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THE RAMSEY NUMBER FOR STRIPES

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If $G_1, \cdots, G_c$ are graphs without loops or multiple edges there is a smallest integer $r(G_1, \cdots, G_c)$ such that if the edges of a complete graph K_n, with $n \geqq r(G_1, \cdots, G_c)$, are painted arbitrarily with c colours the ith coloured subgraph contains G_i as a subgraph for at least one i. $r(G_1, \cdots, G_c)$ is called the Ramsey number of the graphs $G_1, \cdots, G_c$.

Ramsey graph theory was formulated in Cockayne (1970) and independently in Harary and Chratal (to appear). There has been considerable interest in the topic recently. Some properties of the numbers are mentioned in Cockayne (1972) and an extensive bibliography may be found in Harary (1972).

In this paper we determine $r(n_1P_2, \cdots, n_cP_2)$ where, in the notation of Harary (1969), nP_2 is the graph consisting of $2n$ vertices and n independent edges, called here a stripe.

By a circuit we shall mean a graph consisting of a finite number of vertices with each vertex joined to the next and with the last vertex joined to the first. By the complement of a subgraph we mean the vertex complement.

We shall prove

THEOREM. *If $n_1, \cdots, n_c$ are positive integers and $n_1 = \max(n_1, \cdots, n_c)$ then*

$$r(n_1P_2, \cdots, n_cP_2) = n_1 + 1 + \sum_{i=1}^{c} (n_i - 1).$$

This theorem has the following consequence.

COROLLARY. *If a complete graph on n vertices has its edges coloured by c colours then it has a monochromatic subgraph isomorphic to wP_2 where w is the largest integer not greater than*

$$\frac{n + c - 1}{c + 1}.$$

The corollary is proved by noticing that if $\dfrac{n+c-1}{c+1}$ is substituted for each n_i in the expression

$$n_1 + 1 + \sum_{i=1}^{c} (n_i - 1)$$

the number n is obtained.

Now we prove the theorem.

We show first that

$$r(n_1P_2, \cdots, n_cP_2) \geqq n_1 + 1 + \sum_{i=1}^{c} (n_i - 1).$$

Consider a complete graph on

$$n_1 + \sum_{i=1}^{c} (n_i - 1)$$

vertices and partition the vertices into sets $V_1, \cdots, V_c$ where V_1 has $2n_1 - 1$ members and each other V_i has $n_i - 1$. Paint with the first colour all edges which are incident with two vertices of V_1. For each $i = 2, \cdots, c$ paint with the ith colour the edges having two vertices in V_i or one vertex in V_i and one in V_j where $j < i$. For each $i = 1, \cdots, c$ the ith coloured subgraph does not contain a subgraph isomorphic to n_iP_2 and so

$$r(n_1P_2, \cdots, n_cP_2) \geqq n_1 + 1 + \sum_{i=1}^{c} (n_i - 1).$$

The rest of this paper is devoted to proving the opposite inequality, that is,

$$r(n_1P_2, \cdots, n_cP_2) \leqq n_1 + 1 + \sum_{i=1}^{c} (n_i - 1).$$

Suppose that counterexamples exist; that is, there are positive integers c, $n_1, \cdots, n_c$ with $n_1 = \max(n_1, \cdots, n_c)$ and a complete graph on at least

$$n_1 + 1 + \sum_{i=1}^{c} (n_i - 1)$$

vertices which does not have a subgraph isomorphic to n_iP_2 and coloured by the ith colour for any $i = 1, \cdots, c$. Among the counterexamples let G be a minimal one in the following sense:

(1) G is coloured with c colours and no counterexample is coloured with less than c colours.

(2) Among the counterexamples satisfying (1), G is one having a minimal number of vertices.

It is easily seen that these conditions imply that G has exactly

$$n_1 + 1 + \sum_{i=1}^{c} (n_i - 1)$$

vertices, for any subgraph of G having this number of vertices is also a counter-example and so, because of the minimal properties of G, it coincides with G.

The analysis now depends on the study of certain circuits and trees in G. These are the ones having no two adjacent edges of the same colour. If C is one of these subgraphs which has α_i edges coloured by the ith colour for each $i = 1, \cdots, c$ then it has

$$\varepsilon + \sum_{i=1}^{c} \alpha_i$$

vertices where $\varepsilon = 0$ if C is a circuit and $\varepsilon = 1$ if C is a tree. For each i the α_i edges coloured by the ith colour form a subgraph of C isomorphic to $\alpha_i P_2$. If C' is the complementary graph to C in G the number of vertices in C' is

$$n_1 + 1 + \sum_{i=1}^{c} (n_i - 1) - \left(\varepsilon + \sum_{i=1}^{c} \alpha_i\right) = (n_1 - \varepsilon) + 1 + \sum_{i=1}^{c} (n_i - \alpha_i - 1).$$

Suppose that the numbers $n_i - \alpha_i$, $i = 1, \cdots, c$ are $m_1, \cdots, m_c$ in decreasing order. If C is a circuit, $\varepsilon = 0$ and

$$\begin{aligned} n_1 - \varepsilon &= n_1 \\ &\geqq n_i - \alpha_i \text{ for each } i = 1, \cdots, c. \end{aligned}$$

If C is a tree having the additional property that it has at least one edge coloured by each colour then $\alpha_i > 0$ and

$$\begin{aligned} n_1 - \varepsilon &= n_1 - 1 \\ &\geqq n_i - \alpha_i \text{ for each } i = 1, \cdots, c. \end{aligned}$$

In either case $n_1 - \varepsilon \geqq m_1$ and the number of vertices in C' is at least

$$m_1 + 1 + \sum_{i=1}^{c} (m_i - 1).$$

As $m_1 = \max(m_1, \cdots, m_c)$ and C' is a proper subgraph of G we deduce from the minimality of G that C' has a subgraph isomorphic to $(n_i - \alpha_i)P_2$ coloured by the ith colour for one of the $i = 1, \cdots, c$. Combining this with the subgraph of C isomorphic to $\alpha_i P_2$ and coloured by the same colour we obtain in G a subgraph isomorphic to $n_i P_2$ and coloured by this colour. As this is assumed impossible we deduce that any circuit in G, and also any tree in G having at least one edge of each colour, must have two adjacent edges of the same colour.

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Now consider the set of subgraphs of G which are trees having no two adjacent edges of the same colour. As any edge is such a tree this set is not empty. In the set choose a tree, say T, having the maximum number of edges possible. We have shown above that not every colour occurs among the edges of T. Let pq be an edge having a colour different from that of any edge of T. If both p and q are vertices of T the edge pq could be adjoined to a chain linking p and q in T to form a circuit in which no two adjacent edges have the same colour. If exactly one of p and q is a vertex of T then pq can be adjoined to T to form a tree having the same properties as T but with one more vertex. In either case we have a contradiction so that neither p nor q are vertices of T.

We now work towards finding a contradiction to our assumption that a counterexample to the result exists. This is done by proving that the finite tree T contains an infinite number of vertices. We prove that T has an infinite sequence $r_1, r_2, \cdots$ of vertices with the properties:

(1) for each $i \geqq 1$, $r_i r_{i+1}$ is an edge of T,

(2) for each $i \geqq 2$ the vertices r_{i-1}, r_i, and r_{i+1} are all different.

(3) For each $i \geqq 1$ the edges $r_i r_{i+1}$ and $r_i p$ have the same colour.

As T is a tree it has no circuits so condition (2) here is sufficient to prove that the sequence $r_1, r_2, \cdots$ is infinite.

First let r_1 be any vertex of T having valency one in T. Such vertices exist because T is a tree. r_2 is defined by the condition that $r_1 r_2$ is an edge of T. If the edge $r_1 p$ is added to T we get a tree U having one more vertex than T. From the maximal property of T, U must have two adjacent edges of the same colour. These can only be $r_1 r_2$ and $r_1 p$.

If the edge $r_2 p$ is adjoined to T we get a tree having one more vertex than T and hence having two adjacent edges of the same colour. One of these edges must be $r_2 p$ and then r_3 is defined by the property that $r_2 r_3$ is an edge of T having the same colour as $r_2 p$. If $r_3 = r_1$ we can form a tree from T by deleting $r_1 r_2$ and adding the edges $r_2 p$ and pq. This tree has one more vertex than T but has no adjacent edges of the same colour, a contradiction. Hence $r_3 \neq r_1$.

Finally suppose that $i \geqq 2$, that $r_1, \cdots, r_{i+1}$ have been defined and that $r_1, \cdots, r_i$ have the required properties. We need to find an edge $r_{i+1} r_{i+2}$ of T different from $r_i r_{i+1}$ having the same colour as $r_{i+1} p$.

Form a tree U in G by subtracting from T any edge incident with r_i and by adding the edges pq, $r_{i-1} p$, $r_{i+1} p$ and edges qx whenever $r_i x$ is an edge of T different from $r_i r_{i-1}$ or $r_i r_{i+1}$. As U has one more vertex than T two of its adjacent edges are of the same colour. As T does not have two adjacent edges with this property the only possibilities are:

(a) $r_{i-1} p$ and an edge $y r_{i-1}$ of T with $y \neq r_i$.

(b) pq and $r_{i-1} p$, $r_{i+1} p$ or an edge qx where x is a vertex of T.

(c) $r_{i+1}p$ with an edge $r_{i+1}r_{i+2}$ of U with r_{i+2} different from r_i.

(d) qx and an edge xy of T where x is different from r_{i-1}, r_{i+1}, and r_ix, xy are edges of T.

We eliminate possibilities (a), (b) and (d).

If yr_{i-1} is an edge of T with $y \neq r_{i-1}$ the edges yr_{i-1} and $r_{i-1}r_i$ being adjacent have different colours. By assumption the edges $r_{i-1}p$ and $r_{i-1}r_i$ have the same colour. Hence (a) cannot occur.

If (b) occurs we can adjoin to T whichever edge $r_{i-1}p$, $r_{i+1}p$ or qx has the same colour as pq to obtain a tree having no two adjacent edges of the same colour yet having one more vertex than T. The impossibility of this excludes (b).

Suppose that (d) holds. Because of the properties of T and pq, the edges $r_{i-1}r_i$, r_ir_{i+1}, and pq all have different colours, say red, blue, and green respectively. By the induction assumption $r_{i-1}p$ is red and r_ip is blue. Consider the circuit r_ixqp. It must have two adjacent edges of the same colour. As r_ix and r_ir_{i+1} are adjacent edges of T, r_ix is not blue. As in the previous paragraph pq and qx have different colours. Hence r_ix and qx have the same colour. The assumption in (d) then gives two adjacent edges r_ix and xy of T having the same colour. This eliminates (d).

Only the possibility (c) remains and this serves to define r_{i+2} and establish the existence of the infinite sequence $r_1, r_2, \cdots$ in the finite tree T. This contradiction finishes the proof of the theorem.

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