM_T)$ in the laboratory frame,

$$
\begin{aligned}
p^2 &= -M_T^2 \\
(p\cdot k_1)(p\cdot k_2) &= M_T^2\varepsilon_1\varepsilon_2
\end{aligned}
\tag{11.34}
$$

Hence

$$
\begin{aligned}
\eta_{\mu\nu}W_{\mu\nu} &= 4\varepsilon_1\varepsilon_2\sin^2\frac{\theta}{2}W_1 + 2\varepsilon_1\varepsilon_2(1-\sin^2\frac{\theta}{2})W_2 \\
&= 2\varepsilon_1\varepsilon_2\left(W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\right)
\end{aligned}
\tag{11.35}
$$

The double differential cross section in the laboratory frame in Eq. (11.20) can therefore be written

$$
\frac{d^2\sigma}{d\varepsilon_2 d\Omega_2} = \frac{\alpha^2}{4\varepsilon_1^2\varepsilon_2^2\sin^4\theta/2}\left(\frac{\varepsilon_2^2}{2\varepsilon_2}\right)\left(\frac{1}{M_T\varepsilon_1}\right)2\varepsilon_1\varepsilon_2\left(W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\right)
\tag{11.36}
$$

Introduce the *Mott cross section* for the scattering of a relativistic (massless) Dirac electron from a point charge[5]

$$
\sigma_M \equiv \frac{\alpha^2\cos^2(\theta/2)}{4\varepsilon_1^2\sin^4(\theta/2)}
\tag{11.37}
$$

The double differential cross section in the laboratory frame for the scattering of a relativistic Dirac electron from an arbitrary hadronic target to order α^2 then takes the form

$$
\frac{d^2\sigma}{d\Omega_2 d\varepsilon_2} = \sigma_M\frac{1}{M_T}[W_2(q^2, q\cdot p) + 2W_1(q^2, q\cdot p)\tan^2\frac{\theta}{2}]
\tag{11.38}
$$

This is a central result.

It is useful at this point to demonstrate the relation to the *photoabsorption* cross section. The process is illustrated in Fig. 11.4. This cross section

[5] Two factors of Eq. (10.27) restore the correct dimensions (recall $\varepsilon = \hbar kc$).

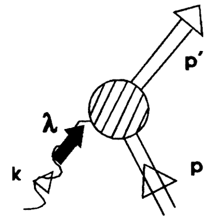

Fig. 11.4. The process of photoabsorption by a hadronic target.

measures one slice of the two-dimensional response surface $W_1(q^2, q \cdot p)$. In fact

$$\sigma_\gamma = \frac{(2\pi)^2\alpha}{\sqrt{(k \cdot p)^2}} W_1(k^2, -k \cdot p) \qquad ; \; k^2 = 0 \tag{11.39}$$

Here $k = (\mathbf{k}, i\omega_k)$ is the four-momentum of the incoming photon. This result is derived as follows.

Start from the interaction with the transverse quantized radiation field

$$H' = -e_p \int \mathbf{J}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x})\, d^3x \tag{11.40}$$

Everything is now in the Schrödinger representation. The quantized radiation field is expanded in normal modes according to Eq. (8.8) with helicity unit vectors $\lambda = \pm 1$ defined in Eq. (8.4) and Fig. 8.1. The scattering operator is again given in lowest order by Eq. (11.4) where now everything is in the interaction representation. The appropriate S-matrix element of this scattering operator is

$$\langle f|\hat{S}|i\rangle = ie_p \left(\frac{1}{2\omega_k \Omega}\right)^{1/2} \mathbf{e}_{\mathbf{k}\lambda} \cdot \langle f|\hat{\mathbf{J}}(0)|i\rangle\, (2\pi)^4 \delta^{(4)}(p + k - p') \tag{11.41}$$

The system is again quantized in a big box with periodic boundary conditions so that Eq. (11.12) should actually be employed. The T-matrix is identified as in Eq. (11.13)

$$\begin{aligned} S_{fi} &= -2\pi i \delta(W_f - W_i) T_{fi} \\ T_{fi} &= -e_p \left(\frac{1}{2\omega_k \Omega}\right)^{1/2} \mathbf{e}_{\mathbf{k}\lambda} \cdot \langle f|\hat{\mathbf{J}}(0)|i\rangle\, \Omega\, \delta_{\mathbf{p}',\mathbf{p}-\mathbf{k}} \end{aligned} \tag{11.42}$$

The cross section is again given by

$$\begin{aligned} \sigma_\gamma &= \frac{\text{Rate}}{\text{Flux}} = 2\pi |T_{fi}|^2 \delta(W_f - W_i) \frac{1}{\text{Flux}} \\ \text{Flux} &= \frac{1}{\Omega} \frac{\sqrt{(k \cdot p)^2}}{\omega_k E} \end{aligned} \tag{11.43}$$

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With unpolarized and unobserved targets, one must again average over initial states and sum over final states and with an unpolarized beam, one must average over photon polarizations. With the use of Eq. (11.15) and the identification of the target response tensor in Eq. (11.20), one finds

$$\sigma_\gamma = \frac{2\pi^2\alpha}{\sqrt{(k\cdot p)^2}} \sum_{\lambda=\pm 1} (\mathbf{e}^\dagger_{\mathbf{k},\lambda})_i W_{ij} (\mathbf{e}_{\mathbf{k},\lambda})_j \tag{11.44}$$

It is now necessary to carry out the polarization sums, and with the insertion of the expressions for the helicity unit vectors one has

$$\begin{aligned}
&\sum_{\lambda\pm 1} (\mathbf{e}^\dagger_{\mathbf{k}\lambda})_i (\mathbf{e}_{\mathbf{k}\lambda})_j \\
&\qquad = \frac{1}{2}[(\mathbf{e}_{\mathbf{k}1} - i\mathbf{e}_{\mathbf{k}2})_i(\mathbf{e}_{\mathbf{k}1} + i\mathbf{e}_{\mathbf{k}2})_j + (\mathbf{e}_{\mathbf{k}1} + i\mathbf{e}_{\mathbf{k}2})_i(\mathbf{e}_{\mathbf{k}1} - i\mathbf{e}_{\mathbf{k}2})_j] \\
&\qquad = (\mathbf{e}_{\mathbf{k}1})_i(\mathbf{e}_{\mathbf{k}1})_j + (\mathbf{e}_{\mathbf{k}2})_i(\mathbf{e}_{\mathbf{k}2})_j \\
&\qquad = \delta_{ij} - \frac{\mathbf{k}_i\mathbf{k}_j}{\mathbf{k}^2}
\end{aligned} \tag{11.45}$$

The last relation follows since the set of unit vectors in Fig. 8.1 is complete. Current conservation can now be employed on the last term

$$\begin{aligned}
k_\mu \langle f|\hat{J}_\mu(0)|i\rangle &= 0 \\
\mathbf{k}\cdot\langle f|\hat{\mathbf{J}}(0)|i\rangle &= |\mathbf{k}|\,\langle f|\hat{J}_0(0)|i\rangle
\end{aligned} \tag{11.46}$$

The required expression in Eq. (11.44) can therefore be written as a covariant polarization sum

$$\sum_\lambda (\mathbf{e}^\dagger_{\mathbf{k},\lambda})_i W_{ij} (\mathbf{e}_{\mathbf{k},\lambda})_j = W_{\mu\mu} \tag{11.47}$$

One has to be careful with the limit $k^2 \to 0$ of the general expression for the target response tensor in Eq. (11.27). From its definition in terms of matrix elements of the current in Eq. (11.20), $W_{\mu\nu}$ cannot be singular in this limit. Thus, by inspection

$$\begin{aligned}
W_2 &\to O(q^2) \qquad ; q^2 \to 0 \\
-W_1 + \frac{(p\cdot q)^2}{M_T^2 q^2} W_2 &\to O(q^2)
\end{aligned} \tag{11.48}$$

The trace of the response tensor is given in general by

$$W_{\mu\mu} = 3W_1 + W_2 \frac{1}{M_T^2}\left[p^2 - \frac{(p\cdot q)^2}{q^2}\right] \tag{11.49}$$

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With the use of Eqs. (11.48) one has

$$\begin{aligned} W_{\mu\mu} &\rightarrow 3W_1 - W_2 - W_1 + O(q^2) \qquad ; q^2 \rightarrow 0 \\ W_{\mu\mu} &\rightarrow 2W_1 + O(q^2) \\ W_{\mu\mu}(k^2 = 0) &= 2W_1(k^2 = 0) \end{aligned} \tag{11.50}$$

This is the desired result, and Eq. (11.39) holds as claimed.