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A NOTE ON MEASURES DETERMINED BY CONTINUOUS FUNCTIONS

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1. **Introduction.** Ellis and Jeffery [2] studied Borel measures determined in a certain way by real valued functions of a real variable which have finite left and right hand limits at each point. If f is such a function and is of bounded variation on an interval I, then the associated measure μ_f has the property that $\mu_f(I)$ equals the total variation of f on I. The authors then indicated in [3] how some of these measures permit the definition of generalized integrals of Denjoy type. In [1], the authors construct an example of a continuous function f, not of bounded variation, such that the associated measure μ_f is the zero measure. The purpose of this note is to show that "most" continuous functions give rise to the zero measure in the sense that there is a residual subset R of $C[a, b]$ such that for each $f \in R$, the associated measure μ_f is the zero measure.

2. **Preliminaries.** For convenience, we shall deal with continuous functions defined on the interval [0, 1] rather than on the whole real line. Let $C[0, 1]$ denote, as usual, the space of all such functions furnished with the sup norm.

Now let $f \in C[0, 1]$. Following [1], [2], or [3] we define an outer measure μ_f^* by Munroe's method II [5]. Thus, for each positive integer n, let C_n denote the class of closed intervals of length less than $1/n$. We now obtain an outer measure $\mu_{f,n}^*$ defined for each subset A of [0, 1] by the equation

$$\mu_{f,n}^*(A) = \inf\left\{\sum_{k=1}^{\infty} |f(b_k)-f(a_k)| : [a_k, b_k] \in C_n ; \bigcup_{k=1}^{\infty} [a_k, b_k] \supset A\right\}.$$

Finally, we define μ_f^* by the equation

$$\mu_f^*(A) = \lim_{n\to\infty} \mu_{f,n}^*(A).$$

The measure μ_f which is the restriction of μ_f^* to its class of measurable sets is then the required measure.

3. **Main result.** In this section we prove that the class of continuous functions f for which μ_f is the zero measure, is residual in $C[0, 1]$. We begin with a lemma.

LEMMA. *Let $f \in C[0, 1]$ and let λ denote Lebesgue measure. Then $\mu_f(A) = 0$ for each $A \subset [0, 1]$ for which $\lambda(f(A)) = 0$.*

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Proof. Let A be any set in [0, 1] for which $\lambda(f(A))=0$. It suffices to show that for each positive integer n, $\mu^*_{f,n}(A)=0$, for in that case

$$\mu^*_f(A) = \lim_{n\to\infty} \mu^*_{f,n}(A) = 0,$$

whence $\mu_f(A)=0$ since μ_f is a complete measure. Thus, fix n and decompose [0, 1] into finitely many intervals each of length less than $1/n$. Let I be one of these intervals. We shall show that $\mu^*_{f,n}(I\cap A)=0$ from which it follows (by the subadditivity of $\mu^*_{f,n}$) that $\mu^*_{f,n}(A)=0$.

Towards this end let M denote the set of points in I at which f attains a strict relative maximum or minimum. The set M is denumerable [7, p. 261], so $\mu^*_{f,n}(M)=0$. Let

$$B = \{x\in A\cap I : f(x) = f(x') \quad \text{for some} \quad x'\in I, x'\neq x\},$$

then $\mu^*_{f,n}(B)=0$. To see this, associate with each $x\in B$ an $x'\in I$, $x'\neq x$ such that $f(x)=f(x')$. The family of intervals $[x, x']$ thus obtained covers B. There exists a denumerable subfamily of this family, $\{[x_k, x'_k]\}_{k=1}^\infty$, which also covers B. Since $\lambda(I)<1/n$, we infer

$$\mu^*_{f,n}(B) \leq \sum_{k=1}^{\infty} |f(x_k)-f(x'_k)| = 0.$$

Now let $D=(I\cap A)\sim(B\cup M)$. Each $x\in D$ is isolated in its level set over I: that is, if $x\in D$, then for each $x'\in I(x'\neq x)$, we have $f(x)\neq f(x')$. Furthermore, since $x\notin M$, $f(t)>f(x)$ for all $t\in I$ on one side of x and $f(t)<f(x)$ for all $t\in I$ on the other side of x. For definiteness, suppose $f(t)>f(x)$ if $t>x$, $t\in I$ and $f(t)<f(x)$ if $t<x$, $t\in I$. It follows from the intermediate value property for continuous functions that the sense of the inequalities is preserved in all $x\in D$. In particular, f is strictly increasing on D. It now follows directly from the definition of $\mu^*_{f,n}$ and the fact that $\lambda(f(D))=0$ that $\mu^*_{f,n}(D)=0$. Specifically, for $\epsilon>0$, let $\bigcup_{k=1}^\infty (a_k, b_k)$ be an open cover of the set $f(D)$ such that $\sum_{k=1}^\infty (b_k-a_k)<\epsilon$. Since f is monotonic on D, each of the sets $D\cap f^{-1}((a_k, b_k))$ is a relative interval of D: that is, there exist numbers c_k and d_k such that $f(c_k)=a_k$, $f(b_k)=d_k$ and $D\cap f^{-1}((a_k, b_k))=D\cap (c_k, d_k)$. Then $D\subset\bigcup_{k=1}^\infty (c_k, d_k)$ and

$$\mu^*_{f,n}(D) \leq \mu^*_{f,n}\left(\bigcup_{k=1}^{\infty} (c_k, d_k)\right) \leq \sum_{k=1}^{\infty} |f(c_k)-f(d_k)| = \sum_{k=1}^{\infty} (b_k-a_k) < \epsilon.$$

Since ϵ was arbitrary, we infer $\mu^*_{f,n}(D)=0$.

We have shown that $\mu^*_{f,n}(M)=0$, $\mu^*_{f,n}(B)=0$, and $\mu^*_{f,n}(D)=0$. Since $I\cap A=M\cup B\cup D$, it follows that $\mu^*_{f,n}(I\cap A)=0$, and the proof of the lemma is complete.

THEOREM. *The class of functions f in* $C[0, 1]$ *for which μ_f is the zero measure, is residual in* $C[0, 1]$.

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Proof. Marcus [4] has proved that if the set of values a continuous function f takes at points where the derivative exists, (finite or infinite), forms a set of Lebesgue measure zero, then for almost every y in the range of f, the level set $L_y \equiv \{x : f(x) = y\}$ is perfect. Let f be such a function. Let $B = \{x : L_{f(x)} \text{ is perfect}\}$. Then, as in the proof of the lemma, $\mu_f(B) = 0$. Let $A = [0, 1] \sim B$. Then $\lambda(f(A)) = 0$ so, by the lemma, $\mu_f(A) = 0$.

Now [6] the class of continuous functions which at no point have a finite or infinite derivative forms a residual subset of $C[0, 1]$. Each function in this class satisfies the conditions of Marcus' theorem, thus each such function gives rise to the zero measure.

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