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CHARACTER CORRESPONDENCES AND π-SPECIAL CHARACTERS IN π-SEPARABLE GROUPS

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0. Introduction. Let π be a set of primes and let G be a π-separable group (all groups considered are finite). Two subsets $X_\pi(G)$ and $B_\pi(G)$ of the set $\mathrm{Irr}(G)$ of irreducible characters of G play an important role in the character theory of π-separable groups and particularly solvable groups. If p is prime and π is the set of all other primes, then the B_π characters of G give a natural one-to-one lift of the Brauer characters of G into $\mathrm{Irr}(G)$. More generally, they have been used to define Brauer characters for sets of primes.

The π-special characters of G (i.e., $X_\pi(G)$) restrict irreducibly and in a one-to-one fashion to a Hall-π-subgroup of G. If an irreducible character χ is quasi-primitive, it factors uniquely as a product of a π-special character an a π'-special character. This is a particularly useful tool in solvable groups.

Assume that A acts on G via automorphisms, and $(|A|, |G|) = 1$. The Glauberman-Isaacs correspondence ρ defines a "natural" bijection between

$$\mathrm{Irr}_A(G) = \{\alpha \in \mathrm{Irr}(G) \,|\alpha \text{ is } A\text{-invariant}\}$$

and $\mathrm{Irr}(C)$, where $C = C_G(A)$. In this paper we investigate the images of π-special and B_π characters under ρ. This seems a natural question to ask since quasi-primitive characters factor into special characters and since ρ respects Clifford correspondence (see Lemma 2.4 below). If $|G|$ is odd or $2 \in \pi$, the image of π-special or B_π-characters are again π-special or B_π, generalizing a result of Uno for B_π characters and $\pi' = \{p\}$. In the general case, if we assume that $\chi \in X_\pi(G)$ is quasi-primitive, then there exists $\lambda \in \mathrm{Irr}(C)$ such that $\lambda^2 = 1_C$ and $\lambda(\chi\rho)$ is π-special. The images of π-special characters behave much like π-special characters. They restrict irreducibly to Hall-π-subgroups. Also for A-invariant $\chi \in X_\pi(G)$ and $\chi' \in X_{\pi'}(G)$, we have

$$(\chi\chi')\rho = (\chi\rho)(\chi'\rho) \in \mathrm{Irr}(C).$$

For a p-solvable group G admitting a coprime operator group A, Uno **[10]** has proved that there is a bijection between the A-invariant Brauer characters of G and the Brauer characters of C (for the prime p). In the last section, we give a similar, yet shorter, proof which also generalizes to

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sets of primes. We conclude the last section by investigating the interaction between ρ and π-blocks. While there is not a one-to-one correspondence between the A-invariant π-blocks of G and the π-blocks of C, we do show that if $\beta_1, \beta_2 \in \mathrm{Irr}(C)$ lie in the same π-block, then $\beta_1\rho^{-1}$ and $\beta_2\rho^{-1}$ belong to the same π-block of G.

1. Special characters. In this section, we give the essential facts about π-special and B_π-characters.

1.1 LEMMA. *Let G be π-separable. Then*

$$X_\pi(G) = \{\chi \in B_\pi(G)\,|\chi(1) \textit{ is a } \pi\textit{-number}\}.$$

Proof. See Lemma 5.4 of [**6**].

1.2 THEOREM. *Let $N \trianglelefteq G$ with G/N a π-group. Let $\chi \in \mathrm{Irr}(G)$ and let θ be an irreducible constituent of χ_N. Suppose G is π-separable.*

(i) *$\chi \in X_\pi(G)$ if and only if $\theta \in X_\pi(N)$, and*

(ii) *$\chi \in B_\pi(G)$ if and only if $\theta \in B_\pi(N)$.*

Proof. Since $\chi(1)$ is a π-number if and only if $\theta(1)$ is, part (i) follows from part (ii), which is Theorem 7.1 of [**6**].

Theorem 1.2 above and Theorem 1.3 below completely determine $X_\pi(G)$. Note that $O(\chi)$ is the order of the linear character $\det(\chi)$.

1.3 THEOREM. *Let G be π-separable, let $N \trianglelefteq G$ with G/N a π'-group. Let $\theta \in B_\pi(N)$. Then*

(i) *There is a unique irreducible constituent χ of θ^G satisfying $\chi \in B_\pi(G)$;*

(ii) *If $I_G(\theta) = G$, then χ extends θ;*

(iii) *χ is π-special if and only if $I_G(\theta) = G$ and θ is π-special. In this case, χ is the unique extension of θ with $O(\chi)$ a π-number.*

We will conclude this section by mentioning some interesting and useful facts regarding π-special characters.

1.4 THEOREM. *If P is a Hall-π-subgroup of a π-separable group and $P \leqq H \leqq G$, then $\chi \to \chi_H$ is a bijection from $X_\pi(G)$ into $X_\pi(H)$. In particular, $\chi \to \chi_P$ is a bijection from $X_\pi(G)$ into $\mathrm{Irr}(P)$.*

Proof. This is [**1**, Proposition 1.6].

1.5 THEOREM. *If G is π-separable, if α and β are π-special and π'-special respectively, then $\alpha\beta \in \mathrm{Irr}(G)$. Furthermore, the factorization of $\alpha\beta$ as a product of a π-special character and π'-special character is unique.*

Proof. This is [**1**, Proposition 7.1].

It is well known that if $\chi \in \mathrm{Irr}(G)$ is quasi-primitive and G is

π-separable, then χ may be factored as above. We let Q_π denote the field extension of Q obtained by adjoining to Q all complex n-th roots of unity for all π-numbers n. The following is Corollary 12.1 of [**6**].

1.6 THEOREM. *Let* $\chi \in B_\pi(G)$ *for a* π*-separable group* G. *Then* $Q(\chi) \subseteq Q_\pi$.

2. Character correspondence. We next look at the character correspondence ρ. For convenience, we make the following hypothesis.

2.1 *Hypothesis.* Let A act on G with $(|A|, |G|) = 1$ and let $C = C_G(A)$.

The map ρ in Theorem 2.2 gives a "natural" correspondence between $\mathrm{Irr}_A(G)$ and $\mathrm{Irr}(C)$, i.e., it is completely determined by the action of A on G and is choice-free. Theorem 2.2, which gives an algorithm for computing ρ, is Corollary 5.2 of [**11**]. Of course, $\rho = \rho(G, A)$ depends on the action of A on G. Also $\rho(H, B)$ exists whenever $B \leqq A$ and $H \leqq G$ is A-invariant. We will drop the indexing and will write ρ or ρ_A when it is obvious what is meant.

2.2 THEOREM. *Assume Hypothesis* 2.1. *Let* $\chi \in \mathrm{Irr}_A(G)$. *Then*

(i) *If* $T \trianglelefteq A$ *and* $D = C_G(T)$, *then*

$$\rho(G, A) = \rho(G, T)\rho(D, A/T);$$

(ii) *If* A *is a* q*-group, then* $\chi\rho_A$ *is the unique* $\beta \in \mathrm{Irr}(C)$ *satisfying* $q \nmid [\chi_C, \beta]$; *and*

(iii) *If* $|G|$ *is odd and* H *is an* A*-invariant subgroup of* G *with* $[G, A]'C \leqq H$, *then there is a unique* $\psi \in \mathrm{Irr}_A(H)$ *such that* $[\chi_H, \psi]$ *is odd. Furthermore,* $\chi\rho = \psi\rho$.

The following proposition is an easy consequence of the above theorem.

2.3 COROLLARY. *Assume the hypotheses of Theorem* 2.2. *Then the fields* $Q(\chi)$ *and* $Q(\chi\rho)$ *are equal. If* G *has odd order and* ψ *is as in part* (iii) *of Theorem* 2.2, *then* $Q(\chi) = Q(\psi)$.

2.4 LEMMA. *Assume Hypothesis* 2.1 *and that* $N \trianglelefteq G$ *is* A*-invariant. Let* $\theta \in \mathrm{Irr}_A(N)$, *let* $I = I_G(\theta)$, *and let* $\phi = \theta\rho$. *Then*

(i) $I \cap C = I_C(\phi)$;

(ii) *If* $\psi \in \mathrm{Irr}_A(I|\theta)$, *then* $(\psi^G)\rho = (\psi\rho)^C$; *and*

(iii) $\psi^G \to (\psi\rho)^C$ *is a bijection from* $\mathrm{Irr}_A(G|\theta)$ *onto* $\mathrm{Irr}(C|\phi)$.

Proof. See Lemma 2.5 of [**12**].

The following theorem is a key tool developed in [**12**, Theorem 2.12].

2.5 THEOREM. *Assume Hypothesis* 2.1, *that* $M \trianglelefteq G$ *is* A*-invariant and* $MC = G$. *Let* $M \leqq H \leqq G$ *and* $\chi \in \mathrm{Irr}_A(G)$. *Then every irreducible* *constituent of* χ_H *is* A*-invariant and*

$$(\chi\rho)_{H\cap C} = \chi_H\rho,$$

extending the map $\rho(H, A)$ *linearly.*

The following will often allow us to reduce to the case where $MC = G$ so that we can employ the above theorem.

2.6 LEMMA. *Assume Hypothesis* 2.1 *and that* G *is* π*-separable. Let* $M \trianglelefteq G$ *be* A*-invariant and assume that* G/M *is a* π*-group or* π'*-group. Let*

$$\rho_0 = \rho(G, A)(\rho(MC, A))^{-1}$$

and let $\chi \in \mathrm{Irr}_A(G)$. *Then*

(i) $\rho_0:\mathrm{Irr}_A(G) \to \mathrm{Irr}_A(MC)$ *is a bijection*;
(ii) $\chi\rho_0$ *is an irreducible constituent of* χ_{MC};
(iii) $\chi \in B_\pi(G)$ *if and only if* $\chi\rho_0 \in B_\pi(MC)$; *and*
(iv) *If* $\chi \in X_\pi(G)$, *then* $\chi\rho_0 \in X_\pi(MC)$.

Proof. Part (i) is immediate from Theorem 2.2. If A is solvable, then Part (ii) follows from [**12**, Lemma 3.1]. To prove (ii), we may assume that $|G|$ is odd and $MC < G$. Then

$$H/M = [G/M, A]'(MC/M)$$

is a proper A-invariant subgroup of G/M and $[G, A]'C \leqq H$. When $|G|$ is odd, part (ii) follows from Theorem 2.2 via an easy inductive argument.

When G/M is a π-group, part (iii) follows from part (ii) and two applications of Theorem 1.2. Part (iv) is then proved analogously. We thus assume that G/M is a π'-group. If $\chi \in B_\pi(G)$, it follows from [**5**, Corollary 6.6] that every irreducible constituent of χ_{MC} is a B_π-character, whence part (ii) implies $\chi\rho_0 \in B_\pi(MC)$. On the other hand, if $\chi\rho_0 \in B_\pi(MC)$, it follows from [**5**, Corollary 6.4] and Theorem 1.3 that exactly one irreducible constituent η of $(\chi\rho_0)^G$ is a B_π-character. Then $\eta \in \mathrm{Irr}_A(G)$ and we have just seen that $\eta\rho_0 \in B_\pi(MC)$. It follows from Theorem 1.3, part (ii) and part (i) that $\eta\rho_0 = \chi\rho_0$ and thus $\eta = \chi$. This completes part (iii). If $\chi \in X_\pi(G)$, then $\chi_{MC} \in X_\pi(MC)$ by Theorem 1.4 and part (iv) now follows from part (ii).

2.7 LEMMA. *Assume Hypothesis* 2.1 *and* $\chi \in \mathrm{Irr}_A(G)$. *Let* $M \trianglelefteq G$ *be* A*-invariant and suppose* $\chi_M \in \mathrm{Irr}(M)$. *Then* $\chi\rho$ *extends* $(\chi_M)\rho$.

Proof. Since $\chi_{MC} \in \mathrm{Irr}(MC)$, it is a routine argument using Theorem 2.2 to prove that $\chi\rho = (\chi_{MC})\rho$. Thus we may assume $MC = G$. The lemma now follows from Theorem 2.5.

2.8 PROPOSITION. *Assume Hypothesis* 2.1 *and that* $N \trianglelefteq G$ *is* A*-invariant. Let* $\chi \in \mathrm{Irr}_A(G)$ *and* $\theta \in \mathrm{Irr}_A(N)$. *Then*

(i) χ_N *has an A-invariant irreducible constituent*;
(ii) θ^G *has an A-invariant irreducible constituent*; *and*
(iii) *If G is π-separable and $\theta \in B_\pi(N)$, there exists an A-invariant $\psi \in B_\pi(G|\theta)$.*

Proof. Parts (i) and (ii) follow from [**4**, Theorems 13.27 and 13.28]. To prove (iii), we may assume that G/N is a π-group or π'-group. If G/N is a π-group, part (iii) follows from part (ii) and Theorem 1.2. If G/N is a π'-group, the unique $\psi \in B_\pi(G|\theta)$ is necessarily A-invariant.

Assume Hypothesis 2.1. If $\chi \in \mathrm{Irr}_A(G)$ and $\chi\rho = \beta$, then

$$\chi(1) \mid |G:C|\beta(1).$$

It has been conjectured that $\beta(1) \mid \chi(1)$, and the conjecture is valid for solvable G. We need the following lemma.

2.9 Lemma. *Assume Hypothesis* 2.1 *and that G is π-separable. Let $\chi \in \mathrm{Irr}_A(G)$. If $\chi(1)$ is a π-number, then so is $(\chi\rho)(1)$.*

Proof. We argue by induction on $|G|$. Choose $M \lhd G$ such that M is A-invariant and G/M is a π-group or π'-group. Let $\theta \in \mathrm{Irr}_A(M)$ be a constituent of χ_M and let $\phi = \theta\rho$. By induction, $\phi(1)$ is a π-number. By Lemma 2.3, $\chi\rho \in \mathrm{Irr}(C|\phi)$ and so we may assume G/M is a π'-group. Then χ extends θ and by Lemma 2.7, $(\chi\rho)(1) = \phi(1)$.

2.10 Lemma. *Assume Hypothesis* 2.1 *and that G is π-separable. Let $Q \leqq G$ be an A-invariant π-subgroup of G. Then*
(i) *Q is contained in an A-invariant Hall-π-subgroup P of G*;
(ii) *If $C \leqq H \leqq G$ and H is A-invariant, then $P \cap H$ is a Hall-π-subgroup of H.*

Proof. Part (i) can be proved via a standard induction argument using the Schur-Zassenhaus Theorem. By [**4**, Theorem 13.8], the A-invariant Hall-π-subgroups of G are C-conjugate. Consequently part (ii) follows from part (i).

3. Images of special characters. We investigate here $\chi\rho$ when χ is π-special or $\chi \in B_\pi(G)$.

3.1 Lemma. *Assume that G is π-separable, $L \trianglelefteq G$, $\chi \in \mathrm{Irr}(G)$, and $\chi_L \in B_\pi(L)$. Then*
(a) *There is a unique linear $\lambda \in \mathrm{Irr}(G)$ satisfying*

$$LO^{\pi'}(G) \leqq \ker(\lambda) \quad \textit{and} \quad \lambda\chi \in B_\pi(G); \quad \textit{and}$$

(b) *If $Q(\chi) \subseteq Q_\pi$, then $\lambda^2 = 1_G$.*

Proof. First, for uniqueness, assume that λ_1 also satisfies the conclusion of part (a) and set $N = LO^{\pi'}(G)$. Then

$$\chi_N \in \mathrm{Irr}(N) \quad \text{and} \quad \lambda\chi, \lambda_1\chi \in B_\pi(G|\chi_N).$$

Theorem 1.3 implies that $\lambda\chi = \lambda_1\chi$ and consequently $\lambda = \lambda_1$. Hence existence implies uniqueness.

Let M be a maximal normal subgroup of G containing L. Arguing by induction on $|G/L|$, we may assume there is a unique linear $\gamma \in \mathrm{Irr}(M)$ satisfying

$$\gamma\chi_M \in B_\pi(M) \quad \text{and} \quad LO^{\pi'}(M) \leqq \ker(\gamma).$$

Since $G = I_G(\chi_M)$, so is

$$G = I_G(\gamma) = I_G(\gamma\chi_M).$$

If G/M is a π'-group, Theorem 1.3 implies there exists $\phi \in B_\pi(G)$ extending $\gamma\chi_M$ and χ_L. It necessarily follows that $\phi = \tau\chi$ for an extension τ of γ. Since $O^{\pi'}(G) = O^{\pi'}(M)$, we finish this case by setting $\lambda = \tau$. We thus assume G/M is a π-group. Since $I_G(\gamma) = G$ and $(|G/M|, O(\gamma)) = 1$, there is an extension $\lambda \in \mathrm{Irr}(G|\gamma)$ with $O(\lambda) = O(\gamma)$ (see Corollary 6.27 of [**4**]). Since $O(\lambda)$ is a π'-number and λ is linear,

$$LO^{\pi'}(G) \leqq \ker(\lambda).$$

By Theorem 1.2, $\lambda\chi \in B_\pi(G)$, proving part (a).

By Lemma 1.6, $Q(\lambda\chi) \subseteq Q_\pi$. If also $Q(\chi) \subseteq Q_\pi$, it follows from the uniqueness of λ that $Q(\lambda) \subseteq Q_\pi$. Since λ is linear and $O(\lambda)$ is a π'-number, $Q(\lambda) = Q$. Thus $\lambda^2 = 1_G$.

3.2 THEOREM. *Assume Hypothesis* 2.1 *and that G is π-separable. Suppose that* $2 \in \pi$ *or* $|C|$ *is odd. Let* $\chi \in \mathrm{Irr}_A(G)$. *Then*

(a) *If* $\chi \in B_\pi(G)$, *then* $\chi\rho \in B_\pi(C)$;

(b) *If* $\chi \in X_\pi(C)$, *then* $\chi\rho \in X_\pi(C)$.

Proof. Part (b) follows from Lemma 1.1, Lemma 2.9, and part (a). To prove (a), we argue by induction on $|G|$. Choose an A-invariant $M \lhd G$ such that G/M is a π-group or π'-group. Let θ be an A-invariant irreducible constituent of χ_M (see Proposition 2.8) and let $\phi = \theta\rho$. By Theorem 1.2 or 1.3 and induction, $\phi \in B_\pi(M \cap C)$. Since $\chi\rho \in \mathrm{Irr}(C|\phi)$ by Lemma 2.4, we may assume that G/M and $C/M \cap C$ are π'-groups. Let $I = I_G(\theta)$ so that $I \cap C = I_C(\phi)$. By Theorem 1.3, there is a unique $\alpha \in B_\pi(I|\theta)$ and furthermore α extends θ. By [**6**, Corollary 6.3], $\alpha^G = \chi$. Similarly there is a unique $\delta \in B_\pi(I \cap C|\phi)$, δ extends ϕ, and $\delta^C \in B_\pi(C)$. If $I < G$, we apply the induction hypothesis and Lemma 2.4 to conclude $\alpha\rho = \delta$ and to show

$$\chi\rho = \delta^C \in B_\pi(C).$$

We thus assume $I = G$, $\chi_M = \theta$, and $\delta^C = \delta$ extends ϕ. By Lemma 2.7, $\chi\rho$ extends ϕ. By Corollary 2.3 and Lemma 1.6,

$$Q(\chi\delta) = Q(\chi) \subseteq Q_\pi.$$

Applying Lemma 3.1, there is a linear $\lambda \in \mathrm{Irr}(C/C \cap M)$ such that $\lambda^2 = 1_C$ and

$$\lambda(\chi\rho) \in B_\pi(G).$$

Since $C/M \cap C$ is a π'-group, the hypotheses imply that $\lambda = 1_C$.

3.3 COROLLARY. *Assume Hypothesis* 2.1 *and that G is π-separable. Assume $2 \in \pi$ or $|C|$ is odd. Then $\chi \to \chi\rho$ is a bijection from $B_\pi(G) \cap$* $\mathrm{Irr}_A(G)$ *onto $B_\pi(C)$.*

Proof. Let $\beta \in B_\pi(C)$. We need just show that

$$\beta\rho^{-1} \in B_\pi(G).$$

Let $M \lhd G$ be A-invariant with G/M a π-group or π'-group. Let $\theta \in \mathrm{Irr}_A(M)$ be a constituent of $(\beta\rho^{-1})_M$. By Lemma 2.4 and an induction argument, we may assume that $\theta \in B_\pi(M)$ and hence that G/M is a π'-group. Then θ^G has a unique irreducible constituent $\chi \in B_\pi(G)$. Since $\chi \in \mathrm{Irr}_A(G)$, Theorem 3.2 and Lemma 2.4 imply that

$$\chi\rho \in B_\pi(C|\theta\rho).$$

Then $\chi\rho = \beta$ by Theorem 1.3.

For $\pi' = \{p\}$ and $|G|$ odd, the above corollary was proven by Uno [**10**]. The converse of part (b) of Theorem 3.2 is not true. To demonstrate this, one can construct an example where G has a normal Hall-π-subgroup M containing C and there exists $\theta \in \mathrm{Irr}_A(M)$ not invariant in G. We next look at the even case.

3.4 THEOREM. *Assume Hypothesis* 2.1 *and that G is π-separable. Let $\chi \in \mathrm{Irr}_A(G)$ be π-special and $\beta = \chi\rho$. In addition, assume there exists a GA-invariant series*

$$G = G_0 \geqq G_1 \geqq \dots \geqq G_k = 1$$

such that each factor group is a π-group or π'-group and such that the restriction of χ to G_i is homogeneous for all i. Then there exists $\lambda \in \mathrm{Irr}(C)$ with $\lambda^2 = 1_C$ such that $\lambda\beta \in X_\pi(C)$. Furthermore, if $\beta \notin X_\pi(C)$, then $2 \notin \pi$ and λ is unique.

Proof. By Lemma 1.6 and Corollary 2.3, $Q(\beta) \subseteq Q_\pi$. First assume the existence of λ and set $N = \ker(\lambda)$. Then $|C:N| \leqq 2$ and $\beta_N \in X_\pi(N)$. If $\beta \notin X_\pi(C)$, then $|C:N| = 2 \notin \pi$. The uniqueness of λ follows from Lemma 3.1 with $L = O^{\pi'}(C)$. Hence the second conclusion follows from the first.

We proceed by induction on $|G|$. Choose $M \lhd G$ such that $M = G_r$ for some r and such that G/M is a π-group or π'-group. Let θ be the unique irreducible constituent of χ_M, so that $\theta \in X_\pi(M)$ is GA-invariant. Let

$$\phi = \theta\rho \in \mathrm{Irr}(M \cap C).$$

By Lemma 2.4, $I_C(\phi) = C$ and $\beta \in \mathrm{Irr}(C|\phi)$.

By the induction hypothesis, there exists $D \trianglelefteq M \cap C$ such that

$$\phi_D \in X_\pi(D) \quad \text{and} \quad |M \cap C/D| \leqq 2.$$

If $\phi \in X_\pi(M)$, we assume that $D = M$. The uniqueness part of the induction hypothesis and the fact that $I_C(\phi) = C$ permit us to assume $D \trianglelefteq C$. We claim that C/D has a normal Hall-π-subgroup L/D. This is trivial if $M \cap C = D$. For the claim, we may assume that

$$|M \cap C/D| = 2 \notin \pi$$

and $C/M \cap C$ is a π-group. Since

$$M \cap C/D \leqq Z(C/D),$$

C/D has a normal Hall-π-subgroup L/D.

By Lemma 2.7, $\beta_L \in \mathrm{Irr}(L|\phi_D)$ and thus $\beta_L \in X_\pi(L)$ by Theorem 1.2. The existence of λ follows from Lemma 3.1.

3.5 COROLLARY. *Assume Hypothesis* 2.1, *that G is π-separable and $\chi \in X_\pi(G)$. Then there exist $J \leqq C$, $\alpha \in X_\pi(J)$ and $\lambda \in \mathrm{Irr}(J)$ such that $\lambda^2 = 1_J$ and $(\lambda\alpha)^C = \chi\rho$.*

Proof. By Theorem 3.4 and Proposition 2.8, we may choose an A-invariant $M \trianglelefteq G$ and $\theta \in \mathrm{Irr}_A(M)$ such that $I = I_G(\theta) < G$. Let $\psi \in \mathrm{Irr}(I|\theta)$ with $\psi^G = \chi$. Since G and I are A-invariant, it follows from [**1**, Theorem 5.10] that there exists $\delta \in \mathrm{Irr}_A(I)$ such that $\delta\psi \in X_\pi(I)$ and $\delta^2 = 1_I$. By induction, we may argue the existence of $J \leqq I \cap C$, $\alpha \in \mathrm{Irr}(J)$, and $\gamma \in \mathrm{Irr}(J)$ such that

$$\gamma^2 = 1_J \quad \text{and} \quad (\gamma\alpha)^{I \cap C} = (\delta\psi)\rho = \delta_{C \cap I}(\psi\rho).$$

Since $(\psi\rho)^C = \chi$ by Lemma 2.4, we finish by setting $\lambda = \delta_{C \cap J}\gamma$.

Let G be π-separable, let $\chi \in \mathrm{Irr}(G)$, and suppose there is a normal series $G = G_0 \geqq G_1 \geqq \ldots \geqq G_k = 1$ such that each factor group is a π-group or π'-group and such that χ restricted to G_i is homogeneous for each i. Then χ can be factored uniquely as a product of a π-special character and π'-special character. Thus the hypotheses of the above theorem are not unreasonable, particularly in light of Lemma 2.4. Furthermore, Lemma 2.4, the above theorem, and Theorem 3.10 imply that ρ is locally determined for odd G and possibly solvable groups of even order.

It is not hard to construct examples where, in Theorem 3.4, $\lambda \neq 1$. In fact many examples exist where G has an extra-special subgroup E of index 2 and $C_E(A) = Z(E)$. Using such an example, one can construct further examples showing the necessity of the primitivity condition in

Theorem 3.4. On the other hand, the next two theorems (as well as Theorem 4.4 below) show that images (under ρ) of π-special characters behave somewhat like π-special characters. This next proposition is known.

3.6 PROPOSITION. *Assume that* $G = KH$, $K \trianglelefteq G$, *and* $K \cap H = M$. *Let* $\theta \in \mathrm{Irr}(K)$ *and* $\theta_M \in \mathrm{Irr}(M)$. *If*

$$I_G(\theta) \cap H = I_H(\theta_M),$$

then $\chi \to \chi_H$ *defines a bijection from* $\mathrm{Irr}(G|\theta)$ *onto* $\mathrm{Irr}(H|\theta_M)$.

Proof. Let $I = I_G(\theta)$, so that

$$I \cap H = I_H(\theta_M).$$

By [**6**, Corollary 4.2], $\xi \to \xi_{I \cap H}$ is a bijection from $\mathrm{Irr}(I|\theta)$ onto $\mathrm{Irr}(I \cap H|\theta_M)$. Since $IH = G$, we have that

$$(\xi^G)_H = \xi_{I \cap H}^H \in \mathrm{Irr}(H|\theta_M) \quad \text{for } \xi \in \mathrm{Irr}(I|\theta).$$

The proposition follows from Clifford's Theorem [6.11 of **4**].

3.7 LEMMA. *Assume Hypotheses* 2.1 *and that* G *is* π-*separable. Let* $\chi \in X_\pi(G)$ *be* A-*invariant and let* $\beta = \chi\rho$. *If* Q *is a Hall*-π-*subgroup of* C, *then* β_Q *is irreducible. In fact the map* $\chi \to (\chi\rho)_Q$ *is a one-to-one map from the set of* A-*invariant* π-*special characters of* G *into* $\mathrm{Irr}(Q)$.

Proof. We argue by induction on $|G|$. Let $M \triangleleft G$ be A-invariant with G/M a π-group or π'-group. In light of Lemma 2.6, we may assume that $G = MC$. By induction, $\theta \to (\theta\rho)_{Q \cap M}$ is a one-to-one map from the set of A-invariant π-special characters of M into $\mathrm{Irr}(Q \cap M)$. For $\theta \in \mathrm{Irr}_A(G)$, the A-invariant G-conjugates of θ are precisely the C-conjugates of θ (see [**4**, Corollary 13.9]), and it is routine to see that

$$(\theta^c)\rho = (\theta\rho)^c \quad \text{for } c \in C.$$

Thus it suffices to fix

$$\theta \in X_\pi(M) \quad \text{and} \quad \phi = \theta\rho \in \mathrm{Irr}(M \cap C)$$

and to show that $\chi \to (\chi\rho)_Q$ is a one-to-one map from $X_\pi(G|\theta)$ into $\mathrm{Irr}(Q|\phi_{M \cap Q})$. If G/M is a π'-group, then

$$X_\pi(G|\theta) = \{\chi\}.$$

Also χ extends θ by Theorem 1.3. By Lemma 2.7, $(\chi\rho)_Q$ extends $\phi_{M \cap Q}$, whence $(\chi\rho)_Q$ is irreducible. We thus assume that G/M is a π-group. By Lemma 2.4, ρ induces a bijection between $\mathrm{Irr}(G|\theta)$ and $\mathrm{Irr}(C|\phi)$. Since $(M \cap C)Q = C$, the proof may be completed by applying Proposition 3.6, once we establish that

$$I_C(\phi) \cap Q = I_Q(\phi_{M \cap Q}).$$

By the uniqueness part of the induction argument,

$$I_Q(\phi_{M \cap Q}) = I_G(\theta) \cap Q$$

and by Lemma 2.4,

$$I_C(\phi) \cap Q = I_G(\theta) \cap C \cap Q = I_G(\theta) \cap Q.$$

We can actually strengthen Lemma 3.7 by showing that ρ and restriction commute. We do this in Theorem 3.9. First we need a proposition to be used in Theorem 3.9 and Theorem 3.10.

3.8 PROPOSITION. *Assume Hypothesis* 2.1 *and that* $|G|$ *is odd. Let* $\chi \in \mathrm{Irr}_A(G)$ *be* π*-special. Let* $H \leqq G$ *be* A*-invariant with* $[G, A]'C \leqq H$*. Let* $\psi \in \mathrm{Irr}_A(H)$ *with* $[\chi_H, \psi]$ *odd. Then* ψ *is* π*-special.*

Proof. By Theorem 2.2, ψ exists and is unique. Let $K = [G, A]$ and $L = H \cap K$. Then $HK = G$, K/L is abelian and $L \trianglelefteq G$. If $K = G$, then $H \trianglelefteq G$ and the result is immediate. Without loss of generality, choose $K \leqq M \lhd G$ with M maximal. Since A centralizes $G/K \simeq H/L$, M is A-invariant. Let $\theta \in \mathrm{Irr}(M)$ be such that $[\chi_M, \theta] \neq 0$. Then θ is A-invariant and $\theta \in X_\pi(M)$. Since

$$[M, A]'(C \cap M) \leqq H \cap M,$$

there exists a unique $\phi \in \mathrm{Irr}_A(H \cap M)$ such that

$$[\theta_{H \cap M}, \phi] \not\equiv 0 \pmod 2.$$

By the induction hypothesis, $\phi \in X_\pi(H \cap M)$. We may assume that $G/M \simeq H/H \cap M$ is a π'-group. Then $\chi_M = \theta$ and $I_H(\phi) = H$. Hence there is a unique $\mu \in X_\pi(H|\phi)$. Since μ extends ϕ and $H/H \cap M$ is abelian, $\psi = \lambda\mu$ for a unique linear $\lambda \in \mathrm{Irr}(H/H \cap M)$. Since $Q(\chi) = Q(\psi)$ and χ is π-special, it follows that $\lambda = 1_H$ by Proposition 3.1.

3.9 THEOREM. *Assume Hypothesis* 2.1 *and that G is* π*-separable. Let P be an* A*-invariant Hall-*π*-subgroup and let* $\chi \in X_\pi(G)$ *be* A*-invariant. Then*

$$(\chi\rho)_{P \cap C} = (\chi_P)\rho.$$

Proof. We argue by induction $|G|\,|A|$. We may assume that $C < G$. We first assume that A is solvable. Choose $T \lhd A$ such that A/T is a q-group for a prime q and let $D = C_G(T)$. Let

$$\eta = \chi\rho_T \in \mathrm{Irr}_A(D) \quad \text{and} \quad \beta = \chi\rho_A.$$

By Theorem 2.2, $\beta = \eta\rho_{A/T}$ and $\eta_C = a\beta + q\Lambda$ for an integer $a \not\equiv 0 \pmod q$ and (possibly zero) character Λ of C. By Lemma 3.7, $\eta_{P \cap D}$ and $\beta_{P \cap C}$ are irreducible. Since

$$[(\eta_{P \cap D})_{P \cap C}, \beta_{P \cap C}] \not\equiv 0 \pmod q,$$

it follows from Theorem 2.2 that

$$(\eta_{P\cap D})\rho_{A/T} = \beta_{P\cap C}.$$

Employing the induction hypothesis and Theorem 1.2,

$$(\chi_P)\rho = (\chi_P)\rho_T\rho_{A/T} = \eta_{P\cap D}\rho_{A/T} = \beta_{P\cap C}.$$

We are done in the case where A is solvable. By the Odd-Order Theorem, we may assume that G is solvable of odd order.

Let $K = [G, A]$, $L = K'$, and $H = LC$. Then $KH = G$ and $K \cap H = L \trianglelefteq G$. Since $C < G$, also $H < G$. Lemma 2.10 implies that $P \cap H$ is a Hall-π-subgroup of H. By Theorem 2.2,

$$\chi_H = \psi + 2\Lambda + \Xi$$

where Λ, $\Xi \in \mathrm{Char}(H)$, $\psi \in \mathrm{Irr}_A(H)$, $\psi\rho = \chi\rho$, and no irreducible constituent of Ξ is A-invariant. If H/L is a π-group, then two applications of Theorem 1.2 yield that every irreducible constituent of χ_H is π-special. Hence, by Theorem 1.4, $\alpha \to \alpha_{H\cap P}$ is a bijection from the irreducible constituents of χ_H into $\mathrm{Irr}(P \cap H)$. Hence

$$[(\chi_P)_{P\cap H}, \psi_{P\cap H}] = [\chi_{P\cap H}, \psi_{P\cap H}]$$

is odd. Since $[P, A]'(P \cap C) \leqq P \cap H$,

$$(\chi_P)\rho = \psi_{P\cap H}\rho.$$

Since $H < G$, we use induction to conclude

$$(\chi_P)\rho = \psi_{P\cap H}\rho = (\psi\rho)_{P\cap C} = (\chi\rho)_{P\cap C}.$$

Hence we may assume that $H/L \simeq G/K$ is not a π-group.

Now KP/K and $(KP \cap H)/L = L(P \cap H)/L$ are Hall-π-subgroups of G/K and H/L. Also $KP < G$ and $L(P \cap H) < H$. Let $Q = L(P \cap H)$. By Proposition 3.8 and Theorem 1.4,

$$\psi_Q \in X_\pi(Q) \quad \text{and} \quad \chi_{KP} \in X_\pi(KP).$$

It suffices by the induction hypothesis, to show $\chi_{KP}\rho = \psi_Q\rho$. Since

$$[KP, A]'(C \cap P) \leqq Q,$$

we need just show that $[\chi_Q, \psi_Q]$ is odd (see Theorem 2.2). Since $LC = H$, two applications of [**4**, Exercise 13.13], no irreducible constituent of Ξ_Q is A-invariant. Since

$$\chi_H = \psi + 2\Lambda + \Xi,$$

we have that

$$[\chi_Q, \psi_Q] \equiv 1 \pmod 2.$$

3.10 THEOREM. *Let π_1 and π_2 be disjoint sets of primes. Assume Hypothesis* 2.1 *and that G is π_i-separable for each i. Let $\chi_i \in \mathrm{Irr}_A(G)$ be*

π_1-special and $\beta_i = \chi_i\rho$ (*for* $i = 1, 2$). *Then*

$$(\chi_1\chi_2)\rho = \beta_1\beta_2.$$

Proof. We argue by induction on $|G|\,|A|$. Choose $M \triangleleft G$ maximal such that M is A-invariant. Without loss of generality, G/M is a π'_2-group. Let $\gamma_i \in \mathrm{Irr}(MC)$ such that $\gamma_i\rho = \beta_i$. By Lemma 2.6, γ_i is π_i-special for each i. By the induction hypothesis, we may assume that $\beta_1\beta_2 \in \mathrm{Irr}(C)$ or that $MC = G$.

First assume $|G|$ is odd. Let $L = [G, A]'$ and $H = LC$. We may assume $C < G$ and hence that $H < G$ (using the solvability of G). Write

$$(\chi_i)_H = \psi_i + 2\Lambda_i + \Xi_i$$

with

$$\Lambda_i, \Xi_i \in \mathrm{Char}(G), \quad \psi_i \in \mathrm{Irr}_A(G),$$

and no irreducible constituent of Ξ_i is A-invariant. By [**4**, Exercise 13.13] and the fact that $LC = H$, every irreducible constituent of $(\psi_i)_L$ is A-invariant and no irreducible constituent of $(\Xi_i)_L$ is A-invariant. Since $L \trianglelefteq G$ and χ_i is π_i-special, all irreducible constituents of $(\psi_i)_L$ and $(\Xi_i)_L$ are π_i-special. By the uniqueness part of Theorem 1.5, it follows that

$$[(\psi_1\psi_2)_L, (\Xi_1\Xi_2)_L] = 0.$$

By Proposition 3.8 and Theorem 1.5, each ψ_i is π_i-special and $\psi_1\psi_2 \in \mathrm{Irr}(H)$. It now follows that

$$[(\chi_1\chi_2)_H, \psi_1\psi_2]$$

is odd. Applying the inductive hypothesis and Theorem 2.2, we conclude that

$$(\chi_1\chi_2)\rho = (\psi_1\psi_2)\rho = (\psi_1\rho)(\psi_2\rho) = \beta_1\beta_2.$$

We are done if $|G|$ is odd. By the Odd-Order Theorem, we can assume that $A \neq 1$ is solvable.

Choose $T \triangleleft A$ so that A/T is a q-group for a prime q. Let $D = C_G(T)$ and $\xi_i = \chi_i\rho_T$. By induction

$$\xi_1\xi_2 \in \mathrm{Irr}(D) \quad \text{and} \quad (\chi_1\chi_2)\rho_A = (\xi_1\xi_2)\rho_{A/T}.$$

By Theorem 1.3, we may write

$$(\xi_i)_C = a_i\beta_i + q\Lambda_i$$

for integers $a_i \not\equiv 0 \pmod q$ and characters Λ_i of C. Then

$$(\xi_1\xi_2)_C = a_1a_2\beta_1\beta_2 + q\Gamma$$

for a character Γ of C. If $\beta_1\beta_2$ is irreducible, then

$$(\chi_1\chi_2)\rho_A = (\xi_1\xi_2)\rho_{A/T} = \beta_1\beta_2.$$

We may thus assume $\beta_1\beta_2$ reduces.

By the first paragraph, $MC = G$. For $M \leqq H \leqq G$, Theorems 1.2 and 1.3 imply that every constituent of $(\chi_i)_H$ is π_i-special $(i = 1, 2)$, and Theorem 2.5 implies that every constituent of $(\chi_i)_H$ is A-invariant. Also for $\eta \in \mathrm{Irr}_A(G)$,

$$(\eta\rho)_{H\cap C} = \eta_H\rho.$$

Applying the inductive hypothesis, we conclude that

$$((\chi_1\chi_2)\rho)_{H\cap C} = ((\chi_1\chi_2)_H)\rho = (\beta_1\beta_2)_{H\cap C}$$

whenever $M \leqq H < G$. We may thus assume that $G/M \simeq C/C \cap M$ is cyclic.

Let $\theta_i \in \mathrm{Irr}(M)$ be an irreducible constituent of $(\chi_i)_M$ and let $\phi_i = \theta_i\rho(i = 1, 2)$. Since G/M is a π_2'-group, we have

$$(\chi_2)_M = \theta_2 \quad \text{and} \quad (\beta_2)_{M\cap C} = \phi_2$$

by Lemma 2.7. Let $I = I_G(\theta_1)$ and choose $\alpha \in \mathrm{Irr}(I|\theta_1)$ with $\alpha^G = \chi_1$. Then α is π_1-special. If $I < G$ then induction yields that

$$(\alpha(\chi_2)_I)\rho = (\alpha\rho)(\beta_2)_{I\cap C}$$

and, since $I = I_G(\phi_1\phi_2)$, we apply Lemma 2.4 to yield

$$(\chi_1\chi_2)\rho = (\alpha\rho(\beta_2)_{I\cap C})^C = (\alpha\rho)^C\beta_2 = \beta_1\beta_2.$$

We may assume that $I = G$. Since G/M is cyclic, we have that β_i extends ϕ_i. Since $\beta_1\beta_2$ reduces, we have $\phi_1\phi_2$ reduces, contradicting the induction hypothesis.

4. Brauer characters and blocks. Let G be π-separable. For a class function μ of G, let μ^* be the restriction of μ to the π-elements of G. If $\mu \in \mathrm{Char}(G)$ and μ^* cannot be written as

$$\mu^* = \alpha_1^* + \ldots + \alpha_n^*$$

for some $n \geqq 2$ and $\alpha_i \in \mathrm{Char}(G)$, then μ^* is called an irreducible π'-Brauer character ($\mu^* \in IBr_\pi'(G)$). This coincides with the usual definition of p-Brauer characters. This definition, due to Isaacs [**5**], has been used by Slattery [**8** or **9**] to define π'-blocks for a π-separable G. This is consistent with the usual theory of p-blocks as well as the notions of π'-blocks put forward by Iizuki [**2**] and Robinson [**7**].

4.1 THEOREM. *If G is π-separable then $\chi \to \chi^*$ is a bijection then $B_\pi(G)$ onto $IBr_{\pi'}(G)$.*

Proof. This is Theorem A of [**6**] and note there $I^\pi(G) = IBr_\pi(G)$.

4.2 PROPOSITION. *Let G be π-separable and let $M \trianglelefteq G$ with G/M a π-group. Let τ and σ be irreducible Brauer-π'-characters of G and M respectively. The multiplicity of σ in τ_M equals the multiplicity of τ in σ^G.*

Proof. This follows from Theorems 4.1 and 1.2 and Frobenius reciprocity for ordinary characters.

When $\pi' = \{p\}$, the above is Theorem 2.2 of [**3**] and the following is Theorem 3.1 of [**3**].

4.3 LEMMA. *Let $M \trianglelefteq G$ where G/M is a π-group and G is π-separable. Assume that $\phi \in \mathrm{Irr}(M)$ and ϕ^* is an irreducible Brauer-π'-character of G. If $I_G(\phi) = I_G(\phi^*)$, then $\chi \to \chi^*$ defines a bijection from $\mathrm{Irr}(G|\phi)$ onto $IBr(G|\phi^*)$.*

Proof. Mimic the proof of [**3**, Theorem 3.1], using the above proposition in place of [**3**, Theorem 2.2].

The next theorem was proven by Uno [**10**] when $\pi' = \{p\}$. Our proof, while similar, is a little shorter and removes the need for what he calls "the projective lifting property".

4.4 THEOREM. *Assume Hypothesis 2.1 and that G is π-separable. Then $\chi \to (\chi\rho)^*$ is a bijection from $B_\pi(G) \cap \mathrm{Irr}_A(G)$ onto $IBr_{\pi'}(C)$.*

Proof. We argue by induction on $|G|$. Let $M \triangleleft G$ with G/M a π-group or π'-group. By Lemma 2.6, we may assume that $MC = G$. By the induction hypothesis, $\theta \to (\theta\rho)^*$ is a bijection from $B_\pi(M) \cap \mathrm{Irr}_A(M)$ onto $IBr(M \cap C)$. Since the map is one-to-one, it follows that

$$I_G(\theta) \cap C = I_C(\,(\theta\rho)^*)$$

for all $\theta \in B_\pi(M) \cap \mathrm{Irr}_A(M)$.

Fix $\theta \in \mathrm{Irr}_A(M)$ and let $\phi = \theta\rho \in \mathrm{Irr}(M \cap C)$. We claim that $\xi \to (\xi\rho)^*$ is a bijection from $\mathrm{Irr}(G|\theta)$ onto $IBr(C|\phi^*)$. By Lemma 2.4, ρ is a bijection from $\mathrm{Irr}(G|\theta)$ onto $\mathrm{Irr}(C|\phi)$. By the last paragraph $I_C(\phi) = I_C(\phi^*)$. Hence, by Lemma 4.3, we may assume that G/M is a π'-group. In this case, $B_\pi(G|\theta)$ and $IBr(C|\phi^*)$ are singletons, say

$$B_\pi(G|\theta) = \{\xi\}.$$

Since $(\,(\xi\rho)^*)_{M\cap C}$ has ϕ^* as a constituent, it suffices for the claim to show that $(\xi\rho)^*$ is irreducible. Let $\theta = \theta_1, \theta_2, \dots, \theta_t$ be the distinct G-conjugates of θ. Since $MC = G$, it follows that each θ_i is A-invariant. For $y \in C$, note that

$$(\theta^y)\rho = (\theta\rho)^y.$$

Hence it follows that if ϕ_i denotes $\theta_i\rho$, then $\phi_1^*, \dots, \phi_t^*$ are distinct C-conjugates of ϕ^*. Since $\xi \in B_\pi(G)$ and G/M is a π'-group,

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$$\xi_M = \theta_1 + \ldots + \theta_t$$

by [**6**, Corollary 6.5]. By Theorem 2.5,

$$(\xi\rho)^*_{M \cap C} = (\xi_M \rho)^* = \phi_1^* + \ldots + \phi_t^*.$$

Since the ϕ_i are C-conjugates, $(\xi\rho)^*$ is irreducible. This establishes the claim.

By the last paragraph and Theorems 1.2 and 1.3, $\chi \to (\chi\rho)^*$ defines a map from $B_\pi(G) \cap \mathrm{Irr}_A(G)$ into $IBr(C)$. By the induction hypothesis and the last paragraph, this map is onto. If

$$\chi_1, \chi_2 \in B_\pi(G) \cap \mathrm{Irr}_A(G)$$

and if

$$(\chi_1\rho)^* = (\chi_2\rho)^*,$$

then the induction argument yields that $(\chi_1)_M$ and $(\chi_2)_M$ have a common (A-invariant) irreducible constituent, whence the last paragraph yields that $\chi_1 = \chi_2$.

The above theorem, along with Theorem 4.1, of course yields a bijection between $IBr_A(G)$ and $IBr(C)$. We will also use ρ to denote this map. Then for $\alpha \in IBr_A(G)$, $\alpha\rho = (\chi\rho)^*$ where $\chi \in B_\pi(G)$ and $\chi^* = \alpha$. In essence, the same algorithm works for computing ρ on $IBr_A(G)$ as on $\mathrm{Irr}_A(G)$.

4.5 THEOREM. *Assume Hypothesis* 2.1 *and that G is π-separable. Let $\mu \in IBr_A(G)$. Then*

(i) *If $T \trianglelefteq A$, then*

$$\mu\rho(G, A) = \mu\rho(G, T)\rho(C_G(T), A/T);$$

(ii) *If A is a q-group, $\mu\rho$ is the unique irreducible constituent of μ_C with multiplicity prime to q;*

(iii) *If $|G|$ is odd and $[G, A]'C \leqq H \leqq G$ with H A-invariant, then there is a unique $\alpha \in IBr_A(H)$ with odd multiplicity in μ_H and furthermore $\mu\rho = \alpha\rho$.*

Proof. This easily follows from Proposition 3.7, Theorems 2.2 and 4.4.

One question that arises is whether there exists a one-to-one correspondence between the set of A-invariant π'-blocks of G and the set of π'-blocks of C. The answer is no, since one can easily find an example with G solvable, $O_\pi(G) = 1$ and $O_\pi(C) \neq 1$. Here G has a unique π'-block, but C does not. But we do get one "direction", Theorem 4.8. The next theorem, due to Slattery, completely characterizes the π'-blocks of G.

Another question that is at present unanswered is the following. Assuming Hypothesis 2.1, is there a one-to-one correspondence between $IBr_A(G)$ and $IBr(C)$ for a fixed prime p. Of course, the answer is yes if G is p-solvable, but the general case is not known.

4.6 THEOREM. *Let B be a π'-block of a π-separable group G and let N be a normal π-subgroup of G. Let $\chi \in B \cap \mathrm{Irr}(G)$, let θ be an irreducible constituent of χ_N and $I = I_G(\theta)$. Then*

(a) $B \subseteq \mathrm{Irr}(G|\theta) \cup IBr(G|\theta)$;

(b) *Equality holds in part* (a) *if $N = O_\pi(G)$ and $I = G$; and*

(c) *There is a unique π'-block b of I such that*

$$b \subseteq \mathrm{Irr}(I|\theta) \cup IBr(I|\theta)$$

and $\alpha \to \alpha^G$ is a bijection from the ordinary (Brauer, respectively) characters of b to those of B.

Proof. This follows from [**8**, Theorems 2.9 and 2.11].

4.7 LEMMA. *Assume Hypothesis* 2.1 *and that G is π-separable. For a π'-block B of G, the following are equivalent.*

(i) *B is A-invariant*;

(ii) *There exists an A-invariant $\chi \in B \cap \mathrm{Irr}(G)$*;

(iii) *There exists an A-invariant $\phi \in B \cap IBr(G)$.*

Proof. By Theorem 4.1, (iii) implies (ii) and trivially (ii) implies (i). We show that (i) implies (iii). Theorem 4.6 yields there is a G-conjugacy class $\theta = \theta_1, \dots, \theta_t$ of irreducible characters of $O_\pi(G)$ such that

$$B \subseteq \mathrm{Irr}(G|\theta) \cup IBr(G|\theta).$$

Since A permutes the θ_i, it follows from Glauberman's Lemma [13.8 of **4**] that we may choose θ to be A-invariant. By Theorem 4.6 and an inductive argument, we may assume that $I = G$. By Proposition 2.9, there exists an A-invariant $\chi \in B_\pi(G|\phi)$. Then χ is A-invariant and $\chi \in B \cap IBr(G)$.

For an A-invariant π'-block B, we let

$$B\rho = \{\chi\rho | \chi \in B \text{ and } \chi \in \mathrm{Irr}_A(G) \cup IBr_A(G)\}.$$

4.8 THEOREM. *Assume Hypothesis* 2.1 *and that G is π-separable. Let B be an A-invariant π'-block of G. Then $B\rho$ is a union of π'-blocks of C.*

Proof. For $\xi \in \mathrm{Irr}(G)$, all the irreducible constituents of ξ^* lie in the same block as ξ. Thus, given $\chi_1, \chi_2 \in \mathrm{Irr}_A(G)$ and $\beta_i = \chi_i\rho$, it suffices to show that χ_1 and χ_2 lie in the same π'-block of G if β_1 and β_2 lie in the same π'-block of C. To prove this, we use induction on $|G:O_\pi(G)|$. Let $N = O_\pi(G)$. By Theorem 4.6, $(\beta_1)_{N \cap C}$ and $(\beta_2)_{N \cap C}$ have a common irreducible constituent ϕ. Let

$$\theta = \phi\rho^{-1} \in \mathrm{Irr}_A(N)$$

and let $I = I_G(\theta)$. By Lemma 2.4,

$$\chi_1, \chi_2 \in \mathrm{Irr}(G|\theta).$$

Let $\psi_i \in \mathrm{Irr}(I|\theta)$ with $\psi_i^G = \chi_i$. By Lemma 2.4 and Theorem 4.6 (c), $\psi_1\rho$

and $\psi_2\rho$ lie in the same block of $I \cap C$. If $I < G$, we use induction and Theorem 4.6 (c) to conclude χ_1 and χ_2 lie in the same block of G. We can thus assume $I = G$ and conclude the proof by applying Theorem 4.6 (b).

For each π'-block B of a π-separable group G, there is a conjugacy class of π'-groups that are called defect groups. When $\pi' = \{p\}$, these are the usual defect groups. In the situation of Theorem 4.6 (c), the defect groups for b are also defect groups for B. Under the hypotheses of Theorem 4.6 (b), the defect groups of B are Hall-π'-subgroups of G. These last two statements follow from [**8**, Lemma 3.7 and Corollary 3.9].

4.9 THEOREM. *Assume Hypothesis* 2.1 *and that G is π-separable. Let B be an A-invariant π'-block of G and write*

$$B\rho = b_1 \cup \ldots \cup b_t$$

for π'-blocks b_i of C. Then

(i) *The defect groups of b_i are contained in defect groups of B, and*

(ii) *For some j, $P \cap C$ is a defect group for b_j whenever P is an A-invariant defect group for B.*

Proof. Let $N = O_\pi(G)$. As in Lemma 4.7, choose $\theta \in \mathrm{Irr}_A(N)$ such that

$$B \subseteq \mathrm{Irr}(G|\theta) \cup IBr(G|\theta).$$

Let $\phi = \theta\rho \in \mathrm{Irr}(N \cap C)$ and note that

$$b_1 \cup \ldots \cup b_t \subseteq \mathrm{Irr}(C|\phi) \subseteq IBr(C|\phi).$$

By Lemma 2.4 and Theorem 4.6, and comments preceding this theorem, we may argue by induction on $|G|$ to assume

$$I_G(\theta) = G,$$

$$B = \mathrm{Irr}(G|\theta) \cup IBr(G|\theta),$$

$$b_1 \cup \ldots \cup b_t = \mathrm{Irr}(C|\phi) \cup IBr(C|\phi),$$

and the defect groups of B are Hall-π-subgroups of G. Part (i) now follows from Lemma 2.10.

Let $M = O_\pi(C)$ and Q be a Hall-π'-subgroup of C. By Proposition 2.8, there exists $\eta \in \mathrm{Irr}(M|\phi)$ that is Q-invariant. This implies that

$$M = O_\pi(I_C(\eta)).$$

Then Theorem 4.6 yields that $\mathrm{Irr}(C|\eta) \cup IBr(C|\eta)$ is a block b' with Q as a defect group. Since $[\eta_N, \phi] \neq 0$, $b' = b_j$ for some j. Since defect groups of B are Hall-π-subgroups of G, the proof is complete (see Lemma 2.10).

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