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RESEARCH ARTICLE

Abelian absolute Galois groups

In Erinnerung an Wulf-Dieter Geyer (1939–2019)

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Abstract

Generalizing a result of Wulf-Dieter Geyer in his thesis, we prove that if K is a finitely generated extension of transcendence degree r of a global field and A is a closed abelian subgroup of $\mathrm{Gal}(K)$, then $\mathrm{rank}(A) \leq r+1$. Moreover, if $\mathrm{char}(K)=0$, then $\hat{\mathbb{Z}}^{r+1}$ is isomorphic to a closed subgroup of $\mathrm{Gal}(K)$.

1. Introduction

A consequence of class field theory appearing in [9, p. 302, Thm. 8.8(b)(iii)] says that the cohomological dimension of every number field K which is not embeddable in $\mathbb{R}$ is 2. On the other hand, $\mathrm{cd}(\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}) = 2$ [9, p. 217, Cor. 3.2 and p. 221, Prop. 4.4] and the group $\hat{\mathbb{Z}}$ occurs as a closed subgroup of $\mathrm{Gal}(\mathbb{Q})$ in many ways [3, p. 379, Thm. 18.5.6]. One may therefore wonder whether $\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$ is isomorphic to a closed subgroup of $\mathrm{Gal}(\mathbb{Q})$.

A somewhat surprising result of Geyer's thesis says that this is not the case. Indeed, every closed abelian subgroup of $\mathrm{Gal}(\mathbb{Q})$ is procyclic [4, p. 357, Satz 2.3] (see also [9, p. 306, Thm. 9.1]).

We generalize this result for every finitely generated extension K of transcendence degree r of a global field. We prove that if a profinite group A is isomorphic to a closed abelian subgroup of $\mathrm{Gal}(K)$, then $\mathrm{rank}(A) \leq r+1$. In particular, $\hat{\mathbb{Z}}^{r+2}$ is not a subgroup of $\mathrm{Gal}(K)$ (Proposition 4.3).

In the rest of this note, we abuse our language and write "A is a closed subgroup of $\mathrm{Gal}(K)$" rather than "A is isomorphic to a closed subgroup of $\mathrm{Gal}(K)$."

It turns out that the latter inequality is sharp. Indeed, if $\mathrm{char}(K)=0$, then $\hat{\mathbb{Z}}^{r+1}$ is a closed subgroup of $\mathrm{Gal}(K)$, while if $\mathrm{char}(K)=p>0$, then $\hat{\mathbb{Z}}$ is a closed subgroup of $\mathrm{Gal}(K)$, $\prod_{l\neq p} \mathbb{Z}_l^{r+1}$ is a closed subgroup of $\mathrm{Gal}(K)$ if $r \geq 0$ (Theorem 5.7), but $\hat{\mathbb{Z}}^{r+1}$ is not a closed subgroup of $\mathrm{Gal}(K)$ if $r \geq 1$ (Remark 5.8). Here l ranges over the prime numbers. The exclusion of the factor $\mathbb{Z}_p$ in the case when $p>0$ and $r\geq 1$ follows from the rule $\mathrm{cd}_p(\mathrm{Gal}(F)) \leq 1$ for each field F of characteristic p [9, p. 256, Thm. 3.3].

2. Preliminaries

One of the basic tools needed in the proof of the generalization of Geyer's result is a special case of the renowned Pontryagin–van Kampen theorem. Here, and in the rest of this note, l stands for a prime number, $\mathbb{Z}_l$ is the ring of l-adic numbers, viewed as a profinite abelian group or as a principal ideal domain. We also write $\hat{\mathbb{Z}} := \prod_l \mathbb{Z}_l$ for the Prüfer group [3, p. 12]. Thus, $\mathbb{Z}_l$ is the free pro-l cyclic group and $\hat{\mathbb{Z}}$ is the free procyclic group.

Proposition 2.1 ([10, p. 129, Thm. 4.3.3]). *Let A be a torsion-free abelian profinite group. Then $A \cong \prod_l \mathbb{Z}_l^{r_l}$, where r_l is a cardinal number for each l.*

The proof of Proposition 2.1 uses a special case of the Pontryagin–van Kampen duality theorem saying that every locally compact abelian topological group A is canonically isomorphic to its double dual group A^{**}, where $A^* = \mathrm{Hom}(A, \mathbb{R}/\mathbb{Z})$. The proof of that special case needed in our proposition, dealing only with abelian profinite groups, appears in [10, Section 2.9]. It is much simpler than the proof of the general theorem [5, p. 376, Thm. 24.2].

We denote the algebraic closure of a field K by $\tilde{K}$ and its separable algebraic closure by K_{sep}. We write $\mathrm{Gal}(K)$ for the absolute Galois group $\mathrm{Gal}(K_{\mathrm{sep}}/K)$ of K. If A is a closed subgroup of $\mathrm{Gal}(K)$, then $K_{\mathrm{sep}}(A)$ denotes the fixed field of A in K_{sep}.

Lemma 2.2. *Let K be a field and A a nontrivial finite subgroup of $\mathrm{Gal}(K)$. Then, $A \cong \mathbb{Z}/2\mathbb{Z}$, $\mathrm{char}(K) = 0$, and the fixed field $\tilde{K}(A)$ of A in $\tilde{K}$ is real closed. In addition, A is the centralizer of itself in $\mathrm{Gal}(K)$.*

Proof. Let $R = K_{\mathrm{sep}}(A)$. Then, a theorem of Artin says that $\mathrm{char}(K) = 0$, $K_{\mathrm{sep}} = \tilde{K}$, and $\tilde{K} = R(\sqrt{-1})$ [7, p. 299, Cor. 9.3]. Let τ be the unique element of order 2 of $\mathrm{Gal}(R)$ defined by $\tau(\sqrt{-1}) = -\sqrt{-1}$.

By [7, p. 452, Prop. 2.4], R is real closed. Let $<$ be the ordering of K induced by the unique ordering of R. If R' is a real closed field extension of K in $\tilde{K}$ whose ordering extends $<$, then by [7, p. 455, Thm. 2.9], there exists a unique K-isomorphism $R \to R'$.

Let σ be an element of the centralizer $C_{\mathrm{Gal}(K)}(A)$ of A in $\mathrm{Gal}(K)$. Then, σR is a real closure of $(K, <)$ and $\mathrm{Gal}(\sigma R) \cong \mathbb{Z}/2\mathbb{Z}$. Also, $\tau(\sigma R) = \tau\sigma R = \sigma\tau R = \sigma R$. By the preceding paragraph applied to σR rather than to R, the restriction of τ to σR is the identity map. In other words, $\tau \in \mathrm{Gal}(\sigma R)$. Since $\mathrm{ord}(\tau) = 2$, the element τ generates $\mathrm{Gal}(\sigma R)$, so $R = \sigma R$. The uniqueness of the K-isomorphism of R into R implies that $\sigma \in \mathrm{Gal}(R) = A$, as desired. □

Corollary 2.3. *Let K be a field and A a closed abelian subgroup of $\mathrm{Gal}(K)$. Then, $A \cong \mathbb{Z}/2\mathbb{Z}$ or $A \cong \prod_l \mathbb{Z}_l^{r_l}$, where l ranges over all prime numbers and r_l is a cardinal number.*

Proof. If A has a non-unit element α of a finite order, then by Lemma 2.2, $\langle\alpha\rangle \cong \mathbb{Z}/2\mathbb{Z}$ and $\langle\alpha\rangle$ is its own centralizer in $\mathrm{Gal}(K)$. Since A is abelian, A is contained in that centralizer. Therefore, $A = \langle\alpha\rangle$.

Otherwise, A is torsion-free. Hence, by Proposition 2.1, A has the desired structure. □

Given a profinite group G and a prime number l, we write $\mathrm{cd}_l(G)$ for the l**th cohomology dimension of G** [9, p. 196, Def. 1.1]. Also, we write ζ_n for a primitive root of unity of order n.

Lemma 2.4. *The following statements hold for prime numbers p, l, and a finite extension E of $\mathbb{Q}_p$:*

(a) *E contains only finitely many roots of unity.*
(b) $l^\infty | [E(\zeta_{l^j})_{j\geq 1} : E]$.
(c) $\mathrm{cd}_l(\mathrm{Gal}(E(\zeta_{l^j})_{j\geq 1})) \leq 1$.

Proof of (a). Let O be the ring of integers of E, $\bar{E}$ the residue field of E, π a prime element of O, U the group of invertible elements of O, and $U^{(1)} = 1 + \pi O$ the subgroup of 1-units of O. Reduction modulo πO yields the following short exact sequence

$$\mathbf{1} \longrightarrow U^{(1)} \longrightarrow U \longrightarrow \bar{E}^\times \longrightarrow \mathbf{1},$$

where $\mathbf{1}$ is the trivial group. By [11, p. 213, Chap. XIV, Prop. 10], $U^{(1)}$ is isomorphic to a direct product of a finite abelian group with a free abelian group. Since $\bar{E}^\times$ is also finite, the torsion group of U is finite. That group is the group of roots of unity in E. □

Proof of (b). By (a), E has only finitely many roots of unity of order l^j with $j \geq 1$. Thus, there exists a non-negative integer j with $\zeta_{l^j} \in E$ and $\zeta_{l^{j+1}} \notin E$. By [7, p. 297, Thm. 9.1], $[E(\zeta_{l^{j+1}}) : E(\zeta_{l^j})] = l$. Apply the

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same argument to the field $E_1 := E(\zeta_{l^{j+1}})$ to find an integer $j_2 > j_1 := j$ such that $\zeta_{l^{j_2}} \in E_1$ and $\zeta_{l^{j_2+1}} \notin E_1$, so $[E_2 : E_1] = l$ with $E_2 := E(\zeta_{l^{j_1+1}}, \zeta_{l^{j_2+1}})$. Continue to find a sequence $j_1 < j_2 < j_3 < \dots$ and fields $E \subset E_1 \subset E_2 \subset E_3 \subset \cdots$ such that $\zeta_{l^{j_{n+1}}} \in E_n := E(\zeta_{l^{j_i+1}})_{i=1}^n$ and $\zeta_{l^{j_{n+1}+1}} \notin E_n$, so $[E_{n+1} : E_n] = l$, for each $n \geq 1$. Hence, $l^\infty | [E(\zeta_{l^j})_{j\geq 1} : E]$. □

Proof of (c). The claim follows from (b) and [9, p. 291, Cor. 7.4(i),(ii)]. □

Note that the citation in the proof of (c) relies on local class field theory.

3. Geyer's theorem

We generalize Geyer's theorem which asserts that every closed abelian subgroup of $\mathrm{Gal}(\mathbb{Q})$ is procyclic [4, p. 357, Satz 2.3].

Lemma 3.1. *Let F be a field of positive characteristic p. Then, no pro-p closed subgroup of* $\mathrm{Gal}(F)$ *is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.*

Proof. Let G be a closed pro-p subgroup of $\mathrm{Gal}(F)$. By [9, p. 256, Thm. 3.3], $\mathrm{cd}(G) \leq 1$. On the other hand, $\mathbb{Z}_p$ is a free pro-p group of rank 1. Hence, by [9, p. 217, Cor. 3.2], $\mathrm{cd}(\mathbb{Z}_p) = 1$. It follows from [9, p. 221, Prop. 4.4] that $\mathrm{cd}(\mathbb{Z}_p \times \mathbb{Z}_p) = \mathrm{cd}(\mathbb{Z}_p) + \mathrm{cd}(\mathbb{Z}_p) = 2$. Therefore, $G \not\cong \mathbb{Z}_p \times \mathbb{Z}_p$, as claimed. □

Lemma 3.2. *Let K be a global field, $l \neq$* $\mathrm{char}(K)$ *a prime number, and M a separable algebraic extension of K. Suppose that M contains all of the roots of unity of order l^i for $i = 1, 2, 3, \dots$. Then,* $\mathrm{cd}_l(\mathrm{Gal}(M)) \leq 1$. *In particular,* $\mathrm{Gal}(M) \not\cong \mathbb{Z}_l \times \mathbb{Z}_l$.

Proof. We distinguish between two cases:

Case A: K is a number field. We assume without loss that $K = \mathbb{Q}$. By assumption, $\zeta_{l^2} \in M \smallsetminus \mathbb{R}$. Thus, M cannot be embedded into $\mathbb{R}$, that is M is **totally imaginary**. Hence by [9, p. 302, Thm. 8.8(a)], $\mathrm{cd}_l(\mathrm{Gal}(M)) \neq \infty$.

Now we consider a prime number p, a valuation v of M lying over p, and the completion $\hat{M}_v$ of M at v. Then, $\zeta_{l^i} \in M \subseteq \hat{M}_v$ for each i. Hence, by Lemma 2.4(b), $l^\infty | [\hat{M}_v : \mathbb{Q}_p]$. Therefore, by [9, p. 302, Thm. 8.8(b)], $\mathrm{cd}_l(\mathrm{Gal}(M)) \leq 1$.

Finally, by [9, p. 217, Cor. 3.2 and p. 221, Prop. 4.4] and [9, p. 217, Cor. 3.2],

$$\mathrm{cd}_l(\mathbb{Z}_l \times \mathbb{Z}_l) = \mathrm{cd}_l(\mathbb{Z}_l) + \mathrm{cd}_l(\mathbb{Z}_l) = 1 + 1 = 2.$$

Hence, $\mathrm{Gal}(M) \not\cong \mathbb{Z}_l \times \mathbb{Z}_l$, as claimed.

Case B: K is a finite separable extension of $\mathbb{F}_p(t)$ with t transcendental over $\mathbb{F}_p$. We assume without loss that $K = \mathbb{F}_p(t)$. By assumption, M contains the field $L := \mathbb{F}_p(\zeta_{l^i})_{i\geq 1}$, so $L(t) \subseteq M$. Since there are infinitely many roots of unity ζ_{l^i} in $\tilde{\mathbb{F}}_p$ and only finitely many of them belong to each finite field, L is an infinite field. In addition, for each $i \geq 1$ the extension $\mathbb{F}_p(\zeta_{l^{i+1}})/\mathbb{F}_p(\zeta_{l^i})$ is cyclic of degree l or trivial. Hence, $\mathrm{Gal}(L/\mathbb{F}_p(\zeta_l)) \cong \mathbb{Z}_l$. Therefore, L is contained in the maximal extension L' of $\mathbb{F}_p(\zeta_l)$ of an l'th power degree. Since $\mathrm{Gal}(L'/\mathbb{F}_p(\zeta_l)) \cong \mathbb{Z}_l$, the restriction map $\mathrm{Gal}(L'/\mathbb{F}_p(\zeta_l)) \to \mathrm{Gal}(L/\mathbb{F}_p(\zeta_l))$ is surjective, and $\mathbb{Z}_l$ is generated by one element, that map is an isomorphism [3, p. 331, Cor. 16.10.8]. It follows that $L = L'$. Therefore, l does not divide the order of $\mathrm{Gal}(L)$.

By [9, p. 208, Cor. 2.3], $\mathrm{cd}_l(\mathrm{Gal}(L)) = 0$. Hence, by [9, p. 272, Prop. 5.2], $\mathrm{cd}_l(\mathrm{Gal}(L(t))) = 1$. Since $\mathrm{Gal}(M) \leq \mathrm{Gal}(L(t))$, we have by [9, p. 204, Prop. 2.1(a)], that $\mathrm{cd}_l(\mathrm{Gal}(M)) \leq 1$. As in Case A, this inequality implies that $\mathrm{Gal}(M) \not\cong \mathbb{Z}_l \times \mathbb{Z}_l$, as claimed. □

Here is the promised result of Geyer.

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Theorem 3.3. *Let K be a global field and A a closed abelian subgroup of* $\mathrm{Gal}(K)$*. Then, A is procyclic.*

Proof. We start the proof with the special case where the torsion group A_{tor} of A is nontrivial. In this case, there exists a non-unit $\tau \in A$ of finite order. By Lemma 2.2, $\mathrm{char}(K) = 0$ and $A \cong \mathbb{Z}/2\mathbb{Z}$. In particular, A is procyclic.

We may therefore assume that A is a nontrivial torsion-free abelian profinite group. By Proposition 2.1, $A \cong \prod_l \mathbb{Z}_l^{r_l}$, where l ranges over all prime numbers and for each l, r_l is a cardinal number, so we may assume that $A \cong \mathbb{Z}_l^r$ for a prime number l and a positive cardinal number r and prove that $A \cong \mathbb{Z}_l$.

Otherwise, A contains a closed subgroup which is isomorphic to $\mathbb{Z}_l \times \mathbb{Z}_l$. Thus, we may assume that $A \cong \mathbb{Z}_l \times \mathbb{Z}_l$ and prove that this assumption leads to a contradiction.

To this end, we denote the fixed field of A in K_{sep} by M and identify $\mathrm{Gal}(M)$ with A. By Lemma 3.1, $l \neq \mathrm{char}(K)$.

Claim: M contains a root of unity ζ_l of order l. Indeed, if $l = 2$, then $\zeta_l = -1 \in M$. Otherwise, $l > 2$ and if $\zeta_l \notin M$, then $[M(\zeta_l) : M]$ is a divisor of $l - 1$ which is greater than 1. On the other hand, $[M(\zeta_l) : M]$ divides the (profinite) order of A which is l^∞, a contradiction.

Since $\mathrm{Gal}(M) \cong \mathbb{Z}_l \times \mathbb{Z}_l$, Lemma 3.2 implies that not all roots of unity of order l^i with $i \geq 1$ belong to M. Let n be the smallest positive integer such that M contains a root of unity of order l^{n-1} but does not contain a root of unity of order l^n. Choose a root of unity ζ_{l^n} and set $M_1 = M(\zeta_{l^n})$. Then, $\zeta_{l^n}^l \in M$ but $\zeta_{l^n} \notin M$. Hence, $[M_1 : M] | l$ and $[M_1 : M] \neq 1$ (by the Claim and [7, p. 289, Thm. 6.2(ii)]), so $[M_1 : M] = l$.

Let U be the open subgroup of $\mathbb{Z}_l$ of index l. Then, the index of each of the subgroups $\mathbb{Z}_l \times U$ and $U \times \mathbb{Z}_l$ of $\mathrm{Gal}(M)$ is l. We choose one of them which is different from $\mathrm{Gal}(M_1)$ and denote its fixed field in K_{sep} by M_2. Then, M_2 is a cyclic extension of M of degree l and $M_1 \neq M_2$.

Since $\zeta_l \in M$, [7, p. 289, Thm. 6.2(i)] implies the existence of $a, x \in K_{\mathrm{sep}}$ with $M_2 = M(x)$ and $a := x^l \in M$. Choose $b \in K_{\mathrm{sep}}$ with $b^{l^{n-1}} = x$, so $b^{l^n} = a$. In particular, $M_2 = M(b^{l^{n-1}}) \subseteq M(b)$ and $[M(b) : M_2] \leq l^{n-1}$. It follows from the preceding paragraph that

$$[M(b) : M] \leq l^n. \tag{3.1}$$

Next choose $\sigma \in A$ such that $\sigma|_{M_1} = \mathrm{id}$ and $\sigma|_{M_2} \neq \mathrm{id}$. In particular, $\sigma x \neq x$, so $\zeta := (\sigma b) b^{-1}$ satisfies

$$\zeta^{l^n} = \sigma b^{l^n} \cdot b^{-l^n} = \sigma a \cdot a^{-1} = a a^{-1} = 1 \text{ and } \zeta^{l^{n-1}} = \sigma b^{l^{n-1}} \cdot b^{-l^{n-1}} = \sigma x \cdot x^{-1} \neq 1,$$

thus ζ is a primitive root of 1 of order l^n.

The definition of M_1 implies that $M_1 = M(\zeta)$. But $M(b)$ is a Galois extension of M (because $\mathrm{Gal}(M)$ is abelian). Hence, $\zeta = (\sigma b) b^{-1} \in M(b)$, so $M_1 \subseteq M(b)$. Since $[M_1 : M] = l$, we have by (3.1) that $[M(b) : M_1] \leq l^{n-1}$. Since σ is the identity on M_1, the latter inequality implies that $\mathrm{ord}(\sigma|_{M(b)}) \leq l^{n-1}$.

On the other hand, the relation $\sigma b = b\zeta$ implies by induction on i that $\sigma^i b = b\zeta^i \neq b$ for each $1 \leq i \leq l^{n-1}$. Hence, $\mathrm{ord}(\sigma|_{M(b)}) > l^{n-1}$. This contradicts the conclusion of the preceding paragraph, as required. □

4. Generalization of Geyer's theorem

The central part of the proof of Geyer's theorem says that for each prime number l, the largest positive integer n for which $\mathbb{Z}_l^n$ is a closed subgroup of $\mathrm{Gal}(\mathbb{Q})$ or of $\mathrm{Gal}(\mathbb{F}_p(t))$ is 1. The next lemma will allow us to generalize that statement to each finitely generated extension of a global field.

Remark 4.1. *Let A be a finitely generated torsion-free abelian pro-l group for a prime number l. [3, p. 519, Prop. 22.7.12(a)] allows us to also consider A as a finitely generated $\mathbb{Z}_l$-module. Since $\mathbb{Z}_l$ is a principal ideal domain, [7, p. 147, Thm. 7.3] implies that $A = \mathbb{Z}_l^n$ is a finitely generated free $\mathbb{Z}_l$-module of rank n for some non-negative integer n. Since $\mathbb{Z}_l$ is generated, as a profinite group, by one element, n is also the rank,* $\mathrm{rank}(A)$*, of A as a profinite group. In other words,* $\mathrm{rank}(A) = \mathrm{rank}_{\mathbb{Z}_l}(A)$.

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Lemma 4.2. *Let K be a field, t an indeterminate, and l a prime number. Suppose that n is the largest positive integer for which $\mathbb{Z}_l^n$ is a closed subgroup of $\mathrm{Gal}(K)$. Then, the largest positive integer m for which $\mathbb{Z}_l^m$ is a closed subgroup of $\mathrm{Gal}(K(t))$ does not exceed $n+1$.*

Proof. Suppose that $A := \mathbb{Z}_l^{n'}$ is a closed subgroup of $\mathrm{Gal}(K(t))$ for some positive integer n'. Let $\varphi : \mathrm{Gal}(K(t)) \to \mathrm{Gal}(K)$ be the restriction map. Then, $\mathrm{Ker}(\varphi) = \mathrm{Gal}(K_{\mathrm{sep}}(t))$. Setting $\bar{A} = \varphi(A)$ and $A_0 = \mathrm{Ker}(\varphi) \cap A$, we get the following commutative diagram of profinite groups:

$$\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & \mathrm{Gal}(K_{\mathrm{sep}}(t)) & \longrightarrow & \mathrm{Gal}(K(t)) & \stackrel{\varphi}{\longrightarrow} & \mathrm{Gal}(K) & \longrightarrow & \mathbf{1} \\
 & & \Big| & & \Big| & & \Big| & & \\
\mathbf{0} & \longrightarrow & A_0 & \longrightarrow & A & \longrightarrow & \bar{A} & \longrightarrow & \mathbf{0},
\end{array}$$

where $\mathbf{0}$ stands for the trivial group of an additive abelian group. Since $\mathbb{Z}_l$ is a principal ideal domain and A is a free $\mathbb{Z}_l$-module of rank n', A_0 is a free $\mathbb{Z}_l$-module, by [7, p. 146, Thm. 7.1]. Also, by [7, p. 148, Lemma 7.4], $\bar{A}$ is a free $\mathbb{Z}_l$-module and $n' = \mathrm{rank}(A_0) + \mathrm{rank}(\bar{A})$.

By [9, p. 272, Prop. 5.2], $\mathrm{Gal}(K_{\mathrm{sep}}(t))$ is a projective group, so also A_0 is a projective group. In other words, $\mathrm{rank}(A_0) \le 1$. Also, by Corollary 2.3 and the assumption of the lemma, $\bar{A} = \mathbb{Z}_l^m$ with $m \le n$ or $l = 2$ and $\bar{A} \cong \mathbb{Z}/2\mathbb{Z}$. In each case, $\mathrm{rank}(\bar{A}) \le n$, hence $\mathrm{rank}(A) = \mathrm{rank}(\bar{A}) + \mathrm{rank}(A_0) \le n+1$, as claimed. □

Proposition 4.3. *Let K be a finitely generated extension with transcendence degree r of a global field K_0 and let A be a closed abelian subgroup of $\mathrm{Gal}(K)$. Then, $A \cong \mathbb{Z}/2\mathbb{Z}$ or $A \cong \prod_l \mathbb{Z}_l^{r_l}$, where l ranges over all prime numbers and $r_l \le r+1$ for each prime number l.*

Proof. By Corollary 2.3, $A \cong \mathbb{Z}/2\mathbb{Z}$ or $A \cong \prod_l \mathbb{Z}_l^{r_l}$, with cardinal numbers r_l. Assume the latter case. If K is a global field, then $r = 0$. Hence, by Theorem 3.3, $r_l \le 0 + 1$ for each l.

Otherwise, $r \ge 1$ and K is a finitely generated extension of transcendence degree 1 of a finitely generated extension K'_0 of transcendence degree $r-1$ of K_0. By induction, for each prime number l, r is the largest positive integer such that $\mathbb{Z}_l^r$ is a closed subgroup of $\mathrm{Gal}(K'_0)$. Hence, by Lemma 4.2, $r+1$ is the largest positive number for which $\mathbb{Z}_l^{r+1}$ is a closed subgroup of $\mathrm{Gal}(K)$. In particular, $r_l \le r+1$, as claimed. □

5. Realizing $\hat{\mathbb{Z}}^{r+1}$ as a closed subgroup of $\mathrm{Gal}(K)$

Let K be a finitely generated extension of $\mathbb{Q}$ of transcendence degree r. We complete Proposition 4.3 in this section by proving that $\hat{\mathbb{Z}}^{r+1}$ is a closed subgroup of $\mathrm{Gal}(K)$. An analogous result holds for a finitely generated extension K of transcendence degree r of $\mathbb{F}_p(t)$, in which case $\prod_{l \ne p} \mathbb{Z}_l^{r+1}$ replaces $\hat{\mathbb{Z}}^{r+1}$.

Remark 5.1 (Valued fields). *We denote the residue field of a valued field (F, v) by $\bar{F}_v$ and its value group by $v(F^\times)$. In addition, we extend v to a valuation of F_{sep} that we also denote by v, consider its valuation ring $O_{v,\mathrm{sep}}$, and let $D_{v,\mathrm{sep}} = \{\sigma \in \mathrm{Gal}(F) \mid \sigma O_{v,\mathrm{sep}} = O_{v,\mathrm{sep}}\}$ be the corresponding **decomposition group**. Then, we let F_v be the fixed field of $D_{v,\mathrm{sep}}$ in F_{sep}. Abusing our notation, we also let v be the restriction of v to F_v. Then, (F_v, v) is the **Henselization** of (F, v).*

One knows that (F_v, v) has the same residue field and value group as those of (F, v) [2, p. 138, Prop. 15.3.7]. Moreover, the valued fields (F_{sep}, v) and (F_v, v) depend on the extension of v to F_{sep} up to isomorphism [2, p. 138, Cor. 15.3.6].

If v is a rank-1 valuation, then so is its extension to F_v. In this case, the completion $(\hat{F}_v, v)$ of (F, v) is also discrete with the same value group and residue field as those of (F, v). Moreover, $(\hat{F}_v, v)$ is also the completion of (F_v, v). By Hensel's lemma, $(\hat{F}_v, v)$ is also Henselian [2, p. 167, Cor. 18.3.2]. We embed F_{sep} into $\hat{F}_{v,\mathrm{sep}}$ and observe that $F_{\mathrm{sep}} \cap \hat{F}_v = F_v$ (since $(F_{\mathrm{sep}} \cap \hat{F}_v, v)$ is an immediate separable algebraic

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extension of (F_v, v)) and $F_{\mathrm{sep}}\hat{F}_v = \hat{F}_{v,\mathrm{sep}}$ (by the Krasner-Ostrowski lemma [2, p. 172, Cor. 18.5.3]). Thus, restriction gives an isomorphism $\mathrm{Gal}(\hat{F}_v) \cong \mathrm{Gal}(F_v)$ of the corresponding absolute Galois groups.

We denote the maximal unramified extension of F_v (resp. $\hat{F}_v$) by $F_{v,\mathrm{ur}}$ (resp. $\hat{F}_{v,\mathrm{ur}}$) and the maximal tamely ramified extension by $F_{v,\mathrm{tr}}$ (resp. $\hat{F}_{v,\mathrm{tr}}$). These fields are Galois extensions of F_v (resp. $\hat{F}_v$). As in [2, p. 133, p. 141, and p. 145], we set $Z(v) = \mathrm{Gal}(F_v)$ for the ***decomposition group****, $T(v) = \mathrm{Gal}(F_{v,\mathrm{ur}})$ for the* ***inertia group****, and $V(v) = \mathrm{Gal}(F_{v,\mathrm{tr}})$ for the* ***ramification group*** *of (F, v). The letters Z, T, and V are borrowed from the German translations Zerlegsungruppe, Trägheitsgruppe, and Verzweigungsgruppe of the English expressions decomposition group, inertia group, and ramification group,*

$$F \;\text{———}\; F_v \;\text{———}\; F_{v,\mathrm{ur}} \;\text{———}\; F_{v,\mathrm{tr}} \;\underset{V(v)}{\text{———}}\; F_{\mathrm{sep}}. \qquad \overbrace{F_v \cdots F_{\mathrm{sep}}}^{Z(v)},\ \overbrace{F_{v,\mathrm{ur}} \cdots F_{\mathrm{sep}}}^{T(v)} \tag{5.1}$$

Each of the fields $F_{v,\mathrm{ur}}$, $F_{v,\mathrm{tr}}$, and F_{sep} is a Galois extension of F_v. By [2, p. 199, Thm. 22.1.1] and [6, Thm. 2.2] (resp. [2, p. 203, Thm. 22.2.1]) both restriction maps

$$\mathrm{Gal}(F_{v,\mathrm{tr}}/F_v) \to \mathrm{Gal}(F_{v,\mathrm{ur}}/F_v) \quad \text{and} \quad \mathrm{Gal}(F_v) \to \mathrm{Gal}(F_{v,\mathrm{tr}}/F_v)$$

split. In particular, each closed subgroup of $\mathrm{Gal}(F_{v,\mathrm{ur}}/F_v)$; hence, each closed subgroup of $\mathrm{Gal}(\bar{F}_v)$ is isomorphic to a closed subgroup of $\mathrm{Gal}(F_{v,\mathrm{tr}}/F_v)$. Also, each closed subgroup of $\mathrm{Gal}(F_{v,\mathrm{tr}}/F_v)$ is isomorphic to a closed subgroup of $\mathrm{Gal}(F_v)$.

Note that E in Theorem 22.1.1 of [2] is F_{sep}, in our notation, so it satisfies the condition $E = E^l$ for all prime numbers $l \neq \mathrm{char}(\bar{F}_v)$ needed in that theorem.

Notation 5.2. *We denote the group of roots of unity in a field F by $\mu(F)$. If $\mathrm{char}(F) = p > 0$ and F is separably closed, then $\mu(F) = \tilde{\mathbb{F}}_p^\times$. If $\mathrm{char}(F) = 0$ and F is algebraically closed, then $\mu(F) = \mu(\tilde{\mathbb{Q}})$ and $\mathbb{Q}_{\mathrm{ab}} := \mathbb{Q}(\mu(\tilde{\mathbb{Q}}))$ is the maximal abelian extension of $\mathbb{Q}$ (by the theorem of Kronecker–Weber [8, p. 324, Thm. 110]).*

Remark 5.3. *Given a field K, the* ***field of formal power series*** *$K((t))$ in the variable t with coefficients in K, also called the* ***field of Laurent series over*** *K, is the field of all formal power series $\sum_{i=m}^\infty a_i t^i$ with $m \in \mathbb{Z}$ and $a_i \in K$ for all $i \geq m$. If $l < m$, then $\sum_{i=m}^\infty a_i t^i$ is identified with $\sum_{i=l}^\infty a_i t^i$ with $a_i = 0$ for each $l \leq i < m$. Summation and multiplication in $K((t))$ are defined by the following rules:*

$$\sum_{i=m}^{\infty} a_i t^i + \sum_{i=m'}^{\infty} a'_i t^i = \sum_{i=\min(m,m')}^{\infty} (a_i + a'_i) t^i,$$

$$\Big(\sum_{i=m}^{\infty} a_i t^i\Big)\Big(\sum_{j=m'}^{\infty} a'_j t^j\Big) = \sum_{k=m+m'}^{\infty} \Big(\sum_{i+j=k} a_i a'_j\Big) t^k.$$

Let v be the unique discrete valuation of $K(t)$ with $v(a) = 0$ for each $a \in K$ and $v(t) = 1$. Then, $(K((t)), v)$ is the completion of $(K(t), v)$, where $v(\sum_{i=m}^\infty a_i t^i) = m$ whenever $a_m \neq 0$. By [2, p. 167, Cor. 18.3.2], $K((t))$ is Henselian with respect to v.

By [1, p. 28, Cor. 2] (or [2, p. 141, Thm. 16.1.1]),

$$\mathrm{Gal}(K((t))_{\mathrm{ur}}/K((t))) \cong \mathrm{Gal}(K).$$

Replacing K by K_{sep}, we have that $K_{\mathrm{sep}}((t))_{\mathrm{ur}} = K_{\mathrm{sep}}((t))$. Since the roots of unity of order n with $\mathrm{char}(K) \nmid n$ are in K_{sep}, we have that $K_{\mathrm{sep}}((t))$ has a cyclic extension of degree n in $K_{\mathrm{sep}}((t))_{\mathrm{tr}}$. Indeed, that extension is $K_{\mathrm{sep}}((t^{1/n}))$.

Going to the limit of these extensions, we obtain with $p := \mathrm{char}(K)$ that $K_{\mathrm{sep}}((t))_{\mathrm{tr}} = \bigcup_{p \nmid n} K_{\mathrm{sep}}((t^{1/n}))$ and $\mathrm{Gal}(K_{\mathrm{sep}}((t))_{\mathrm{tr}}/K_{\mathrm{sep}}((t))) \cong \prod_{l \neq p} \mathbb{Z}_l$.

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Moreover, if $\mathrm{char}(K)=0$*, then the ramification group* $\mathrm{Gal}(\tilde{K}((t))_{\mathrm{tr}})$ *of* $\tilde{K}((t))$ *is trivial [2, p. 145, Thm. 16.2.3], so* $\tilde{K}((t))_{\mathrm{tr}}=\widetilde{K((t))}$*. Thus, by the preceding paragraph, in this case,* $\mathrm{Gal}(\tilde{K}((t)))\cong\hat{\mathbb{Z}}$*.*

Lemma 5.4. *Let* K_0 *be a field of characteristic p, t an indeterminate, and r a positive integer. Suppose that* $\mu(K_{0,\mathrm{sep}})\subseteq K_0$ *and* $\prod_{l\neq p}\mathbb{Z}_l^r$ *is a closed subgroup of* $\mathrm{Gal}(K_0)$*. Then,* $\prod_{l\neq p}\mathbb{Z}_l^{r+1}$ *is a closed subgroup of* $\mathrm{Gal}(K_0(t))$*.*

Proof. By assumption, the field K_0 has a separable algebraic extension K with $\mathrm{Gal}(K)\cong\prod_{l\neq p}\mathbb{Z}_l^r$. Let v be the discrete K-valuation of $K(t)$ with $v(t)=1$ and choose a Henselization $M:=K(t)_v$ of $K(t)$ with respect to v. Then,

$$\bar{M}:=\overline{K(t)_v}=K \tag{5.2}$$

is the residue field of both $K(t)$ and M with respect to v.

Claim: M is linearly disjoint from $\tilde{K}$ over K. Indeed, let $\tilde{K}_1,\dots,\tilde{K}_n$ be linearly independent elements of $\tilde{K}$ over K. Assume toward contradiction that there exist $m_1,\dots,m_n\in M$ not all zero with $\sum_{i=1}^n m_i\tilde{K}_i=0$. Dividing $m_1,\dots,m_n$ by the element with the least v-value, we may assume that the v-residues $\bar{m}_1,\dots,\bar{m}_n$ are elements of K and one of them is non-zero. Thus, $\Sigma_{i=1}^n\bar{m}_i\tilde{K}_i=0$, contradicting the assumption on $\tilde{K}_1,\dots,\tilde{K}_n$. This proves our claim.

By [2, p. 200, Cor. 22.1.2],

$$Z(v)/V(v)\cong\chi(v)\rtimes\mathrm{Gal}(\bar{M})\overset{(5.2)}{=}\chi(v)\rtimes\mathrm{Gal}(K), \tag{5.3}$$

where $Z(v)=\mathrm{Gal}(M)$ and $V(v)$ are respectively the corresponding decomposition and the ramification groups of M and

$$\chi(v)=\mathrm{Hom}(v(M_{\mathrm{sep}}^{\times})/v(M^{\times}),\mu(K_{0,\mathrm{sep}})). \tag{5.4}$$

See [2, last line of page 144] with $\bar{\mu}$ in that line being $\mu(K_{0,\mathrm{sep}})$, as introduced in the first paragraph of [2, p. 143, Sec. 16.2].

The action of $\mathrm{Gal}(K)$ on $\chi(v)$ is given for each $\tau\in\mathrm{Gal}(K)$, each homomorphism $h:v(M_{\mathrm{sep}}^{\times})/v(M^{\times})\to\mu(K_{0,\mathrm{sep}})$, and every $\gamma\in v(M_{\mathrm{sep}}^{\times})$, by

$$\tau(h)(\gamma+v(M^{\times}))=\tau(h(\gamma+v(M^{\times})))=h(\gamma+v(M^{\times})),$$

where the latter equality holds because $\mu(K_{0,\mathrm{sep}})\subseteq K_0\subseteq K$. In other words, that action is trivial. It follows that

$$\mathrm{Gal}(M_{\mathrm{tr}}/M)\overset{(5.1)}{\cong}Z(v)/V(v)\overset{(5.3)}{\cong}\chi(v)\times\mathrm{Gal}(K). \tag{5.5}$$

By [2, p. 147, Cor. 16.2.7], there is a short exact sequence

$$\mathbf{1}\longrightarrow V(v)\longrightarrow T(v)\longrightarrow\chi(v)\longrightarrow\mathbf{1}.$$

Hence, $\chi(v)\cong T(v)/V(v)$.

By our choice of v, the completion of $K(t)$ with respect to v (which is also the completion of the Henselian field M) is the field $K((t))$ of formal power series in t with coefficients in K [2, p. 83, Example 9.2.2]. The maximal unramified extension of $K((t))$ is $K_{\mathrm{sep}}((t))$ and by Remark 5.3, $\chi(v)\cong T(v)/V(v)\cong\mathrm{Gal}(M_{\mathrm{tr}}/M_{\mathrm{ur}})\cong\prod_{l\neq p}\mathbb{Z}_l$.

By the definition of K, $\mathrm{Gal}(K)\cong\prod_{l\neq p}\mathbb{Z}_l^r$. Hence, by the preceding paragraph,

$$\mathrm{Gal}(M_{\mathrm{tr}}/M)\overset{(5.5)}{\cong}\chi(v)\times\mathrm{Gal}(K)\cong\prod_{l\neq p}\mathbb{Z}_l\times\prod_{l\neq p}\mathbb{Z}_l^r=\prod_{l\neq p}\mathbb{Z}_l^{r+1}.$$

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Since by [6, Thm. 2.2], the epimorphism $\mathrm{Gal}(M) \to \mathrm{Gal}(M_{\mathrm{tr}}/M)$ splits, $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$ is a closed subgroup of $\mathrm{Gal}(M)$. Since M is a separable algebraic extension of $K_0(t)$ [2, p. 137, Thm. 15.3.5], $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$ is also a closed subgroup of $\mathrm{Gal}(K_0(t))$, as claimed. □

Remark 5.5. *Note that the references that support both (5.3) and (5.4) hold also in the case where* $\mathrm{char}(K_0) = 0$.

The following result will be needed in Theorem 5.7.

Lemma 5.6. *Let L be a set of prime numbers and H an open subgroup of $\prod_{l \in L} \mathbb{Z}_l$. Then, $H \cong \prod_{l \in L} \mathbb{Z}_l$.*

Proof. We set $Z := \prod_{l \in L} \mathbb{Z}_l$ and consider all the groups appearing in this proof as additive groups. Since H is open in Z, its index $n := (Z : H)$ is a positive integer. Since Z is abelian, H is normal in Z, so $nZ \leq H$

By [3, p. 13, Lemma 1.4.2(e)], $n\mathbb{Z}_l \cong \mathbb{Z}_l$ for each $l \in L$. Hence, $nZ = \prod_{l \in L} n\mathbb{Z}_l \cong \prod_{l \in L} \mathbb{Z}_l = Z$.

Let $n = \prod_{l \in L'} l^{i(l)}$ be the decomposition of n into a product of prime powers. If l and l' are distinct prime numbers, then l' is a unit of the ring $\mathbb{Z}_l$, so $l'\mathbb{Z}_l = \mathbb{Z}_l$. Hence, $nZ = \prod_{l \in L \cap L'} l^{i(l)}\mathbb{Z}_l \times \prod_{l \in L \smallsetminus L'} \mathbb{Z}_l$. Therefore, $(Z : nZ) = \prod_{l \in L \cap L'} (\mathbb{Z}_l : l^{i(l)}\mathbb{Z}_l) = \prod_{l \in L \cap L'} l^{i(l)} \leq n = (Z : H)$. Combining this result with the result of the first paragraph of the proof, we have $H = nZ$. Therefore, by the second paragraph of the proof, $H \cong Z$, as claimed. □

This brings us to the main result of the current section.

Theorem 5.7. *Let F be a finitely generated extension of transcendence degree $r \geq 0$ of a global field F_0 of characteristic p and let $F' = F(\mu(F_{0,\mathrm{sep}}))$. Then, $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$ is a closed subgroup of* $\mathrm{Gal}(F')$*, hence also of* $\mathrm{Gal}(F)$.

Proof. In the case where $r = 0$, F itself is a global field, hence Hilbertian [3, p. 242, Thm. 13.4.2]. Since F' is an abelian extension of F, a theorem of Kuyk asserts that F' is also Hilbertian [3, p. 333, Thm. 16.11.3]. Since F is countable, so is F'. By [3, p. 379, Thm. 18.5.6], for almost all $\sigma \in \mathrm{Gal}(F')$ (in the sense of the Haar measure of $\mathrm{Gal}(F')$) the closed subgroup $\langle \sigma \rangle$ of $\mathrm{Gal}(F')$ generated by σ is isomorphic to $\hat{\mathbb{Z}}$. Since $\prod_{l \neq p} \mathbb{Z}_l$ is a closed subgroup of $\prod_l \mathbb{Z}_l$ and $\prod_l \mathbb{Z}_l \cong \hat{\mathbb{Z}}$ [3, p. 15, Lemma 1.4.5], $\prod_{l \neq p} \mathbb{Z}_l$ is a closed subgroup of $\mathrm{Gal}(F')$.

Alternatively, by a theorem of Whaples, for each $l \neq p$ the field F' has a Galois extension F'_l with $\mathrm{Gal}(F'_l/F') \cong \mathbb{Z}_l$ [3, p. 314, Cor. 16.6.7]. Then, $F'' := \prod_{l \neq p} F'_l$ is a Galois extension of F' with $\mathrm{Gal}(F''/F') \cong \prod_{l \neq p} \mathbb{Z}_l$. Since $\prod_{l \neq p} \mathbb{Z}_l$ is projective [3, p. 507, Cor. 22.4.6], the restriction map $\mathrm{Gal}(F') \to \mathrm{Gal}(F''/F')$ splits [3, p. 506, Remark 22.4.2]. Hence, again, $\prod_{l \neq p} \mathbb{Z}_l$ is a closed subgroup of $\mathrm{Gal}(F')$.

Next assume by induction that $r \geq 1$ and the theorem holds for $r - 1$. Choose a finitely generated extension F_{r-1} of transcendence degree $r - 1$ of F_0 in F and let $F'_{r-1} = F_{r-1}(\mu(F_{0,\mathrm{sep}}))$. Since F is finitely generated over F_0 of transcendence degree r, we may choose t in F which is transcendental over F_{r-1} and $[F : F_{r-1}(t)] < \infty$. Then, $F' = F'_{r-1}F$ is a finite extension of $F'_{r-1}(t)$. Let L be the maximal separable extension of $F'_{r-1}(t)$ in F', so F'/L is a purely inseparable extension of L. Then, L is a finite separable extension of $F'_{r-1}(t)$.

$$
\begin{array}{ccccccc}
F_{r-1}(\mu(F_{0,\mathrm{sep}})) = F'_{r-1} & \text{——} & F'_{r-1}(t) & \text{——} & L & \text{——} & F' = F'_{r-1}F \\
| & & | & & & & | \\
F_{r-1} & \text{——} & F_{r-1}(t) & & \text{————} & & F
\end{array}
$$

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Hence,

$$\mathrm{Gal}(L) \text{ is an open subgroup of } \mathrm{Gal}(F'_{r-1}(t)). \tag{5.6}$$

By the induction hypothesis, $\prod_{l\neq p}\mathbb{Z}_l^r$ is a closed subgroup of $\mathrm{Gal}(F'_{r-1})$. Therefore, by (5.6), Lemma 5.4, and Lemma 5.6, $\prod_{l\neq p}\mathbb{Z}_l^{r+1}$ is a closed subgroup of $\mathrm{Gal}(L)$. Since F'/L is a purely inseparable extension (in particular $F'=L$ if $\mathrm{char}(F_0)=0$), $\prod_{l\neq p}\mathbb{Z}_l^{r+1}$ is a closed subgroup of $\mathrm{Gal}(F')$, hence also of $\mathrm{Gal}(F)$, as claimed. □

Remark 5.8. *Let F be a field as in Theorem 5.7. If $p=0$, then $\hat{\mathbb{Z}}^{r+1}=\prod_{l\neq p}\mathbb{Z}_l^{r+1}$. Hence, by that theorem, $\hat{\mathbb{Z}}^{r+1}$ is isomorphic to a closed subgroup of* $\mathrm{Gal}(F)$.

If $p\neq 0$ but $r=0$, then $F=F_0$ is a countable Hilbertian field and again, by [3, p. 379, Thm. 18.5.6], for almost all $\sigma\in\mathrm{Gal}(F)$ we have $\langle\sigma\rangle\cong\hat{\mathbb{Z}}$.

However, by [9, p. 256, Thm. 3.3], $\mathrm{cd}_p(\mathrm{Gal}(F))\leq 1$. *On the other hand, by [9, p. 221, Prop. 4.4],* $\mathrm{cd}_p(\mathbb{Z}_p^{r+1})=r+1\geq 2$ *if $r\geq 1$. Hence, $\mathbb{Z}_p^{r+1}$ is isomorphic to no closed subgroup of* $\mathrm{Gal}(F)$. *Therefore, $\hat{\mathbb{Z}}^{r+1}$ is isomorphic to no closed subgroup of* $\mathrm{Gal}(F)$.

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