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Packing topological entropy for amenable group actions

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Abstract. Packing topological entropy is a dynamical analogy of the packing dimension, which can be viewed as a counterpart of Bowen topological entropy. In the present paper we give a systematic study of the packing topological entropy for a continuous G-action dynamical system (X, G), where X is a compact metric space and G is a countable infinite discrete amenable group. We first prove a variational principle for amenable packing topological entropy: for any Borel subset Z of X, the packing topological entropy of Z equals the supremum of upper local entropy over all Borel probability measures for which the subset Z has full measure. Then we obtain an entropy inequality concerning amenable packing entropy. Finally, we show that the packing topological entropy of the set of generic points for any invariant Borel probability measure μ coincides with the metric entropy if either μ is ergodic or the system satisfies a kind of specification property.

Key words: packing topological entropy, amenable group, variational principle, generic point
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1. *Introduction*

In 1973, in a profound and influential paper [3], Bowen introduced a definition of topological entropy of subsets inspired by Hausdorff dimension, which is now known as Bowen topological entropy or dimensional entropy. For dynamical systems over compact

Hausdorff spaces, Bowen showed that Bowen topological entropy on the whole space coincides with the Adler–Konheim–McAndrew topological entropy defined through open covers.

Bowen topological entropy can be viewed as dynamically analogous to Hausdorff dimension and has exhibited very deep connections with dimension theory in dynamical system and multifractal analysis ever since its appearance (see, for example, [19]). It is natural to consider the analogous concepts in dynamical systems for other forms of dimensions. For pointwise dimension (of a measure), its dynamical correspondence is the Brin–Katok local entropy [4]. For packing dimension, its dynamical correspondence is the packing topological entropy, which was introduced by Feng and Huang [9]. Applying the methods in geometric measure theory, they also provided variational principles for Bowen topological entropy and packing topological entropy. Other works on packing topological entropy can be found in [27], where the packing topological entropy for certain non-compact subsets was considered.

In this paper we focus on packing topological entropy in the framework of countable discrete amenable group actions.

1.1. *Amenable packing entropy and local entropies.* Let (X, G) be a G-action topological dynamical system, where X is a compact metric space with metric d and G is a topological group acting continuously on X. Throughout this paper we assume that G is a countable infinite discrete amenable group unless otherwise specified. Recall that a countable discrete group G is *amenable* if there is a sequence of non-empty finite subsets $\{F_n\}$ of G which are asymptotically invariant, that is,

$$\lim_{n\to+\infty} \frac{|F_n \triangle gF_n|}{|F_n|} = 0 \quad \text{for all } g \in G.$$

Such sequences are called *Følner sequences*. One may refer to [14, 18] for more details on amenable groups and their actions. A Følner sequence $\{F_n\}$ in G is said to be *tempered* if there exists a constant $C > 0$ which is independent of n such that

$$\left| \bigcup_{k<n} F_k^{-1} F_n \right| \le C|F_n| \quad \text{for any } n. \tag{1.1}$$

Let $F(G)$ denote the collection of finite subsets of G. For $F \in F(G)$, let d_F be the metric defined by

$$d_F(x, y) = \max_{g\in F} d(gx, gy) \quad \text{for } x, y \in X.$$

Let $\varepsilon > 0$ and x in X; we denote

$$B_F(x, \varepsilon) = \{y \in X : d_F(x, y) < \varepsilon\}$$

and

$$\overline{B}_F(x, \varepsilon) = \{y \in X : d_F(x, y) \le \varepsilon\},$$

which are respectively the open and closed *(F-)Bowen balls* with center x and radius ε. When we want to clarify the underlying metric d, we also denote the above balls

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by $B_F(x, \varepsilon, d)$ and $\overline{B}_F(x, \varepsilon, d)$. For a $\mathbb{Z}$-action (or $\mathbb{N}$-action) topological dynamical system (X, T) (T is the homeomorphism (or the continuous onto map) on X), let $F = \{0, 1, \ldots, n-1\} := [0, n-1]$. The ($F$-)Bowen balls will be written as $B_n(x, \varepsilon, T)$ or $B_n(x, \varepsilon, d)$.

Definition 1.1. Let $\{F_n\}$ be a sequence of finite subsets of G with $|F_n| \to \infty$ (which need not be Følner). For $Z \subseteq X$, $s \geq 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$, define

$$P(Z, N, \varepsilon, s, \{F_n\}) = \sup \sum_i \exp(-s|F_{n_i}|),$$

where the supremum is taken over all finite or countable pairwise disjoint families $\{\overline{B}_{F_{n_i}}(x_i, \varepsilon)\}$ such that $x_i \in Z$, $n_i \geq N$ for all i. The quantity $P(Z, N, \varepsilon, s, \{F_n\})$ does not increase as N increases, hence the following limit exists:

$$P(Z, \varepsilon, s, \{F_n\}) = \lim_{N \to +\infty} P(Z, N, \varepsilon, s, \{F_n\}).$$

Define

$$\mathcal{P}(Z, \varepsilon, s, \{F_n\}) = \inf \left\{ \sum_{i=1}^{+\infty} P(Z_i, \varepsilon, s, \{F_n\}) : \bigcup_{i=1}^{+\infty} Z_i \supset Z \right\}.$$

It is easy to see that if $Z \subseteq \bigcup_{i=1}^{+\infty} Z_i$, then $\mathcal{P}(Z, \varepsilon, s, \{F_n\}) \leq \sum_{i=1}^{+\infty} \mathcal{P}(Z_i, \varepsilon, s, \{F_n\})$. There exists a critical value of the parameter s, which we will denote by $h^P_{\text{top}}(Z, \varepsilon, \{F_n\})$, where $\mathcal{P}(Z, \varepsilon, s, \{F_n\})$ jumps from $+\infty$ to 0, that is,

$$\mathcal{P}(Z, \varepsilon, s, \{F_n\}) = \begin{cases} 0, & s > h^P_{\text{top}}(Z, \varepsilon, \{F_n\}), \\ +\infty, & s < h^P_{\text{top}}(Z, \varepsilon, \{F_n\}). \end{cases}$$

It is not hard to see that $h^P_{\text{top}}(Z, \varepsilon, \{F_n\})$ increases when ε decreases. We call

$$h^P_{\text{top}}(Z, \{F_n\}) := \lim_{\varepsilon \to 0} h^P_{\text{top}}(Z, \varepsilon, \{F_n\})$$

the *amenable packing topological entropy* (*amenable packing entropy* or *packing entropy*, for short) of Z (with respect to the Følner sequence $\{F_n\}$).

Let $M(X)$ denote the collection of Borel probability measures on X.

Definition 1.2. Let $\{F_n\}$ be a sequence of finite subsets of G with $|F_n| \to \infty$. For $\mu \in M(X)$ and $Z \in \mathcal{B}(X)$ (the Borel σ-algebra on X), denote

$$\overline{h}^{\text{loc}}_\mu(Z, \{F_n\}) = \int_Z \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \varepsilon))\, d\mu$$

and

$$\underline{h}^{\text{loc}}_\mu(Z, \{F_n\}) = \int_Z \lim_{\varepsilon \to 0} \liminf_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \varepsilon))\, d\mu,$$

which are called the *upper local entropy* and the *lower local entropy* of μ over Z (with respect to $\{F_n\}$), respectively.

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In the $\mathbb{Z}$-action or $\mathbb{N}$-action case, if the sequence $\{F_n\}$ is chosen to be $F_n = [0, n-1]$ (hence $\{F_n\}$ is naturally a Følner sequence), the local entropies of a topological dynamical system (X, T) will be denoted by $\underline{h}_\mu^{\mathrm{loc}}(Z, T)$ and $\overline{h}_\mu^{\mathrm{loc}}(Z, T)$, respectively.

By the Brin–Katok entropy formula for amenable group actions (see [26]), if μ is in addition G-invariant and $\{F_n\}$ is a tempered Følner sequence which satisfies the growth condition

$$\lim_{n\to+\infty} \frac{|F_n|}{\log n} = +\infty, \tag{1.2}$$

then the values of the upper and lower local entropies over the whole space X coincide with the measure-theoretic entropy of the system (X, G).

We will prove the following variational principle between amenable packing entropy and upper local entropy.

THEOREM 1.3. *Let (X, G) be a G-action topological dynamical system and G a countable infinite discrete amenable group. Let $\{F_n\}$ be a sequence of finite subsets in G satisfying the growth condition (1.2). Then for any non-empty Borel subset Z of X,*

$$h_{\mathrm{top}}^P(Z, \{F_n\}) = \sup\{\overline{h}_\mu^{\mathrm{loc}}(Z, \{F_n\}) : \mu \in M(X), \mu(Z) = 1\}.$$

1.2. *Amenable packing entropy inequalities via factor maps.* Let (X, G) and (Y, G) be two G-action topological dynamical systems. A continuous map $\pi : (X, G) \to (Y, G)$ is called a homomorphism or a factor map from (X, G) to (Y, G) if it is onto and $\pi \circ g = g \circ \pi$, for all $g \in G$. We also say that (X, G) is an extension of (Y, G) or (Y, G) is a factor of (X, G).

For a subset Z of X, we denote by $h_{\mathrm{top}}^{UC}(Z, \{F_n\})$ the upper capacity topological entropy of Z (defined in §2). We will show in §2 that the packing entropy can be estimated via the upper capacity topological entropy with parameters (Proposition 2.7). This is a dynamical version of the fact that the packing dimension can be defined via the upper Minkowski dimension. Applying Proposition 2.7, we can prove the following packing entropy inequalities for factor maps.

THEOREM 1.4. *Let G be a countable infinite discrete amenable group and $\pi : (X, G) \to (Y, G)$ be a factor map between two G-action topological dynamical systems. Let $\{F_n\}$ be any tempered Følner sequence in G satisfying the growth condition (1.2). Then, for any Borel subset E of X,*

$$\begin{aligned} h_{\mathrm{top}}^P(\pi(E), \{F_n\}) \le h_{\mathrm{top}}^P(E, \{F_n\}) &\le h_{\mathrm{top}}^P(\pi(E), \{F_n\}) \\ &\quad + \sup_{y\in Y} h_{\mathrm{top}}^{UC}(\pi^{-1}(y), \{F_n\}). \end{aligned} \tag{1.3}$$

We remark here that for $\mathbb{Z}$-actions, the inequalities were proved in [24]. But for amenable group actions, except employing Bowen's idea in [2] and the quasi-tiling techniques developed by Ornstein and Weiss [18], we need a crucial covering lemma for amenable groups built by Lindenstrauss while proving pointwise theorems for amenable groups in [15].

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1.3. *Amenable packing entropy for certain subsets.* Let $M(X, G)$ and $E(X, G)$ be the collection of G-invariant and ergodic G-invariant Borel probability measures on X, respectively. Since G is amenable, $M(X, G)$ and $E(X, G)$ are both non-empty. For $\mu \in M(X, G)$, let $h_\mu(X, G)$ denote the measure-theoretic entropy of (X, G) with respect to μ.

For $\mu \in M(X, G)$ and a Følner sequence $\{F_n\}$ in G, let $G_{\mu,\{F_n\}}$ be the set of generic points for μ (with respect to $\{F_n\}$), which is defined by

$$G_{\mu,\{F_n\}} = \left\{x \in X : \lim_{n\to+\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(gx) = \int_X f \, d\mu, \text{ for any } f \in C(X)\right\}.$$

For simplicity, we write $G_{\mu,\{F_n\}}$ as G_μ when there is no ambiguity over $\{F_n\}$. But we should note that for different Følner sequence $\{F_n\}$, the corresponding G_μ may not coincide. When μ is ergodic and the Følner sequence $\{F_n\}$ is tempered, G_μ has full measure for μ (see Remark 5.1). But when μ is not ergodic, the set G_μ could be empty.

For the case $G = \mathbb{Z}$, Bowen [3] proved that the Bowen topological entropy of G_μ equals the measure-theoretic entropy of μ if μ is ergodic. Pfister and Sullivan [20] extended Bowen's result to the system with the so-called g-almost product property for invariant Borel probability measure μ. And the results for the amenable group action version were proved by Zheng and Chen [26] and Zhang [23]. The g-almost product property (see [20]) is an extension of the specification property for $\mathbb{Z}$-systems and was generalized to amenable systems in [23] (called the almost specification property there). It was shown in [23] that weak specification implies almost specification for amenable systems. We defer the detailed definitions of almost specification and weak specification to §5.

We will prove for packing entropy the following theorem.

THEOREM 1.5. *Let (X, G) be a G-action topological dynamical system with G a countable infinite discrete amenable group and let $\mu \in M(X, G)$ and $\{F_n\}$ be a Følner sequence in G satisfying the growth condition (1.2). If either μ is ergodic and $\{F_n\}$ is tempered or (X, G) satisfies the almost specification property, then*

$$h^P_{\text{top}}(G_\mu, \{F_n\}) = h_\mu(X, G). \tag{1.4}$$

In geometric measure theory, a set is said to be *regular* (or 'dimension-regular') if it has equal Hausdorff and packing dimensions [22]. As a counterpart in dynamical systems, we have the following definition.

Definition 1.6. A subset is said to be *regular in the sense of dimensional entropy* (or *regular* for short) if it has equal Bowen entropy and packing entropy.

An affirmative task in the study of packing entropy is to compute the dimensional entropies (including packing entropy and its dual, Bowen entropy) for various subsets and to make clear which subsets are regular. From Theorem 1.5 and results in [23, 26] for Bowen entropy, under the conditions in Theorem 1.5, the set G_μ is regular since both of its dimensional entropies equal the measure-theoretic entropy for μ.

1.4. *Organization of the paper.* In §2 we give some properties of amenable packing entropy including connections with Bowen entropy and upper capacity entropy. We devote

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§3 to the proof of Theorem 1.3, the variational principle between amenable packing entropy and upper local entropy. This extends Feng and Huang's result in [9] from $\mathbb{Z}$-actions to amenable group actions. In §4 we prove the packing entropy inequalities for factor maps (Theorem 1.4). Finally, in §5, we give the proof of Theorem 1.5 and provide some examples to discuss the regularity for certain subsets. These examples include subsets of symbolic dynamical system and fibers of the (T, T^{-1}) transformation. Some detailed computations on these examples are included in Appendix A.

2. *Properties of amenable packing entropy*

Due to the definition of packing entropy in §1, it is not hard to prove that the packing entropy has the following properties.

PROPOSITION 2.1. *Let $\{F_n\}$ be a sequence of finite subsets in G with $|F_n| \to \infty$, and let Z, Z' and $Z_i (i = 1, 2, \ldots)$ be subsets of X.*

(1) *If $Z \subseteq Z'$, then*

$$h^P_{\text{top}}(Z, \{F_n\}) \le h^P_{\text{top}}(Z', \{F_n\}).$$

(2) *If $Z \subseteq \bigcup_{i=1}^{+\infty} Z_i$, then for any $\varepsilon > 0$,*

$$h^P_{\text{top}}(Z, \varepsilon, \{F_n\}) \le \sup_{i \ge 1} h^P_{\text{top}}(Z_i, \varepsilon, \{F_n\}).$$

Hence

$$h^P_{\text{top}}(Z, \{F_n\}) \le \sup_{i \ge 1} h^P_{\text{top}}(Z_i, \{F_n\}).$$

(3) *If $\{F_{n_k}\}$ is a subsequence of $\{F_n\}$, then*

$$h^P_{\text{top}}(Z, \{F_{n_k}\}) \le h^P_{\text{top}}(Z, \{F_n\}).$$

In the following we recall the definition of amenable Bowen topological entropy which was introduced in [25].

Let $\{F_n\}$ be a sequence of finite subsets in G with $|F_n| \to \infty$. For $Z \subseteq X$, $s \ge 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$, define

$$\mathcal{M}(Z, N, \varepsilon, s, \{F_n\}) = \inf \sum_i \exp(-s|F_{n_i}|),$$

where the infimum is taken over all finite or countable families $\{B_{F_{n_i}}(x_i, \varepsilon)\}$ such that $x_i \in X$, $n_i \ge N$ and $\bigcup_i B_{F_{n_i}}(x_i, \varepsilon) \supseteq Z$. The quantity $\mathcal{M}(Z, N, \varepsilon, s, \{F_n\})$ does not decrease as N increases and ε decreases, hence the following limits exist:

$$\mathcal{M}(Z, \varepsilon, s, \{F_n\}) = \lim_{N \to +\infty} \mathcal{M}(Z, N, \varepsilon, s, \{F_n\})$$

and

$$\mathcal{M}(Z, s, \{F_n\}) = \lim_{\varepsilon \to 0} \mathcal{M}(Z, \varepsilon, s, \{F_n\}).$$

The *Bowen topological entropy* $h^B_{\text{top}}(Z, \{F_n\})$ is then defined as the critical value of the parameter s, where $\mathcal{M}(Z, s, \{F_n\})$ jumps from $+\infty$ to 0, that is,

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$$\mathcal{M}(Z, s, \{F_n\}) = \begin{cases} 0, & s > h^B_{\text{top}}(Z, \{F_n\}), \\ +\infty, & s < h^B_{\text{top}}(Z, \{F_n\}). \end{cases}$$

Next we will compare packing topological entropy with Bowen topological entropy.

PROPOSITION 2.2. *Let $\{F_n\}$ be a sequence of finite subsets in G with $|F_n| \to \infty$. For any $Z \subseteq X$,*

$$h^B_{\text{top}}(Z, \{F_n\}) \le h^P_{\text{top}}(Z, \{F_n\}).$$

Proof. The proof for the case $h^B_{\text{top}}(Z, \{F_n\}) = 0$ is obvious. Now assume $h^B_{\text{top}}(Z, \{F_n\}) > 0$ and let $0 < s < h^B_{\text{top}}(Z, \{F_n\})$.

Let $\{Z_i\}_{i=1}^{+\infty}$ be any covering of Z. For each i, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, let $\{\overline{B}_{F_n}(x_{i,j}, \varepsilon)\}_{j=1}^{N_i}$ be a disjoint subfamily of $\{\overline{B}_{F_n}(x, \varepsilon)\}_{x \in Z_i}$ with maximal cardinality N_i. Then

$$\bigcup_{i=1}^{N_i} B_{F_n}(x_{i,j}, 3\varepsilon) \supseteq Z_i.$$

So

$$\mathcal{M}(Z_i, n, 3\varepsilon, s, \{F_n\}) \le N_i e^{-|F_n|s} \le P(Z_i, n, \varepsilon, s, \{F_n\}),$$

and hence

$$\mathcal{M}(Z_i, 3\varepsilon, s, \{F_n\}) \le P(Z_i, \varepsilon, s, \{F_n\}).$$

Thus

$$\mathcal{M}(Z, 3\varepsilon, s, \{F_n\}) \le \sum_{i=1}^{+\infty} M(Z_i, 3\varepsilon, s, \{F_n\}) \le \sum_{i=1}^{+\infty} P(Z_i, \varepsilon, s, \{F_n\}),$$

from which we deduce that

$$\mathcal{M}(Z, 3\varepsilon, s, \{F_n\}) \le \mathcal{P}(Z, \varepsilon, s, \{F_n\}).$$

Since $s < h^B_{\text{top}}(Z, \{F_n\})$, we have $\mathcal{M}(Z, s, \{F_n\}) = +\infty$. So

$$1 \le \mathcal{M}(Z, 3\varepsilon, s, \{F_n\}) \le \mathcal{P}(Z, \varepsilon, s, \{F_n\})$$

whenever ε is sufficiently small. This implies that

$$h^P_{\text{top}}(Z, \varepsilon, \{F_n\}) \ge s.$$

Letting $\varepsilon \to 0$, we have $h^P_{\text{top}}(Z, \{F_n\}) \ge s$. Hence

$$h^B_{\text{top}}(Z, \{F_n\}) \le h^P_{\text{top}}(Z, \{F_n\}).$$

□

COROLLARY 2.3. *Let $\mu \in M(X, G)$, $Z \subseteq X$ with $\mu(Z) = 1$ and $\{F_n\}$ be a Følner sequence in G. Then*

$$h_\mu(X, G) \le h^P_{\text{top}}(Z, \{F_n\}).$$

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Proof. Let $\{F_{n_k}\}$ be a subsequence of $\{F_n\}$ which is tempered and satisfies the growth condition (1.2). By [26], $h_\mu(X, G) \le h^B_{\text{top}}(Z, \{F_{n_k}\})$. Together with Proposition 2.2 and (3) of Proposition 2.1, we have

$$h_\mu(X, G) \le h^P_{\text{top}}(Z, \{F_{n_k}\}) \le h^P_{\text{top}}(Z, \{F_n\}). \qquad \square$$

Let $\varepsilon > 0$, $Z \subseteq X$ and $F \in F(G)$. A subset $E \subseteq Z$ is said to be an (F, ε)-*separated set* of Z if, for any two distinct points $x, y \in E$, $d_F(x, y) > \varepsilon$. Let $s_F(Z, \varepsilon)$ denote the largest cardinality of (F, ε)-separated sets for Z. A subset $E \subseteq X$ is said to be an (F, ε)-*spanning set* of Z if, for any $x \in Z$, there exists $y \in E$ with $d_F(x, y) \le \varepsilon$. Let $r_F(Z, \varepsilon)$ (sometimes we use $r_F(Z, \varepsilon, d)$ to indicate the accompanied metric d) denote the smallest cardinality of (F, ε)-spanning sets for Z. For a sequence of finite subsets $\{F_n\}$ in G with $|F_n| \to \infty$, the *upper capacity topological entropy* of Z is defined as

$$h^{UC}_{\text{top}}(Z, \{F_n\}) = \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{|F_n|} \log s_{F_n}(Z, \varepsilon) = \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{|F_n|} \log r_{F_n}(Z, \varepsilon).$$

The second equality comes from the following simple fact:

$$r_{F_n}(Z, 2\varepsilon) \le s_{F_n}(Z, 2\varepsilon) \le r_{F_n}(Z, \varepsilon). \tag{2.1}$$

For convention we denote

$$h^{UC}_{\text{top}}(Z, \varepsilon, \{F_n\}) = \limsup_{n\to+\infty} \frac{1}{|F_n|} \log s_{F_n}(Z, \varepsilon),$$

and then

$$h^{UC}_{\text{top}}(Z, \{F_n\}) = \lim_{\varepsilon\to 0} h^{UC}_{\text{top}}(Z, \varepsilon, \{F_n\}).$$

We note here that for the case $Z = X$ and where $\{F_n\}$ is a Følner sequence, the quantity $h^{UC}_{\text{top}}(X, \{F_n\})$ coincides with $h_{\text{top}}(X, G)$, the topological entropy of (X, G).

PROPOSITION 2.4. *Let $\{F_n\}$ be a sequence of finite subsets in G satisfying the growth condition (1.2). Then for any subset Z of X and any $\varepsilon > 0$,*

$$h^P_{\text{top}}(Z, \varepsilon, \{F_n\}) \le h^{UC}_{\text{top}}(Z, \varepsilon, \{F_n\}).$$

Hence

$$h^P_{\text{top}}(Z, \{F_n\}) \le h^{UC}_{\text{top}}(Z, \{F_n\}).$$

Proof. Let $\varepsilon > 0$ be fixed. The proposition is obvious for the case $h^P_{\text{top}}(Z, \varepsilon, \{F_n\}) = 0$. Assume $h^P_{\text{top}}(Z, \varepsilon, \{F_n\}) > 0$ and let $0 < t < s < h^P_{\text{top}}(Z, \varepsilon, \{F_n\})$. Then

$$P(Z, \varepsilon, s, \{F_n\}) \ge \mathcal{P}(Z, \varepsilon, s, \{F_n\}) = +\infty.$$

Thus for any N, there exists a countable pairwise disjoint family $\{\overline{B}_{F_{n_i}}(x_i, \varepsilon)\}$ with $x_i \in Z$ and $n_i \ge N$ for all i such that $1 < \sum_i e^{-|F_{n_i}|s}$. For each k, let m_k be the number of i with $n_i = k$. Then we have

$$\sum_i e^{-|F_{n_i}|s} = \sum_{k\ge N} m_k e^{-|F_k|s}.$$

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Since $\{F_n\}$ satisfies the growth condition $\lim_{n\to+\infty}(|F_n|/\log n) = +\infty$, $\sum_{k\geq 1} e^{|F_k|(t-s)}$ converges. Let $M = \sum_{k\geq 1} e^{|F_k|(t-s)}$. There must be some $k \geq N$ such that $m_k > (1/M)e^{|F_k|t}$, otherwise the above sum is at most

$$\sum_{k\geq 1} \frac{1}{M} e^{|F_k|t} e^{-|F_k|s} = 1.$$

So $s_{F_k}(Z, \varepsilon) \geq m_k > (1/M)e^{|F_k|t}$ and hence

$$h^{UC}_{\text{top}}(Z, \varepsilon, \{F_n\}) = \limsup_{k\to+\infty} \frac{1}{|F_k|} \log s_{F_k}(Z, \varepsilon) \geq t,$$

from which we deduce that $h^{P}_{\text{top}}(Z, \varepsilon, \{F_n\}) \leq h^{UC}_{\text{top}}(Z, \varepsilon, \{F_n\})$. □

As a corollary, we have the following result.

COROLLARY 2.5. *If $\{F_n\}$ is a Følner sequence that satisfies the growth condition (1.2), then*

$$h^{P}_{\text{top}}(X, \{F_n\}) = h_{\text{top}}(X, G).$$

Proof. By Corollary 2.3, for any $\mu \in M(X, G)$, $h_\mu(X, G) \leq h^{P}_{\text{top}}(X, \{F_n\})$. Applying the variational principle for amenable topological entropy (cf. [17, 21]), we have $h^{P}_{\text{top}}(X, \{F_n\}) \geq h_{\text{top}}(X, G)$. By Proposition 2.4, we have $h^{P}_{\text{top}}(X, \{F_n\}) \leq h_{\text{top}}(X, G)$. □

Remark 2.6.

(1) By [8] (see also [25]), if $\{F_n\}$ is a tempered Følner sequence and satisfies the growth condition (1.2), then

$$h^{P}_{\text{top}}(X, \{F_n\}) = h_{\text{top}}(X, G) = h^{B}_{\text{top}}(X, \{F_n\}).$$

(2) If we replace X by a G-invariant compact subset, the above equality also holds and hence any G-invariant compact subset is regular (when the Følner sequence $\{F_n\}$ is tempered and satisfies the growth condition (1.2)).

At the end of this section we will give further relations between packing entropy and upper capacity topological entropy.

PROPOSITION 2.7. *Let $\varepsilon > 0$, Z be a subset of X and let $\{F_n\}$ be a sequence of finite subsets in G satisfying the growth condition (1.2).*

(1) *We have*

$$h^{P}_{\text{top}}(Z, \varepsilon, \{F_n\}) \leq \inf\Big\{ \sup_{i\geq 1} h^{UC}_{\text{top}}(Z_i, \varepsilon, \{F_n\}) : Z = \bigcup_{i=1}^{\infty} Z_i \Big\}.$$

Hence

$$h^{P}_{\text{top}}(Z, \{F_n\}) \leq \inf\Big\{ \sup_{i\geq 1} h^{UC}_{\text{top}}(Z_i, \{F_n\}) : Z = \bigcup_{i=1}^{\infty} Z_i \Big\}.$$

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(2) *For any $\delta > 0$, there exists a cover $\bigcup_{i=1}^{\infty} Z_i = Z$ (which depends on both ε and δ) such that*

$$h_{\mathrm{top}}^{P}(Z, \varepsilon, \{F_n\}) + \delta \geq \sup_{i\geq 1} h_{\mathrm{top}}^{UC}(Z_i, 3\varepsilon, \{F_n\}).$$

Proof. For any $Z = \bigcup_{i=1}^{\infty} Z_i$, by Propositions 2.1 and 2.4,

$$h_{\mathrm{top}}^{P}(Z, \varepsilon, \{F_n\}) \leq \sup_{i\geq 1} h_{\mathrm{top}}^{P}(Z_i, \varepsilon, \{F_n\}) \leq \sup_{i\geq 1} h_{\mathrm{top}}^{UC}(Z_i, \varepsilon, \{F_n\}).$$

Hence

$$h_{\mathrm{top}}^{P}(Z, \varepsilon, \{F_n\}) \leq \inf\Big\{ \sup_{i\geq 1} h_{\mathrm{top}}^{UC}(Z_i, \varepsilon, \{F_n\}) : Z = \bigcup_{i=1}^{\infty} Z_i \Big\}.$$

For the opposite direction, we may assume that $h_{\mathrm{top}}^{P}(Z, \varepsilon, \{F_n\}) < \infty$. Let $\delta > 0$ be fixed and set $s = h_{\mathrm{top}}^{P}(Z, \varepsilon, \{F_n\}) + \delta$. By the definition of the amenable packing entropy, we have that $\mathcal{P}(Z, \varepsilon, s, \{F_n\}) = 0$. Then there exists a cover $\bigcup_{i=1}^{\infty} Z_i \supseteq Z$ such that

$$\sum_{i\geq 1} P(Z_i, \varepsilon, s, \{F_n\}) < 1.$$

For each Z_i, when N is large enough, we have $P(Z_i, N, \varepsilon, s, \{F_n\}) < 1$. Let E be any $(F_N, 3\varepsilon)$-separated subset of Z_i. Noting that the closed Bowen balls $\overline{B}_{F_N}(x_i, \varepsilon)$ ($x_i \in E \subset Z_i$) are pairwise disjoint, we have

$$|E| e^{-s|F_N|} = \sum_{x_i \in E} e^{-s|F_N|} \leq P(Z_i, N, \varepsilon, s, \{F_n\}) < 1.$$

Hence $s_{F_N}(Z_i, 3\varepsilon) < e^{s|F_N|}$, which leads to $h_{\mathrm{top}}^{UC}(Z_i, 3\varepsilon, \{F_n\}) \leq s$. Thus we have

$$h_{\mathrm{top}}^{P}(Z, \varepsilon, \{F_n\}) + \delta \geq \sup_{i\geq 1} h_{\mathrm{top}}^{UC}(Z_i, 3\varepsilon, \{F_n\}).$$

□

Remark 2.8. Proposition 2.4 is motivated from the equivalent definition of packing dimension through Minkowski dimension in geometric measure theory, which was due to Tricot [22] (see also [16] for reference). It is unclear to us whether

$$h_{\mathrm{top}}^{P}(Z, \{F_n\}) = \inf\Big\{ \sup_{i\geq 1} h_{\mathrm{top}}^{UC}(Z_i, \{F_n\}) : Z = \bigcup_{i=1}^{\infty} Z_i \Big\}.$$

3. *A variational principle for amenable packing entropy*

In this section we will give the proof of our Theorem 1.3, the variational principle for amenable packing topological entropy. We assume in this section that $\{F_n\}$ is a sequence of finite subsets in G and satisfies growth condition (1.2).

3.1. *Lower bound.*

PROPOSITION 3.1. *Let $Z \subseteq X$ be a Borel set. Then*

$$h_{\mathrm{top}}^{P}(Z, \{F_n\}) \geq \sup\{\overline{h}_{\mu}^{\mathrm{loc}}(Z, \{F_n\}) : \mu \in M(X), \mu(Z) = 1\}.$$

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For the proof, we need the following classical $5r$-lemma in geometric measure theory (cf. [16, Theorem 2.1]).

LEMMA 3.2. ($5r$-lemma) *Let (X, d) be a compact metric space and $\mathcal{B} = \{B(x_i, r_i)\}_{i\in I}$ be a family of closed (or open) balls in X. Then there exists a finite or countable subfamily $\mathcal{B}' = \{B(x_i, r_i)\}_{i\in I'}$ of pairwise disjoint balls in $\mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{i\in I'} B(x_i, 5r_i).$$

We also need the following lemma, which comes directly from the definition of the packing entropy.

LEMMA 3.3. *Let $E \subset X$ and $s > 0$. Then for any $0 < \varepsilon_1 < \varepsilon_2$,*

$$P(\overline{E}, \varepsilon_2, s, \{F_n\}) \le P(E, \varepsilon_1, s, \{F_n\}).$$

Proof of Proposition 3.1. Let $\mu \in M(X)$ with $\mu(Z) = 1$ and assume $\overline{h}_\mu^{\mathrm{loc}}(Z, \{F_n\}) > 0$.

Let $0 < s < \overline{h}_\mu^{\mathrm{loc}}(Z, \{F_n\})$. Then there exist $\varepsilon, \delta > 0$ and a Borel set $A \subset Z$ with $\mu(A) > 0$ such that for every $x \in A$,

$$\overline{h}_\mu(x, \varepsilon, \{F_n\}) > s + \delta,$$

where $\overline{h}_\mu(x, \varepsilon, \{F_n\}) := \limsup_{n\to+\infty} -(1/|F_n|) \log \mu(B_{F_n}(x, \varepsilon))$.

Let $E \subset A$ be any Borel set with $\mu(E) > 0$. Define

$$E_n = \{x \in E : \mu(B_{F_n}(x, \varepsilon)) < e^{-|F_n|(s+\delta)}\}, \quad n \in \mathbb{N}.$$

Then $\bigcup_{n=N}^{+\infty} E_n = E$ for any $N \in \mathbb{N}$, and hence $\mu(\bigcup_{n=N}^{+\infty} E_n) = \mu(E)$. Fix $N \in \mathbb{N}$. There exists $n \ge N$ such that

$$\mu(E_n) \ge \frac{1}{n(n+1)}\mu(E).$$

Fix such n and consider the family of Bowen balls $\{B_{F_n}(x, \varepsilon/5) : x \in E_n\}$. By the $5r$-lemma, Lemma 3.2 (we use metric d_{F_n} instead of d), there exists a finite pairwise disjoint family $\{B_{F_n}(x_i, \varepsilon/5)\}$ with $x_i \in E_n$ such that

$$\bigcup_i B_{F_n}(x_i, \varepsilon) \supset \bigcup_{x\in E_n} B_{F_n}\left(x, \frac{\varepsilon}{5}\right) \supset E_n.$$

Thus

$$P\left(E, N, \frac{\varepsilon}{5}, s, \{F_n\}\right) \ge P\left(E_n, N, \frac{\varepsilon}{5}, s, \{F_n\}\right) \ge \sum_i e^{-|F_n|s} = e^{|F_n|\delta} \sum_i e^{-|F_n|(s+\delta)}$$
$$\ge e^{|F_n|\delta} \sum_i \mu(B_{F_n}(x_i, \varepsilon)) \ge e^{|F_n|\delta}\mu(E_n) \ge \frac{e^{|F_n|\delta}}{n(n+1)}\mu(E).$$

By the growth condition (1.2) of the sequence $\{F_n\}$, we have

$$\frac{e^{|F_n|\delta}}{n(n+1)} \to +\infty \quad \text{as } n \to +\infty.$$

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Letting $N \to +\infty$, we obtain that

$$P\left(E, \frac{\varepsilon}{5}, s, \{F_n\}\right) = +\infty.$$

Note that this equality holds for every Borel set $E \subset A$ with $\mu(E) > 0$.

Let $\{A_i\}_{i=1}^{\infty}$ be any covering of A. Then by Lemma 3.3,

$$\sum_i P\left(A_i, \frac{\varepsilon}{10}, s, \{F_n\}\right) \geq \sum_i P\left(\overline{A}_i, \frac{\varepsilon}{5}, s, \{F_n\}\right).$$

Since there must exist some A_i such that $\overline{A}_i \cap A$ (which is a Borel set now) contains a Borel subset $E \subset \overline{A}_i \cap A$ with $\mu(E) > 0$, we have

$$\sum_i P\left(A_i, \frac{\varepsilon}{10}, s, \{F_n\}\right) \geq P\left(E, \frac{\varepsilon}{5}, s, \{F_n\}\right) = +\infty.$$

Thus

$$\mathcal{P}\left(Z, \frac{\varepsilon}{10}, s, \{F_n\}\right) \geq \mathcal{P}\left(A, \frac{\varepsilon}{10}, s, \{F_n\}\right) = +\infty,$$

from which we deduce that

$$h_{\text{top}}^P(Z, \{F_n\}) \geq h_{\text{top}}^P\left(Z, \frac{\varepsilon}{10}, \{F_n\}\right) \geq s.$$

Since s is chosen arbitrarily in $(0, \overline{h}_\mu(Z, \{F_n\}))$, we finally show that

$$h_{\text{top}}^P(Z, \{F_n\}) \geq \overline{h}_\mu^{\text{loc}}(Z, \{F_n\}).$$

This finishes the proof of Proposition 3.1. □

3.2. *Upper bound.* The following proposition is the upper bound part of the variational principle. In fact it is valid for any analytic set Z. Recall that a set in a metric space is said to be *analytic* if it is a continuous image of the set $\mathcal{N}$ of infinite sequences of natural numbers. In a Polish space, the collection of analytic subsets contains Borel sets and is closed under countable unions and intersections (cf. [11]).

PROPOSITION 3.4. *Let $Z \subseteq X$ be an analytic set with $h_{\text{top}}^P(Z, \{F_n\}) > 0$. For any $0 < s < h_{\text{top}}^P(Z, \{F_n\})$, there exist a compact set $K \subseteq Z$ and $\mu \in M(K)$ such that $\overline{h}_\mu^{\text{loc}}(K, \{F_n\}) \geq s$.*

The following lemma is needed.

LEMMA 3.5. *Let $Z \subseteq X$ and $s, \varepsilon > 0$. Assume that $\mathcal{P}(Z, \varepsilon, s, \{F_n\}) = +\infty$. Then for any given finite interval $(a, b) \subset \mathbb{R}$ with $a \geq 0$ and any $N \in \mathbb{N}$, there exists a finite disjoint collection $\{\overline{B}_{F_{n_i}}(x_i, \varepsilon)\}$ such that $x_i \in Z$, $n_i \geq N$ and $\sum_i \exp(-|F_{n_i}|s) \in (a, b)$.*

Proof. See [9, Lemma 4.1]. □

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Proof of Proposition 3.4. Since Z is analytic, there exists a continuous surjective map $\phi : \mathcal{N} \to Z$. Let $\Gamma_{n_1,n_2,\ldots,n_p} = \{(m_1, m_2, \ldots) \in \mathcal{N} : m_1 \le n_1, m_2 \le n_2, \ldots, m_p \le n_p\}$ and let $Z_{n_1,\ldots,n_p} = \phi(\Gamma_{n_1,\ldots,n_p})$.

For $0 < s < h^P_{\mathrm{top}}(Z, \{F_n\})$, take $\varepsilon > 0$ small enough to make $0 < s < h^P_{\mathrm{top}}(Z, \varepsilon, \{F_n\})$ and take $t \in (s, h^P_{\mathrm{top}}(Z, \varepsilon, \{F_n\}))$. Following Feng and Huang's steps (which are inspired by the work of Joyce and Preiss [12] on packing measures), we will construct inductively the following data:

(D-1) a sequence of finite sets $(K_i)_{i=1}^{+\infty}$ with $K_i \subset Z$;
(D-2) a sequence of finite measures $(\mu_i)_{i=1}^{+\infty}$ with each μ_i supported on K_i;
(D-3) a sequence of integers $(n_i)_{i=1}^{+\infty}$ and a sequence of positive numbers $(\gamma_i)_{i=1}^{+\infty}$;
(D-4) a sequence of integer-valued functions $(m_i : K_i \to \mathbb{N})_{i=1}^{+\infty}$.

Moreover, the sequences (K_i), (μ_i), (n_i), (γ_i) and $(m_i(\cdot))$ will be constructed to satisfy the following conditions.

(C-1) For each i, the family $\mathcal{V}_i := \{B(x, \gamma_i)\}_{x \in K_i}$ is disjoint. Each element in $\mathcal{V}_{i+1}$ is a subset of $B(x, \gamma_i/2)$ for some $x \in K_i$.
(C-2) For each i, $K_i \subset Z_{n_1,\ldots,n_i}$ and $\mu_i = \sum_{y \in K_i} e^{-|F_{m_i(y)}|s} \delta_y$ with $1 < \mu_1(K_1) < 2$.
(C-3) For each $x \in K_i$ and $z \in B(x, \gamma_i)$,

$$\overline{B}_{F_{m_i(x)}}(z, \varepsilon) \cap \bigcup_{y \in K_i \setminus \{x\}} B(y, \gamma_i) = \emptyset \tag{3.1}$$

and

$$\mu_i(B(x, \gamma_i)) = e^{-|F_{m_i(x)}|s} \le \sum_{y \in E_{i+1}(x)} e^{-|F_{m_{i+1}(y)}|s} < (1 + 2^{-(i+1)})\mu_i(B(x, \gamma_i)), \tag{3.2}$$

where $E_{i+1}(x) = B(x, \gamma_i) \cap K_{i+1}$.

We will give the construction later.

Suppose the sequences (K_i), (μ_i), (n_i), (γ_i) and $(m_i(\cdot))$ have been constructed. By (3.2), for $V_i \in \mathcal{V}_i$,

$$\mu_i(V_i) \le \mu_{i+1}(V_i) = \sum_{V \in \mathcal{V}_{i+1}, V \subset V_i} \mu_{i+1}(V) \le (1 + 2^{-(i+1)})\mu_i(V_i).$$

Repeatedly using the above inequalities, we have for any $j > i$ and any $V_i \in \mathcal{V}_i$,

$$\mu_i(V_i) \le \mu_j(V_i) \le \prod_{n=i+1}^{j} (1 + 2^{-n})\mu_i(V_i) \le C\mu_i(V_i), \tag{3.3}$$

where $C := \prod_{n=1}^{+\infty}(1 + 2^{-n}) < +\infty$.

Let $\tilde{\mu}$ be a limit point of (μ_i) in the weak* topology. Let

$$K = \bigcap_{n=1}^{+\infty} \overline{\bigcup_{i \ge n} K_i}.$$

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Then $\tilde{\mu}$ is supported on K. Furthermore,

$$K \subset \bigcap_{p=1}^{+\infty} \overline{Z_{n_1,\dots,n_p}}.$$

By the continuity of ϕ, applying Cantor's diagonal argument, we can show that $\bigcap_{p=1}^{+\infty} Z_{n_1,\dots,n_p} = \bigcap_{p=1}^{+\infty} \overline{Z_{n_1,\dots,n_p}}$. Hence K is a compact subset of Z.

By (3.3), for any $x \in K_i$,

$$e^{-|F_{m_i(x)}|s} = \mu_i(B(x,\gamma_i)) \le \tilde{\mu}(B(x,\gamma_i)) \le C\mu_i(B(x,\gamma_i)) = Ce^{-|F_{m_i(x)}|s}.$$

In particular,

$$1 \le \sum_{x\in K_1} \mu_1(B(x,\gamma_1)) \le \tilde{\mu}(K) \le \sum_{x\in K_1} C\mu_1(B(x,\gamma_1)) \le 2C.$$

Note that $K \subset \bigcup_{x\in K_i} B(x,\gamma_i/2)$. By (3.1), the first part of (C-3), for each $x \in K_i$ and $z \in \overline{B}(x,\gamma_i)$, reads

$$\tilde{\mu}(\overline{B}_{F_{m_i(x)}}(z,\varepsilon)) \le \tilde{\mu}(\overline{B}(x,\gamma_i/2)) \le Ce^{-|F_{m_i(x)}|s}.$$

For each $z \in K$ and $i \in \mathbb{N}$, $z \in \overline{B}(x,\gamma_i/2)$ for some $x \in K_i$. Hence

$$\tilde{\mu}(B_{F_{m_i(x)}}(z,\varepsilon)) \le Ce^{-|F_{m_i(x)}|s}.$$

Define $\mu = \tilde{\mu}/\tilde{\mu}(K)$. Then $\mu \in M(K)$. For each $z \in K$, there exists a sequence $k_i \uparrow +\infty$ such that $\mu(B_{F_{k_i}}(z,\varepsilon)) \le Ce^{-|F_{k_i}|s}/\tilde{\mu}(K)$. Hence $\overline{h}^{\mathrm{loc}}_{\mu}(K,\{F_n\}) \ge s$.

Now the only thing left is to give the inductive construction of the data (K_i), (μ_i), (n_i), (γ_i) and $(m_i(\cdot))$. The inductive steps are as follows.

Step 1. Construct the data K_1, μ_1, n_1, γ_1 and $m_1(\cdot)$.

Since $t \in (s, h^P_{\mathrm{top}}(Z,\varepsilon,\{F_n\}))$, we have that $\mathcal{P}(Z,\varepsilon,t,\{F_n\}) = +\infty$. Let

$$H = \bigcup \{U \subset X : U \text{ is open}, \mathcal{P}(Z\cap U,\varepsilon,t,\{F_n\}) = 0\}.$$

Then by the separability of X, H is a countable union of the open sets U. Hence $\mathcal{P}(Z\cap H,\varepsilon,t,\{F_n\}) = 0$. Let $Z' = Z\setminus H = Z\cap(X\setminus H)$. If $\mathcal{P}(Z'\cap U,\varepsilon,t,\{F_n\}) = 0$ for some open set U, then

$$\mathcal{P}(Z\cap U,\varepsilon,t,\{F_n\}) \le \mathcal{P}(Z'\cap U,\varepsilon,t,\{F_n\}) + \mathcal{P}(Z\cap H,\varepsilon,t,\{F_n\}) = 0.$$

So $U \subset H$ and then $Z'\cap U = \emptyset$. Hence for any open set $U \subset X$, either $Z'\cap U = \emptyset$ or $\mathcal{P}(Z'\cap U,\varepsilon,t,\{F_n\}) > 0$.

Because $\mathcal{P}(Z,\varepsilon,t,\{F_n\}) \le \mathcal{P}(Z',\varepsilon,t,\{F_n\}) + \mathcal{P}(Z\cap H,\varepsilon,t,\{F_n\}) = \mathcal{P}(Z',\varepsilon,t,\{F_n\})$, we have $\mathcal{P}(Z',\varepsilon,t,\{F_n\}) = \mathcal{P}(Z,\varepsilon,t,\{F_n\}) = +\infty$. Then $\mathcal{P}(Z',\varepsilon,s,\{F_n\}) = +\infty$.

By Lemma 3.5, we can find a finite set $K_1 \subset Z'$, an integer-valued function $m_1(x)$ on K_1 such that the collection $\{\overline{B}_{F_{m_1(x)}}(x,\varepsilon)\}_{x\in K_1}$ is disjoint and

$$\sum_{x\in K_1} e^{-|F_{m_1(x)}|s} \in (1,2).$$

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Define $\mu_1 = \sum_{x\in K_1} e^{-|F_{m_1(x)}|s}\delta_x$, where δ_x denotes the Dirac measure at x. Take $\gamma_1 > 0$ sufficiently small such that for any function $z : K_1 \to X$ with $\max_{x\in K_1} d(x, z(x)) \le \gamma_1$, we have for each $x \in K_1$,

$$(\overline{B}(z(x), \gamma_1) \cup \overline{B}_{F_{m_1(x)}}(z(x), \varepsilon)) \cap \left(\bigcup_{y\in K_1\setminus\{x\}} \overline{B}(z(y), \gamma_1) \cup \overline{B}_{F_{m_1(y)}}(z(y), \varepsilon) \right) = \emptyset. \tag{3.4}$$

Since $K_1 \subset Z'$, we have $\mathcal{P}(Z \cap B(x, \gamma_1/4), \varepsilon, t, \{F_n\}) \ge \mathcal{P}(Z' \cap B(x, \gamma_1/4), \varepsilon, t, \{F_n\}) > 0$ for each $x \in K_1$. Therefore we can pick a sufficiently large $n_1 \in \mathbb{N}$ so that $Z_{n_1} \supset K_1$ and $\mathcal{P}(Z_{n_1} \cap B(x, \gamma_1/4), \varepsilon, t, \{F_n\}) > 0$ for each $x \in K_1$.

Step 2. Construct the data K_2, μ_2, n_2, γ_2 and $m_2(\cdot)$.

By (3.4), the family of balls $\{B(x, \gamma_1)\}_{x\in K_1}$ are pairwise disjoint. For each $x \in K_1$, since $\mathcal{P}(Z_{n_1} \cap B(x, \gamma_1/4), \varepsilon, t, \{F_n\}) > 0$, similarly to Step 1, we can construct a finite set $E_2(x) \subset Z_{n_1} \cap B(x, \gamma_1/4)$ and an integer-valued function $m_2 : E_2(x) \to \mathbb{N} \cap [\max\{m_1(y) : y \in K_1\}, +\infty)$ such that:

(2-a) for each open set U with $U \cap E_2(x) \ne \emptyset$, $\mathcal{P}(Z_{n_1} \cap U, \varepsilon, t, \{F_n\}) > 0$;

(2-b) the elements in the family $\{\overline{B}_{F_{m_2(y)}}(y, \varepsilon)\}_{y\in E_2(x)}$ are pairwise disjoint and

$$\mu_1(\{x\}) < \sum_{y\in E_2(x)} e^{-|F_{m_2(y)}|s} < (1 + 2^{-2})\mu_1(\{x\}).$$

To see this, we fix $x \in K_1$. Denote $V = Z_{n_1} \cap B(x, \gamma_1/4)$. Let

$$H_x := \bigcup \{U \subset X : U \text{ is open and } \mathcal{P}(V \cap U, \varepsilon, t, \{F_n\}) = 0\}.$$

Set $V' = V \setminus H_x$. Then as in Step 1, we can show that

$$\mathcal{P}(V', \varepsilon, t, \{F_n\}) = \mathcal{P}(V, \varepsilon, t, \{F_n\}) > 0.$$

Moreover, for any open set $U \subset X$, either $V' \cap U = \emptyset$ or $\mathcal{P}(V' \cap U, \varepsilon, t, \{F_n\}) > 0$. Since $s < t$, we have that $\mathcal{P}(V', \varepsilon, s, \{F_n\}) = +\infty$. By Lemma 3.5, we can find a finite set $E_2(x) \subset V'$ and a map $m_2 : E_2(x) \to \mathbb{N} \cap [\max\{m_1(y) : y \in K_1\}, +\infty)$ such that (2-b) holds. Notice that if an open set U satisfies $U \cap E_2(x) \ne \emptyset$, then $U \cap V' \ne \emptyset$. So $\mathcal{P}(Z_{n_1} \cap U, \varepsilon, t, \{F_n\}) \ge \mathcal{P}(V' \cap U, \varepsilon, t, \{F_n\}) > 0$. Hence (2-a) holds.

Since the family $\{B(x, \gamma_1)\}_{x\in K_1}$ is disjoint, so is the family $\{E_2(x)\}_{x\in K_1}$. Define

$$K_2 = \bigcup_{x\in K_1} E_2(x) \quad \text{and} \quad \mu_2 = \sum_{y\in K_2} e^{-|F_{m_2(y)}|s}\delta_y.$$

By (3.4) and (2-b), the elements in $\{\overline{B}_{F_{m_2(y)}}(y, \varepsilon)\}_{y\in K_2}$ are pairwise disjoint. Hence we can take $0 < \gamma_2 < \gamma_1/4$ such that for any function $z : K_2 \to X$ satisfying $\max_{x\in K_2} d(x, z(x)) < \gamma_2$, we have for $x \in K_2$,

$$(\overline{B}(z(x), \gamma_2) \cup \overline{B}_{F_{m_2(x)}}(z(x), \varepsilon)) \cap \left(\bigcup_{y\in K_2\setminus\{x\}} \overline{B}(z(y), \gamma_2) \cup \overline{B}_{F_{m_2(y)}}(z(y), \varepsilon) \right) = \emptyset. \tag{3.5}$$

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Choose a sufficiently large $n_2 \in \mathbb{N}$ so that $Z_{n_1,n_2} \supset K_2$ and

$$\mathcal{P}(Z_{n_1,n_2} \cap B(x, \gamma_2/4), \varepsilon, t, \{F_n\}) > 0$$

for each $x \in K_2$.

Step 3. Next we suppose that the data $K_i, \mu_i, n_i, \gamma_i$ and $m_i(\cdot)$ $(i = 1, \ldots, p)$ have been constructed. We will construct the data $K_{p+1}, \mu_{p+1}, n_{p+1}, \gamma_{p+1}$ and $m_{p+1}(\cdot)$.

Assume that we have constructed $K_i, \mu_i, n_i, \gamma_i$ and $m_i(\cdot)$ for $i = 1, \ldots, p$. And assume that for any function $z : K_p \to X$ with $d(x, z(x)) < \gamma_p$ for all $x \in K_p$, we have

$$(\overline{B}(z(x), \gamma_p) \cup \overline{B}_{F_{m_p(x)}}(z(x), \varepsilon)) \cap \left(\bigcup_{y \in K_p \setminus \{x\}} \overline{B}(z(y), \gamma_p) \cup \overline{B}_{F_{m_p(y)}}(z(y), \varepsilon) \right) = \emptyset \tag{3.6}$$

for each $x \in K_p$; and $Z_{n_1,\ldots,n_p} \supset K_p$ and $\mathcal{P}(Z_{n_1,\ldots,n_p} \cap B(x, \gamma_p/4), \varepsilon, t, \{F_n\}) > 0$ for each $x \in K_p$.

Note that the family of balls $\{\overline{B}(x, \gamma_p)\}_{x \in K_p}$ are pairwise disjoint. For each $x \in K_p$, since $\mathcal{P}(Z_{n_1,\ldots,n_p} \cap B(x, \gamma_p/4), \varepsilon, t, \{F_n\}) > 0$, similarly to Step 2, we can construct a finite set $E_{p+1}(x) \subset Z_{n_1,\ldots,n_p} \cap B(x, \gamma_p/4)$ and an integer-valued function $m_{p+1} : E_{p+1}(x) \to \mathbb{N} \cap [\max\{m_p(y) : y \in K_p\}, +\infty)$ such that:

(3-a) for each open set U with $U \cap E_{p+1}(x) \neq \emptyset$, $\mathcal{P}(Z_{n_1,\ldots,n_p} \cap U, \varepsilon, t, \{F_n\}) > 0$;

(3-b) the elements in the family $\{\overline{B}_{F_{m_{p+1}(y)}}(y, \varepsilon)\}_{y \in E_{p+1}(x)}$ are pairwise disjoint and

$$\mu_p(\{x\}) < \sum_{y \in E_{p+1}(x)} e^{-|F_{m_{p+1}(y)}|s} < (1 + 2^{-(p+1)})\mu_p(\{x\}).$$

By (3.6) and (3-b), the family $\{\overline{B}_{F_{m_{p+1}(y)}}(y, \varepsilon)\}_{y \in K_{p+1}}$ is disjoint. Hence we can take $0 < \gamma_{p+1} < \gamma_p/4$ such that for any function $z : K_{p+1} \to X$ with $\max_{x \in K_{p+1}} d(x, z(x)) < \gamma_{p+1}$, we have for each $x \in K_{p+1}$,

$$(\overline{B}(z(x), \gamma_{p+1}) \cup \overline{B}_{F_{m_{p+1}(x)}}(z(x), \varepsilon)) \cap \left(\bigcup_{y \in K_{p+1} \setminus \{x\}} \overline{B}(z(y), \gamma_{p+1}) \cup \overline{B}_{F_{m_{p+1}(y)}}(z(y), \varepsilon) \right) = \emptyset. \tag{3.7}$$

Choose a sufficiently large $n_{p+1} \in \mathbb{N}$ so that $Z_{n_1,\ldots,n_{p+1}} \supset K_{p+1}$ and $\mathcal{P}(Z_{n_1,\ldots,n_{p+1}} \cap B(x, \gamma_{p+1}/4), \varepsilon, t, \{F_n\}) > 0$ for each $x \in K_{p+1}$.

Then we finish the required construction and complete the proof of the proposition. □

4. *Packing entropy inequalities for factor maps*

In this section we will prove Theorem 1.4. The proof is a combination of Bowen's method in [2] and quasi-tiling techniques for amenable groups.

4.1. *Preliminaries for amenable groups.*

Let G be a countable infinite discrete amenable group. Let A and K be two non-empty finite subsets of G. Recall that $B(A, K)$, the

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K-boundary of A, is defined by

$$B(A, K) = \{g \in G : Kg \cap A \neq \emptyset \text{ and } Kg \cap (G \setminus A) \neq \emptyset\}.$$

For $\delta > 0$, the set A is said to be (K, δ)-*invariant* if

$$\frac{|B(A, K)|}{|A|} < \delta.$$

We say a sequence of non-empty finite subsets $\{F_n\}$ of G becomes more and more invariant if, for any $\delta > 0$ and any non-empty finite subset K of G, F_n is (K, δ)-invariant for sufficiently large n. An equivalent condition for the sequence $\{F_n\}$ to be a Følner sequence is that $\{F_n\}$ becomes more and more invariant (see [18]).

Let $\tilde{\mathcal{F}}$ be a collection of finite subsets of G. It is said to be δ-disjoint if for every $A \in \tilde{\mathcal{F}}$ there exists an $A' \subset A$ such that $|A'| \geq (1 - \delta)|A|$ and such that $A' \cap B' = \emptyset$ for every $A \neq B \in \tilde{\mathcal{F}}$.

The following is a covering lemma for amenable groups by Lindenstrauss.

LEMMA 4.1. (Lindenstrauss's covering lemma, [15, Corollary 2.7]) *For any $\delta \in (0, 1/100)$, $C > 0$ and finite $D \subset G$, let $M \in \mathbb{N}$ be sufficiently large (depending only on δ, C and D). Let $F_{i,j}$ be an array of finite subsets of G where $i = 1, \ldots, M$ and $j = 1, \ldots, N_i$, with the following two requirements.*

(1) *For every i, $\bar{F}_{i,*} = \{F_{i,j}\}_{j=1}^{N_i}$ satisfies*

$$\left| \bigcup_{k'<k} F_{i,k'}^{-1} F_{i,k} \right| \leq C|F_{i,k}| \quad \text{for } k = 2, \ldots, N_i.$$

(2) *Denote $F_{i,*} = \bigcup \bar{F}_{i,*}$. The finite set sequences $F_{i,*}$ satisfy that for every $1 < i \leq M$ and every $1 \leq k \leq N_i$,*

$$\left| \bigcup_{i'<i} D F_{i',*}^{-1} F_{i,k} \right| \leq (1 + \delta)|F_{i,k}|.$$

Assume that $A_{i,j}$ is another array of finite subsets of G with $F_{i,j} A_{i,j} \subset F$ for some finite subset F of G. Let $A_{i,*} = \bigcup_j A_{i,j}$ and

$$\alpha = \frac{\min_{1 \leq i \leq M} |D A_{i,*}|}{|F|}.$$

Then the collection of subsets of F,

$$\tilde{\mathcal{F}} = \{F_{i,j} a : 1 \leq i \leq M, 1 \leq j \leq N_i \text{ and } a \in A_{i,j}\},$$

has a subcollection $\mathcal{F}$ that is $10\delta^{1/4}$-disjoint such that

$$\left| \bigcup \mathcal{F} \right| \geq (\alpha - \delta^{1/4})|F|.$$

4.2. *Proof of Theorem 1.4.* Let $G, \{F_n\}$ and $\pi : (X, G) \to (Y, G)$ be as in Theorem 1.4 and $E \subset X$ be a Borel set. Let d and ρ be the compatible metrics on X and Y, respectively.

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For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x_1, x_2 \in X$ with $d(x_1, x_2) \le \delta$, one has $\rho(\pi(x_1), \pi(x_2)) \le \varepsilon$. Now let $\{y_i\}_{i=1}^k \subset \pi(E)$ be any (F_n, ε)-separated set of $\pi(E)$ and choose for each i a point $x_i \in \pi^{-1}(y_i) \cap E$. Then $\{x_i\}_{i=1}^k$ forms an (F_n, δ)-separated set of E. Hence

$$s_{F_n}(\pi(E), \varepsilon) \le s_{F_n}(E, \delta)$$

and

$$h_{\text{top}}^{UC}(\pi(E), \varepsilon, \{F_n\}) \le h_{\text{top}}^{UC}(E, \delta, \{F_n\}). \tag{4.1}$$

By Proposition 2.7(2), for any $\eta > 0$, there exists a cover $\bigcup_{i=1}^{\infty} E_i = E$ such that

$$h_{\text{top}}^{P}(E, \delta/3, \{F_n\}) + \eta \ge \sup_{i \ge 1} h_{\text{top}}^{UC}(E_i, \delta, \{F_n\}).$$

Then we have

$$\begin{aligned} h_{\text{top}}^{P}(\pi(E), \varepsilon, \{F_n\}) &\le \sup_{i \ge 1} h_{\text{top}}^{P}(\pi(E_i), \varepsilon, \{F_n\}) \quad \text{(by Proposition 2.1(2))} \\ &\le \sup_{i \ge 1} h_{\text{top}}^{UC}(\pi(E_i), \varepsilon, \{F_n\}) \quad \text{(by Proposition 2.4)} \\ &\le \sup_{i \ge 1} h_{\text{top}}^{UC}(E_i, \delta, \{F_n\}) \quad \text{(noting that 4 also holds for each } E_i) \\ &\le h_{\text{top}}^{P}(E, \delta/3, \{F_n\}) + \eta, \end{aligned}$$

which implies that

$$h_{\text{top}}^{P}(\pi(E), \{F_n\}) \le h_{\text{top}}^{P}(E, \{F_n\}).$$

To get the upper bound, we need to prove the following inequality for amenable upper capacity topological entropy, which extends a result by Bowen [**2**, Theorem 17] to amenable group actions.

THEOREM 4.2. *Let G be a countable infinite discrete amenable group and $\pi : (X, G) \to (Y, G)$ be a factor map between two G-action topological dynamical systems. Let $\{F_n\}$ be any tempered Følner sequence in G satisfying growth condition (1.2). Then for any subset E of X,*

$$h_{\text{top}}^{UC}(E, \{F_n\}) \le h_{\text{top}}^{UC}(\pi(E), \{F_n\}) + \sup_{y \in Y} h_{\text{top}}^{UC}(\pi^{-1}(y), \{F_n\}). \tag{4.2}$$

Proof. If $\sup_{y \in Y} h_{\text{top}}^{UC}(\pi^{-1}(y), \{F_n\}) = \infty$ then there is nothing to prove. So we assume that

$$a := \sup_{y \in Y} h_{\text{top}}^{UC}(\pi^{-1}(y), \{F_n\}) < \infty.$$

To verify (4.2), we need some preparations in the following three steps.

Step 1. Construct $F_{i,j}$, the array of subsets of G.

Fix $\tau > 0$. For any $\varepsilon > 0$, let $0 < \delta < \min\{\varepsilon, 1/100\}$ be small enough. Let $C > 0$ be the constant in the tempered condition (1.1) for the Følner sequence $\{F_n\}$ and let $D = \{e_G\} \subset G$, where e_G is the identity element of G. Let $M > 0$ be large enough as in Lemma 4.1 corresponding to δ, C and D.

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For each $y \in Y$, choose $m(y) \in \mathbb{N}$ such that for any $n \geq m(y)$,

$$\frac{1}{|F_n|} \log r_{F_n}(\pi^{-1}(y), \varepsilon, d) \leq h^{UC}_{\text{top}}(\pi^{-1}(y), \{F_n\}) + \tau \leq a + \tau. \tag{4.3}$$

Here recall that $r_{F_n}(\pi^{-1}(y), \varepsilon, d)$ denotes the smallest cardinality of (F_n, ε)-spanning sets for $\pi^{-1}(y)$. Let E_y be an $(F_{m(y)}, \varepsilon)$-spanning set of $\pi^{-1}(y)$ with the smallest cardinality $|E_y| = r_{F_{m(y)}}(\pi^{-1}(y), \varepsilon, d)$. Denote

$$U_y = \{p \in X : \text{ there exists } q \in E_y \text{ such that } d_{F_{m(y)}}(p, q) < 2\varepsilon\},$$

which is an open neighborhood of $\pi^{-1}(y)$. Since $\bigcap_{\gamma>0} \pi^{-1}(\overline{B(y, \gamma, \rho)}) = \pi^{-1}(y)$, we have $(X \setminus U_y) \bigcap (\bigcap_{\gamma>0} \pi^{-1}(\overline{B(y, \gamma, \rho)})) = \emptyset$. Hence by the finite intersection property of compact sets, there is a $W_y := B(y, \gamma_y, \rho)$ for some $\gamma_y > 0$ such that $U_y \supset \pi^{-1}(W_y)$. Since Y is compact, there exist $y_{1,1}, \ldots, y_{1,r_1}$ such that $W_{y_{1,1}}, \ldots, W_{y_{1,r_1}}$ cover Y. List the Følner sets in the collection $\{F_{m(y_{1,k})} : 1 \leq k \leq r_1\}$ by

$$F_{n_{1,1}}, F_{n_{1,2}}, \ldots, F_{n_{1,N_1}} \quad \text{where } n_{1,1} < n_{1,2} < \cdots < n_{1,N_1}.$$

Note that $N_1 = \#\{F_{m(y_{1,k})} : 1 \leq k \leq r_1\} \leq r_1$.

For each $y \in Y$, choose $m(y) > n_{1,N_1}$ such that (4.3) holds for any $n \geq m(y)$. Repeating the above process, we can obtain $y_{2,1}, \ldots, y_{2,r_2} \in Y$ such that $W_{y_{2,1}}, \ldots, W_{y_{2,r_2}}$ cover Y. We then list the Følner sets in the collection $\{F_{m(y_{2,k})} : 1 \leq k \leq r_2\}$:

$$F_{n_{2,1}}, F_{n_{2,2}}, \ldots, F_{n_{2,N_2}} \quad \text{where } n_{2,1} < n_{2,2} < \cdots < n_{2,N_2}.$$

Note that $N_2 = \#\{F_{m(y_{2,k})} : 1 \leq k \leq r_2\} \leq r_2$.

Repeating the above steps inductively, we can obtain, for each $1 \leq i \leq M$:

(1) a collection of points $y_{i,1}, \ldots, y_{i,r_i} \in Y$ such that $W_{y_{i,1}}, \ldots, W_{y_{i,r_i}}$ cover Y;
(2) a collection of Følner sets $\{F_{n_{i,1}}, F_{n_{2,2}}, \ldots, F_{n_{i,N_i}}\}$ $(= \{F_{m(y_{i,k})} : 1 \leq k \leq r_i\})$ with $n_{i,1} < n_{i,2} < \cdots < n_{i,N_i}$ and $N_i \leq r_i$.

From the above construction, for each $1 \leq i \leq M-1$, we have $n_{i,N_i} < n_{i+1,1}$. Moreover, $n_{i+1,1}$ can be chosen sufficiently large compared with n_{i,N_i} such that, for every $1 < i \leq M$ and every $1 \leq k \leq N_i$,

$$\left| \bigcup_{i'<i} F^{-1}_{n_{i',*}} F_{n_i,k} \right| \leq (1+\delta)|F_{n_i,k}|. \tag{4.4}$$

For simplification, we denote $F_{i,j} = F_{n_{i,j}}$ for each $1 \leq i \leq M$ and $1 \leq j \leq N_i$, which is the array of G we required.

Step 2. Produce quasi-tilings from $F_{i,j}$.

Let η_1 be a common Lebesgue number of the family of open covers $\{W_{y_{i,1}}, \ldots, W_{y_{i,r_i}}\}$ with respect to the metric ρ. Denote $\eta = (\eta_1/2)$.

Let N be large enough such that, for every $n > N$, F_n is $(F_{i,*} \cup \{e_G\}, \delta)$-invariant for all $1 \leq i \leq M$.

For each $y \in Y$ and $n > N$, let

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$$A_{i,j} = \{a \in F_n : F_{i,j}a \subset F_n \text{ and there exists } 1 \le k \le r_i \text{ such that } F_{m(y_{i,k})} = F_{i,j} \text{ and } \overline{B}(ay, \eta, \rho) \subseteq W_{y_{i,k}}\}.$$

We note here that $A_{i,j}$ depends on y.

For any $g \in F_n \setminus B(F_n, F_{i,*} \cup \{e_G\})$, we have $F_{i,*}g \subset F_n$. Since $\eta_1 (= 2\eta)$ is a Lebesgue number of $\{W_{y_{i,1}}, \ldots, W_{y_{i,r_i}}\}$, $\overline{B}(gy, \eta, \rho)$ is contained in some $W_{y_{i,k}}$ and then $g \in A_{i,*}$. Hence

$$A_{i,*} \supseteq F_n \setminus B(F_n, F_{i,*} \cup \{e_G\})$$

for each $1 \le i \le M$, and

$$\alpha = \frac{\min_{1\le i\le M} |DA_{i,*}|}{|F_n|} = \frac{\min_{1\le i\le M} |A_{i,*}|}{|F_n|} > 1 - \delta.$$

We are now able to apply Lemma 4.1: the temperedness assumption for $\{F_n\}$ makes requirement (1) of that lemma hold and (4.4) makes requirement (2) hold. From the collection of subsets of F_n,

$$\tilde{\mathcal{F}} = \{F_{i,j}a : 1 \le i \le M, 1 \le j \le N_i \text{ and } a \in A_{i,j}\},$$

we can find by Lemma 4.1 a subcollection $\mathcal{F}$ which is $10\delta^{1/4}$-disjoint and

$$\left|\bigcup \mathcal{F}\right| \ge (\alpha - \delta^{1/4})|F_n| \ge (1 - \delta - \delta^{1/4})|F_n|. \tag{4.5}$$

In fact, this subcollection $\mathcal{F}$ is a $10\delta^{1/4}$ quasi-tiling of F_n subordinate to $y \in Y$ (see, for example, [14] for the detail definition of quasi-tiling).

There may exist overlaps between elements in $\mathcal{F}$. Since $\mathcal{F}$ is $10\delta^{1/4}$-disjoint, there exits $T' \subset T$ for each $T \in \mathcal{F}$ such that $|T'|/|T| \ge 1 - 10\delta^{1/4}$ and the collection $\{T' : T \in \mathcal{F}\}$ is disjoint. Denote this new collection by $\mathcal{F}'$. By (4.5),

$$\left|\bigcup \mathcal{F}\right| \le \sum_{T\in\mathcal{F}} |T| \le \frac{1}{1 - 10\delta^{1/4}} \sum_{T\in\mathcal{F}} |T'| \le \frac{1}{1 - 10\delta^{1/4}}|F_n| \tag{4.6}$$

and

$$\left|\bigcup \mathcal{F}'\right| = \sum_{T\in\mathcal{F}} |T'| \ge (1 - 10\delta^{1/4})(1 - \delta - \delta^{1/4})|F_n|. \tag{4.7}$$

Step 3. Cover $\pi^{-1}B_{F_n}(y, \eta, \rho)$ through F_n-Bowen balls in (X, G).

CLAIM. *When δ is sufficiently small, for any $y \in Y$ and $n > N$, there exist $l(y) > 0$ and $v_1(y), v_2(y), \ldots, v_{l(y)}(y) \in X$ such that*

$$\bigcup_{i=1}^{l(y)} B_{F_n}(v_i(y), 4\varepsilon, d) \supseteq \pi^{-1}(B_{F_n}(y, \eta, \rho))$$

and

$$l(y) \le \exp((a + 2\tau)|F_n|).$$

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Proof of the Claim. For each $T = F_{i,j}a \in \mathcal{F}$, since $a \in A_{i,j}$, by the construction of $A_{i,j}$, there exists some point in $\{y_{i,1}, y_{i,2}, \ldots, y_{i,r_i}\}$, denoted by y_T, such that $\overline{B}(ay, \eta, \rho) \subseteq W_{y_T}$ and $F_{m(y_T)} = F_{i,j}$.

In the following we will recover the F_n-orbits of (X, G) from the T-orbits.

Let $E \subset X$ be any finite ε-spanning set under the metric d. For any sequence of points $\{z_T\}_{T\in\mathcal{F}}$ with each $z_T \in E_{y_T}$ and any sequence of points $\{z_g\}_{g\in F_n\setminus\cup\mathcal{F}'}$ with each $z_g \in E$, let

$$V(y; \{z_T\}, \{z_g\}) := \Big\{u \in X : d_{T'}(u, a^{-1}z_T) < 2\varepsilon \text{ for all } T = F_{i,j}a \in \mathcal{F},\\ d(gu, z_g) < 2\varepsilon \text{ for all } g \in F_n \setminus \bigcup \mathcal{F}'\Big\}.$$

It is not hard to verify that

$$\bigcup_{\{z_T\},\{z_g\}} V(y; \{z_T\}, \{z_g\}) \supseteq \pi^{-1}(B_{F_n}(y, \eta, \rho)),$$

that is, the family $\{V(y; \{z_T\}, \{z_g\}) : z_T \in E_{y_T}, T \in \mathcal{F}, z_g \in E, g \in F_n \setminus \bigcup \mathcal{F}'\}$ forms an open cover of $\pi^{-1}(B_{F_n}(y, \eta, \rho))$. We also note that some of the $V(y; \{z_T\}, \{z_g\})$ may be empty.

We pick any point $v(\{z_T\}, \{z_g\})$ in each non-empty $V(y; \{z_T\}, \{z_g\})$. Then

$$B_{F_n}(v(\{z_T\}, \{z_g\}), 4\varepsilon, d) \supseteq V(y; \{z_T\}, \{z_g\}).$$

Enumerate these $v(\{z_T\}, \{z_g\})$ by $y_1, y_2, \ldots, y_{l(y)}$. We then obtain

$$\bigcup_{i=1}^{l(y)} B_{F_n}(y_i, 4\varepsilon, d) \supseteq \pi^{-1}(B_{F_n}(y, \eta, \rho)). \tag{4.8}$$

Now the only thing left is to estimate $l(y)$. Clearly,

$$\begin{aligned} l(y) &\le \prod_{T\in\mathcal{F}} |E_{y_T}| \cdot \prod_{g\in F_n\setminus\cup\mathcal{F}'} |E| = \prod_{T\in\mathcal{F}} r_{F_{m(y_T)}}(\pi^{-1}(y), \varepsilon, d) \cdot |E|^{|F_n|-|\bigcup\mathcal{F}'|} \\ &\le \exp\left(\sum_{T\in\mathcal{F}} |F_{m(y_T)}|(a+\tau) + \left(|F_n| - \left|\bigcup\mathcal{F}'\right|\right)\log|E|\right) \\ &\le \exp\left(\left(\frac{1}{1-10\delta^{1/4}}(a+\tau) + (1-(1-10\delta^{1/4})(1-\delta-\delta^{1/4}))\log|E|\right)|F_n|\right) \\ &\qquad\qquad \text{(by (4.6) and (4.7))} \\ &\le \exp((a+2\tau)|F_n|) \quad \text{(when } \delta \text{ is sufficiently small).} \end{aligned}$$

□

We now proceed to prove (4.2). For any subset E of X, let H be an (F_n, η)-spanning set of $\pi(E)$ with minimal cardinality $r_{F_n}(\pi(E), \eta, \rho)$. Then by the above claim, the set $R = \{v_i(y) : 1 \le i \le l(y), y \in H\}$ forms an $(F_n, 4\varepsilon)$-spanning set of E, since

$$\bigcup_{y\in H}\bigcup_{i=1}^{l(y)} B_{F_n}(v_i(y), 4\varepsilon, d) \supseteq \bigcup_{y\in H} \pi^{-1}(B_{F_n}(y, \eta, \rho)) \supseteq \pi^{-1}\pi(E) \supseteq E.$$

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Hence

$$r_{F_n}(E, 4\varepsilon, d) \le r_{F_n}(\pi(E), \eta, \rho) \cdot \exp((a + 2\tau)|F_n|). \tag{4.9}$$

From this we deduce that

$$h^{UC}_{\text{top}}(E, \{F_n\}) \le h^{UC}_{\text{top}}(\pi(E), \{F_n\}) + a + 2\tau.$$

Letting τ tend to 0, (4.2) is proved. □

Proof of the upper bound of $h^P_{\text{top}}(E, \{F_n\})$. What we need in fact is inequality (4.9) in the proof of Theorem 4.2.

From (2.1), we first convert inequality (4.9) into

$$s_{F_n}(E, 8\varepsilon, d) \le s_{F_n}(\pi(E), \eta, \rho) \cdot \exp((a + 2\tau)|F_n|).$$

This implies that

$$h^{UC}_{\text{top}}(E, 8\varepsilon, \{F_n\}) \le h^{UC}_{\text{top}}(\pi(E), \eta, \{F_n\}) + a + 2\tau.$$

By Proposition 2.7(2) again, for any $\delta > 0$, there exists a cover $\bigcup_{i=1}^{\infty} V_i = \pi(E)$ such that

$$h^P_{\text{top}}(\pi(E), \eta/3, \{F_n\}) + \delta \ge \sup_{i\ge1} h^{UC}_{\text{top}}(V_i, \eta, \{F_n\}).$$

Using an similar argument to that in the proof of the lower bound,

$$\begin{aligned} h^P_{\text{top}}(E, 8\varepsilon, \{F_n\}) &\le \sup_{i\ge1} h^P_{\text{top}}(\pi^{-1}(V_i), 8\varepsilon, \{F_n\}) \le \sup_{i\ge1} h^{UC}_{\text{top}}(\pi^{-1}(V_i), 8\varepsilon, \{F_n\}) \\ &\le \sup_{i\ge1} h^{UC}_{\text{top}}(V_i, \eta, \{F_n\}) + a + 2\tau \\ &\le h^P_{\text{top}}(\pi(E), \eta/3, \{F_n\}) + \delta + a + 2\tau. \end{aligned}$$

Hence

$$h^P_{\text{top}}(E, \{F_n\}) \le h^P_{\text{top}}(\pi(E), \{F_n\}) + a + 2\tau.$$

Since $\tau > 0$ is arbitrary, we finally obtain

$$h^P_{\text{top}}(E, \{F_n\}) \le h^P_{\text{top}}(\pi(E), \{F_n\}) + \sup_{y\in Y} h^{UC}_{\text{top}}(\pi^{-1}(y), \{F_n\}).$$ □

5. *Amenable packing entropy for certain subsets*

5.1. *The set of generic points.* Recall that for $\mu \in M(X, G)$ and a Følner sequence $\{F_n\}$ in G, the set of generic points for μ (with respect to $\{F_n\}$) is defined by

$$G_\mu = \left\{ x \in X : \lim_{n\to+\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(gx) = \int_X f \, d\mu, \text{ for any } f \in C(X) \right\}.$$

Remark 5.1. If $\mu \in E(X, G)$ and $\{F_n\}$ is a tempered Følner sequence then $\mu(G_\mu) = 1$. To show this, let $\{f_i\}_{i=1}^{\infty}$ be a countable dense subset of $C(X)$ and denote

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$$X_i = \left\{ x \in X : \lim_{n\to+\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f_i(gx) = \int_X f_i \, d\mu \right\}.$$

By the pointwise ergodic theorem, $\mu(X_i) = 1$. Hence $G_\mu = \bigcap_{i=1}^{\infty} X_i$ has full measure.

The system (X, G) is said to have the *almost specification* property if there exists a non-decreasing function $g : (0, 1) \to (0, 1)$ with $\lim_{r\to 0} g(r) = 0$ (a mistake-density function) and a map $m : (0, 1) \to F(G) \times (0, 1)$ such that for any $k \in \mathbb{N}$, any $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in (0, 1)$, and any $x_1, x_2, \ldots, x_k \in X$, if F_i is $m(\varepsilon_i)$-invariant, $i = 1, 2, \ldots, k$, and $\{F_i\}_{i=1}^k$ are pairwise disjoint, then

$$\bigcap_{1\le i\le m} B(g; F_i, x_i, \varepsilon_i) \neq \emptyset,$$

where $B(g; F, x, \varepsilon) := \{y \in X : |\{h \in F : d(hx, hy) > \varepsilon\}| \le g(\varepsilon)|F|\}$, the Bowen ball allowing a mistake with density $g(\varepsilon)$.

Remark 5.2. It was shown in [23] that the weak specification implies the almost specification. Recall that in [6] the system (X, G) (the group G need not be amenable) has *weak specification* if for any $\varepsilon > 0$ there exists a non-empty finite subset F of G with the following property: for any finite collection $F_1, \ldots, F_m$ of finite subsets G with

$$FF_i \cap F_j = \emptyset \quad \text{for } 1 \le i, j \le m, i \neq j,$$

and for any collection of points $x_1, \ldots, x_m \in X$, there exists a point $y \in X$ such that

$$d(gx_i, gy) \le \varepsilon \quad \text{for all } g \in F_i, 1 \le i \le m,$$

that is,

$$\bigcap_{1\le i\le m} \overline{B}_{F_i}(x_i, \varepsilon) \neq \emptyset.$$

In this section, we will prove Theorem 1.5, that is, if $\mu \in M(X, G)$, the Følner sequence $\{F_n\}$ satisfies the growth condition (1.2) and either μ is ergodic and $\{F_n\}$ is tempered or (X, G) has almost specification, then

$$h^P_{\text{top}}(G_\mu, \{F_n\}) = h_\mu(X, G). \tag{5.1}$$

The idea of the proof comes from Pfister and Sullivan [20] (see also [23, 26] for amenable group actions).

5.1.1. *Upper bound for* $h^P_{\text{top}}(G_\mu, \{F_n\})$. In the following we are going to prove $h^P_{\text{top}}(G_\mu, \{F_n\}) \le h_\mu(X, G)$ assuming that the Følner sequence $\{F_n\}$ satisfies the growth condition (1.2).

For $\mu \in M(X, G)$, let $\{K_m\}_{m\in\mathbb{N}}$ be a decreasing sequence of closed convex neighborhoods of μ in $M(X)$ such that $\bigcap_{m\in\mathbb{N}} K_m = \{\mu\}$. Let

$$A_{n,m} = \left\{ x \in X : \frac{1}{|F_n|} \sum_{g\in F_n} \delta_x \circ g^{-1} \in K_m \right\} \quad \text{for } m, n \in \mathbb{N},$$

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and

$$R_{N,m} = \left\{ x \in X : \text{ for any } n > N, \frac{1}{|F_n|} \sum_{g \in F_n} \delta_x \circ g^{-1} \in K_m \right\} \quad \text{for } m, N \in \mathbb{N}.$$

Then for any $m, N \geq 1$,

$$R_{N,m} = \bigcap_{n>N} A_{n,m} \quad \text{and} \quad G_\mu \subseteq \bigcup_{k>N} R_{k,m}.$$

For $\varepsilon > 0$ and $Z \subseteq X$, recall that $s_{F_n}(Z, \varepsilon)$ denotes the maximal cardinality of any (F_n, ε)-separated subset of Z. Then we have

$$\limsup_{n\to+\infty} \frac{1}{|F_n|} \log s_{F_n}(R_{N,m}, \varepsilon) \leq \limsup_{n\to+\infty} \frac{1}{|F_n|} \log s_{F_n}(A_{n,m}, \varepsilon) \quad \text{for any } m, N \geq 1. \tag{5.2}$$

By the claim in [26, p. 878] (we note that it also works for non-ergodic invariant measures),

$$\lim_{\varepsilon\to 0} \lim_{m\to+\infty} \limsup_{n\to+\infty} \frac{1}{|F_n|} \log s_{F_n}(A_{n,m}, \varepsilon) \leq h_\mu(X, G).$$

Hence for any $\eta > 0$, there exists $0 < \varepsilon_1$ such that, for any $0 < \varepsilon < \varepsilon_1$, there exists $M = M(\varepsilon) \in \mathbb{N}$ such that

$$\limsup_{n\to+\infty} \frac{1}{|F_n|} \log s_{F_n}(A_{n,m}, \varepsilon) < h_\mu(X, G) + \eta,$$

whenever $m \geq M$. Especially,

$$\limsup_{n\to+\infty} \frac{1}{|F_n|} \log s_{F_n}(A_{n,M}, \varepsilon) < h_\mu(X, G) + \eta. \tag{5.3}$$

Taking (5.2) and (5.3) together, for any $0 < \varepsilon < \varepsilon_1$, we have that for any $N \in \mathbb{N}$,

$$\limsup_{n\to+\infty} \frac{1}{|F_n|} \log s_{F_n}(R_{N,M}, \varepsilon) < h_\mu(X, G) + \eta.$$

Since for any $N' \in \mathbb{N}$, $G_\mu \subseteq \bigcup_{N\geq N'} R_{N,M}$, by Propositions 2.1 and 2.4,

$$h^P_{\text{top}}(G_\mu, \varepsilon, \{F_n\}) \leq \sup_{N\geq N'} h^P_{\text{top}}(R_{N,M}, \varepsilon, \{F_n\}) \leq \sup_{N\geq N'} \limsup_{n\to+\infty} \frac{1}{|F_n|} \log s_{F_n}(R_{N,M}, \varepsilon),$$

from which it follows that

$$h^P_{\text{top}}(G_\mu, \varepsilon, \{F_n\}) \leq h_\mu(X, G) + \eta.$$

Letting $\varepsilon \to 0$ and then $\eta \to 0$, we obtain that $h^P_{\text{top}}(G_\mu, \{F_n\}) \leq h_\mu(X, G)$.

5.1.2. *Lower bound for* $h^P_{\text{top}}(G_\mu, \{F_n\})$. For the case where μ is ergodic and $\{F_n\}$ is tempered, since $\mu(G_\mu) = 1$, Corollary 2.3 gives the lower bound.

For the case when $\mu \in M(X, G)$ and the system (X, G) has almost specification property, the proof of the lower bound becomes rather complicated because of the quasi-tiling techniques for amenable groups. But it was shown in [23] that

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$h^B_{\mathrm{top}}(G_\mu, \{F_n\}) = h_\mu(X, G)$. Hence by Proposition 2.2, we obtain $h^P_{\mathrm{top}}(G_\mu, \{F_n\}) \geq h_\mu(X, G)$.

5.2. *G-symbolic dynamical system.* Let A be a finite set with cardinality $|A| \geq 2$ and let $A^G = \{(x_g)_{g\in G} : x_g \in A\}$ be the G-symbolic space over A. Consider the left action of G on A^G:

$$g'(x_g)_{g\in G} = (x_{gg'})_{g\in G} \quad \text{for all } g' \in G \text{ and } (x_g)_{g\in G} \in A^G.$$

(A^G, G) forms a G-symbolic dynamical system or a G-acting full shift (over A). For any non-empty closed G-invariant subset X of A^G, the subsystem (X, G) is called a subshift. For $x = (x_g)_{g\in G}$ and a finite subset $F \subset G$, denote by $x|_F = (x_g)_{g\in F} \in A^F$ the restriction of x to F and denote $[x|_F] = \{\omega \in A^G : \omega_g = x_g \text{ for all } g \in F\}$ (which is called a *cylinder*).

Fix any tempered Følner sequence $\{F_n\}$ of G with $F_0 = \{e_G\} \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots$ and $\bigcup_n F_n = G$. Note that $\{F_n\}$ satisfies the growth condition (1.2) automatically. We can then define a metric d on A^G associated to $\{F_n\}$ as follows:

$$d(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are not equal on } F_0, \\ e^{-|F_n|} & n = \max\{k : x|_{F_k} = y|_{F_k}\}. \end{cases} \tag{5.4}$$

To discuss the regularity for subsets of (A^G, G), we need to consider the relation between Bowen entropy (packing entropy) and the corresponding Hausdorff dimension (packing dimension). Before that, we recall the definitions of Hausdorff dimension and packing dimension (cf. [16]).

Definition 5.3. Let (X, d) be a compact metric space. Let $0 \leq s < \infty$. For $E \subset X$ and $\varepsilon > 0$, put

$$P^s_\varepsilon(E) = \sup \sum_i (\mathrm{diam} B_i)^s$$

where the supremum is taken over all disjoint families of closed balls $\{\overline{B}_i\}$ such that $\mathrm{diam} B_i \leq \varepsilon$ and the centers of the B_i are in E.

Then set $P^s(E) = \lim_{\varepsilon\to 0} P^s_\varepsilon(E)$ (since $P^s_\varepsilon(E)$ is non-decreasing on ε) and define

$$\mathcal{P}^s(E) = \inf\left\{ \sum_{i=1}^\infty P^s(E_i) : E = \bigcup_{i=1}^\infty E_i \right\}.$$

The *packing dimension* of E is then defined by

$$\dim_P(E) = \inf\{s : \mathcal{P}^s(E) = 0\} = \sup\{s : \mathcal{P}^s(E) = \infty\}.$$

Let $\mathcal{H}^s(E) = \lim_{\varepsilon\to 0} \mathcal{H}^s_\varepsilon(E)$, where

$$\mathcal{H}^s_\varepsilon(E) = \inf\left\{ \sum_{i=1}^\infty (\mathrm{diam} E_i)^s : E \subset \bigcup_{i=1}^\infty E_i, \mathrm{diam} E_i \leq \varepsilon \right\}.$$

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The *Hausdorff dimension* of E is then defined by

$$\dim_H(E) = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

Comparing with the definitions of dimensional entropies, we have the following proposition.

PROPOSITION 5.4. *Let the Følner sequence $\{F_n\}$ satisfy the following two conditions:*

(1) $F_m F_n \subset F_{m+n}$ *for each* $m, n \in \mathbb{N}$;

(2) $\lim_{n\to\infty}((F_{n+1})/F_n) = 1$.

Then for any $E \subset A^G$,

$$h^B_{\text{top}}(E, \{F_n\}) = \dim_H E \quad \textit{and} \quad h^P_{\text{top}}(E, \{F_n\}) = \dim_P E,$$

where the dimensions $\dim_H$ *and* $\dim_P$ *are both under the metric* d *defined in (5.4).*

Proof. See Appendix A.1. □

Remark 5.5.

(1) Due to [**10**], the sequence of finite subsets $\{F_n\}$ of G satisfying condition (1) in Proposition 5.4 is called a *regular system*. If G is a finitely generated group, and letting F_n be the collection of elements in G with word length (with respect to a finite symmetric generating subset) no more than n, then $\{F_n\}$ satisfies condition (1).

(2) There are examples of amenable groups which admit Følner sequences satisfying the conditions in Proposition 5.4. An abelian example of the group G is $\mathbb{Z}^d$, with $F_n = [-n, n]^d$. An non-abelian example of the group G is the dihedral group, with F_n chosen to be the collection of elements with word length no more than n (see [**5**]).

By Proposition 5.4, we have the following result.

PROPOSITION 5.6. *Let the Følner sequence $\{F_n\}$ satisfy the conditions in Proposition 5.4. Then any subset $E \subset A^G$ is regular in the sense of dimensional entropy if and only if E is dimension-regular (under the Følner sequence $\{F_n\}$ and metric d).*

By Remark 2.6, any non-empty closed G-invariant subset X of A^G is regular in the sense of dimensional entropy. Hence we have the following corollary.

COROLLARY 5.7. *Let the Følner sequence $\{F_n\}$ satisfy the conditions in Proposition 5.4. Then any non-empty closed G-invariant subset X of A^G is both regular in the sense of dimensional entropy and dimension-regular (under the Følner sequence $\{F_n\}$ and metric d).*

Let $0 \le \alpha < \beta \le 1$ and $A = \{0, 1\}$. Let $H \subset G$ such that

$$\liminf_{n\to+\infty} \frac{|H \cap F_n|}{|F_n|} = \alpha \quad \text{and} \quad \limsup_{n\to+\infty} \frac{|H \cap F_n|}{|F_n|} = \beta,$$

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that is H is a subset of G with lower density α and upper density β with respect to $\{F_n\}$. Now we define $X_{\alpha,\beta} \subset \{0,1\}^G$ by

$$X_{\alpha,\beta} = \{(x_g)_{g\in G} : x_g = 0 \text{ if } g \notin H\}.$$

Assume in addition that $\{F_n\}$ satisfies the conditions in Proposition 5.4. Then we have the following proposition.

PROPOSITION 5.8. *$h^B_{\text{top}}(X_{\alpha,\beta}, \{F_n\}) = \alpha \log 2$ and $h^P_{\text{top}}(X_{\alpha,\beta}, \{F_n\}) = \beta \log 2$. Hence $X_{\alpha,\beta}$ is not regular.*

Proof. See Appendix A.2. □

5.3. *Fibers of $\{T, T^{-1}\}$ transformation.* Random walk in random scenery (RWRS) is a class of stationary random processes which are well studied in both probability theory and ergodic theory. RWRS provides measure-theoretic models with amazingly rich behavior (see [1, 7]). Among the class of RWRS, $\{T, T^{-1}\}$ transformation, although apparently simple, is possibly the best known in the history of ergodic theory since it is a natural example of a K-automorphism that is not Bernoulli [13]. In spite of its measure-theoretic aspect, we will consider the topological model of the $\{T, T^{-1}\}$ transformation and investigate subsets of the topological system.

Definition 5.9. (Topological $\{T, T^{-1}\}$ transformation) Let $A = \{1, -1\}$, and by convention we denote the shift map on $A^{\mathbb{Z}}$ by T, which is defined by

$$(T(x))_i = x_{i+1} \quad \text{for any } x = (x_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}.$$

The $\{T, T^{-1}\}$ transformation, denoted by S, on $A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ is defined by

$$S(x, y) = \begin{cases} (T(x), T(y)) & \text{if } y_0 = 1, \\ (T^{-1}(x), T(y)) & \text{if } y_0 = -1. \end{cases}$$

Then $S^n(x, y) = (T^{\omega(y,n)}(x), T^n y)$ for $n \in \mathbb{Z}$, where

$$\omega(y, n) := \begin{cases} \sum_{j=0}^{n-1} y_j & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{j=n}^{-1} y_j & \text{if } n < 0. \end{cases}$$

Clearly for the system $(A^{\mathbb{Z}} \times A^{\mathbb{Z}}, S)$, the acting group G here is the integer group $\mathbb{Z}$. The Følner sequence $\{F_n\}$ is chosen naturally to be $F_n = \{0, 1, \ldots, n-1\} := [0, n-1]$. Define a metric ρ on $A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ by

$$\rho((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\},$$

where d is the metric on $A^{\mathbb{Z}}$ defined by

$$d(x, y) = 2^{-n} \quad \text{where } n = \min\{|i| : x_i \neq y_i\}.$$

We note that the metric d here is different from (5.4).

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Let $\pi : A^{\mathbb{Z}} \times A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the projection to the second coordinate. Then it induces a factor map between $(A^{\mathbb{Z}} \times A^{\mathbb{Z}}, S)$ and $(A^{\mathbb{Z}}, T)$. For any $y \in A^{\mathbb{Z}}$, let $E_y := A^{\mathbb{Z}} \times \{y\}$ denote the fiber of y under the factor map π. We denote by

$$h^P_{\text{top}}(E_y, S) = h^P_{\text{top}}(E_y, \{F_n\}) \quad \text{and} \quad h^B_{\text{top}}(E_y, S) = h^B_{\text{top}}(E_y, \{F_n\}),$$

the packing and Bowen entropies of E_y for the $\mathbb{Z}$-system $(A^{\mathbb{Z}} \times A^{\mathbb{Z}}, S)$, respectively.

For $n > 0$, denote

$$M(y, n) = \max_{0 \le i \le n} \omega(y, i) \quad \text{and} \quad m(y, n) = \min_{0 \le i \le n} \omega(y, i).$$

PROPOSITION 5.10. *For the packing and Bowen entropies of E_y, we have:*

(1) $h^P_{\text{top}}(E_y, S) = \limsup_{n \to +\infty}((M(y, n) - m(y, n))/n) h_{\text{top}}(A^{\mathbb{Z}}, T)$;

(2) $h^B_{\text{top}}(E_y, S) = \liminf_{n \to +\infty}((M(y, n) - m(y, n))/n) h_{\text{top}}(A^{\mathbb{Z}}, T)$.

Here $h_{\text{top}}(A^{\mathbb{Z}}, T) (= \log 2)$ is the topological entropy of the symbolic dynamical system $(A^{\mathbb{Z}}, T)$. Hence E_y is regular if and only if $\lim_{n \to +\infty}(M(y, n) - m(y, n))/n$ exists.

Proof. See Appendix A.3. □

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A. *Appendix. Proofs of Propositions 5.4, 5.8 and 5.10*

In this appendix we will give detailed proofs of Propositions 5.4, 5.8 and 5.10.

A.1. *Proof of Proposition 5.4.* Let E be a subset of the compact metric space (A^G, d) as defined in §5.2. Recall that in Proposition 5.4 the Følner sequence $\{F_n\}$ satisfies the following two conditions:

(1) $F_m F_n \subseteq F_{m+n}$ for each $m, n \in \mathbb{N}$;

(2) $\lim_{n \to \infty}((F_{n+1})/F_n) = 1$.

We divide the proof of Proposition 5.4 into two parts.

Part 1. Proof of $h^B_{\text{top}}(E, \{F_n\}) = \dim_H E$. Let $s > h^B_{\text{top}}(E, \{F_n\})$. From the definition of Bowen entropy, we have

$$\mathcal{M}(E, \varepsilon, s, \{F_n\}) = 0 \quad \text{for any } \varepsilon > 0.$$

Hence for any $N > 0$, $\mathcal{M}(E, N, 1, s, \{F_n\}) = 0$. For any $\delta > 0$, there exists a countable family $\{B_{F_{n_i}}(x^i, 1)\}$ with $x^i \in E$, $n_i \ge N$ and $\bigcup_i B_{F_{n_i}}(x^i, 1) \supset E$ such that

$$\sum_i e^{-s|F_{n_i}|} < \delta.$$

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Notice that

$$\begin{aligned} B_{F_{n_i}}(x^i, 1) &= \{y \in A^G : \rho(gx^i, gy) < 1, \text{ for all } g \in F_{n_i}\} \\ &= \{y \in A^G : (gx^i)_{e_G} = (gy)_{e_G}, \text{ for all } g \in F_{n_i}\} \\ &= [x^i|_{F_{n_i}}] \end{aligned} \tag{A.1}$$

and $\operatorname{diam}[x^i|_{F_{n_i}}] = e^{-|F_{n_i}|} \le e^{-|F_N|}$. Since the family of cylinders $\{[x^i|_{F_{n_i}}]\}$ covers E, we have

$$\mathcal{H}^s_{e^{-|F_N|}}(E) \le \sum_i e^{-s|F_{n_i}|} < \delta.$$

Therefore $\mathcal{H}^s_{e^{-|F_N|}}(E) = 0$ and then $\mathcal{H}^s(E) = 0$. This means that $\dim_H E \le s$. Since $s > h^B_{\text{top}}(E, \{F_n\})$ is arbitrary, $h^B_{\text{top}}(E, \{F_n\}) \ge \dim_H E$.

Now we will show $h^B_{\text{top}}(E, \{F_n\}) \le \dim_H E$.

If $h^B_{\text{top}}(E, \{F_n\}) = 0$, then there is nothing to prove. Assume $h^B_{\text{top}}(E, \{F_n\}) > 0$ and let $0 < s < h^B_{\text{top}}(E, \{F_n\})$. Then there exists $0 < \varepsilon < 1$ such that

$$\mathcal{M}(E, \varepsilon, s, \{F_n\}) > 1. \tag{A.2}$$

Assume $\varepsilon \in (e^{-|F_k|}, e^{-|F_{k-1}|}]$ for some $k \in \mathbb{N}$ and let $\eta > 0$ be fixed. By condition (2), there exists $N' > k$ such that

$$\frac{|F_n|}{|F_{n-k}|} < 1 + \eta \quad \text{whenever } n > N'.$$

By (A.2), there exists $N > N'$ such that $\mathcal{M}(E, N, \varepsilon, s, \{F_n\}) > 1$.

Let $\{E_i\}_{i=1}^\infty$ be any countable family that covers E and $\operatorname{diam} E_i < e^{-|F_{N+k}|}$ for each i. By the definition of the metric d, $\operatorname{diam} E_i = e^{-|F_{n_i}|}$ for some $n_i > N + k$. Choose any point $x^i \in E_i$. Then $E_i \subset [x^i|_{F_{n_i}}]$.

Noticing that

$$\begin{aligned} B_{F_{n_i-k}}(x^i, \varepsilon) &= \{y \in A^G : \rho(gx^i, gy) < \varepsilon, \text{ for all } g \in F_{n_i-k}\} \\ &= \{y \in A^G : \rho(gx^i, gy) \le e^{|F_k|}, \text{ for all } g \in F_{n_i-k}\} \\ &= \{y \in A^G : (gx^i)|_{F_k} = (gy)|_{F_k}, \text{ for all } g \in F_{n_i-k}\} \\ &= \{y \in A^G : x^i|_{F_k F_{n_i-k}} = y|_{F_k F_{n_i-k}}\} = [x^i|_{F_k F_{n_i-k}}] \\ &\supseteq [x^i|_{F_{n_i}}] \quad (\text{since } F_k F_{n_i-k} \subseteq F_{n_i}), \end{aligned} \tag{A.3}$$

the family $\{B_{F_{n_i-k}}(x^i, \varepsilon)\}_{i=1}^\infty$ also covers E (with each $n_i - k > N$). So

$$\sum_{i=1}^\infty e^{-s|F_{n_i-k}|} \ge \mathcal{M}(E, N, \varepsilon, s, \{F_n\}) > 1.$$

Hence

$$\sum_{i=1}^\infty e^{-(s/(1+\eta))|F_{n_i}|} = \sum_{i=1}^\infty e^{-(s/(1+\eta))(|F_{n_i}|/|F_{n_i-k}|)|F_{n_i-k}|} > \sum_{i=1}^\infty e^{-s|F_{n_i-k}|} > 1,$$

which implies that $\mathcal{H}^{(s/(1+\eta))}(E) \ge \mathcal{H}^{(s/(1+\eta))}_{e^{-|F_{N+k}|}}(E) \ge 1$. Thus $\dim_H E \ge (s/(1+\eta))$.

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Letting $\eta \to 0$, we have $\dim_H E \geq s$. Since $0 < s < h^B_{\text{top}}(E, \{F_n\})$ is chosen arbitrarily, we obtain that $h^B_{\text{top}}(E, \{F_n\}) \leq \dim_H E$.

Part 2. Proof of $h^P_{\text{top}}(E, \{F_n\}) = \dim_P E$. We first show $h^P_{\text{top}}(E, \{F_n\}) \geq \dim_P E$.

Let $s > h^P_{\text{top}}(E, \{F_n\})$. Then, for any $0 < \varepsilon < 1$, we have that $h^P_{\text{top}}(E, \varepsilon, \{F_n\}) < s$ and hence $\mathcal{P}(E, \varepsilon, s, \{F_n\}) = 0$. So for any $\delta > 0$, there exists a countable covering $\{E_i\}_{i=1}^{\infty}$ of E such that

$$\sum_{i=1}^{\infty} P(E_i, \varepsilon, s, \{F_n\}) < \delta.$$

For each i, we can find $N_i \in \mathbb{N}$ sufficiently large such that

$$P(E_i, N_i, \varepsilon, s, \{F_n\}) < P(E_i, \varepsilon, s, \{F_n\}) + \frac{\delta}{2^i}.$$

Let $\{B_{i,j}\}_{j=1}^{\infty}$ be a family of disjoint closed balls in A^G (with centers $x^{i,j} \in E_i$ and $\text{diam} B_{i,j} \leq e^{-|F_{N_i}|}$) such that

$$P^s_{e^{-|F_{N_i}|}}(E_i) \leq \sum_{j=1}^{\infty} (\text{diam } B_{i,j})^s + \frac{\delta}{2^i}.$$

From the definition of the metric d, $\text{diam } B_{i,j} = e^{-|F_{n_{i,j}}|}$ for some $n_{i,j} \geq N_i$. Noticing that

$$\begin{aligned} B_{i,j} &= B(x^{i,j}, e^{-|F_{n_{i,j}}|}) = [x^{i,j}|_{F_{n_{i,j}}}] \\ &\supseteq \overline{B}_{F_{n_{i,j}}}(x^{i,j}, \varepsilon) \quad \text{(here we have assumed } \varepsilon < 1\text{)}, \end{aligned}$$

$\{\overline{B}_{F_{n_{i,j}}}(x^{i,j}, \varepsilon)\}_{j=1}^{\infty}$ is also a pairwise disjoint family. Hence

$$\begin{aligned} \mathcal{P}^s(E) &\leq \sum_{i=1}^{\infty} P^s(E_i) \leq \sum_{i=1}^{\infty} P^s_{e^{-|F_{N_i}|}}(E_i) \leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} e^{-s|F_{n_{i,j}}|} + \frac{\delta}{2^i} \right) \\ &\leq \sum_{i=1}^{\infty} P(E_i, N_i, \varepsilon, s, \{F_n\}) + \delta < \sum_{i=1}^{\infty} \left(P(E_i, \varepsilon, s, \{F_n\}) + \frac{\delta}{2^i} \right) + \delta \\ &< 3\delta. \end{aligned}$$

Thus $\mathcal{P}^s(E) = 0$ and $\dim_P E \leq s$. Since $s > h^P_{\text{top}}(E, \{F_n\})$ is arbitrary, we obtain $\dim_P E \leq h^P_{\text{top}}(E, \{F_n\})$.

Next we will show $h^P_{\text{top}}(E, \{F_n\}) \leq \dim_P E$.

Assume $h^P_{\text{top}}(E, \{F_n\}) > 0$, otherwise there is nothing to prove. Let $0 < s < h^P_{\text{top}}(E, \{F_n\})$. Then there exists $0 < \varepsilon < 1$ such that $h^P_{\text{top}}(E, \varepsilon, \{F_n\}) > s$. Assume $\varepsilon \in (e^{-|F_k|}, e^{-|F_{k-1}|}]$ for some $k \in \mathbb{N}$ and let $\eta > 0$ be fixed. Similarly to Part 1, by condition (2), there exists $N > k$ such that

$$\frac{|F_n|}{|F_{n-k}|} < 1 + \eta \quad \text{whenever } n > N.$$

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Let $\{E_i\}_{i=1}^{\infty}$ be any countable family that covers E. Then

$$\sum_{i=1}^{\infty} P(E_i, \varepsilon, s, \{F_n\}) \geq \mathcal{P}(E, \varepsilon, s, \{F_n\}) = \infty.$$

For each i, let $N_i > N$ be sufficiently large such that

$$P_{e^{-|F_{N_i}|}}^{(s/(1+\eta))}(E_i) < P^{(s/(1+\eta))}(E_i) + \frac{1}{2^i}.$$

Let $\{\overline{B}_{F_{n_{i,j}}}(x^{i,j}, \varepsilon)\}_{j=1}^{\infty}$ be a disjoint family with $x^{i,j} \in E_i$ and $n_{i,j} \geq N_i$ for each j such that

$$P(E_i, N_i, \varepsilon, s, \{F_n\}) < \sum_{j=1}^{\infty} e^{-s|F_{n_{i,j}}|} + \frac{1}{2^i}.$$

With similar discussion to (A.3), $\overline{B}_{F_{n_{i,j}}}(x^{i,j}, \varepsilon) \supseteq [x^{i,j}|_{F_{n_{i,j}+k}}]$ and hence $\{[x^{i,j}|_{F_{n_{i,j}+k}}]\}_{j=1}^{\infty}$ is a disjoint family of closed balls with

$$\operatorname{diam}[x^{i,j}|_{F_{n_{i,j}+k}}] = e^{-|F_{n_{i,j}+k}|} \leq e^{-|F_{N_i+k}|}.$$

Therefore

$$P_{e^{-|F_{N_i+k}|}}^{(s/(1+\eta))}(E_i) \geq \sum_{j=1}^{\infty} e^{-(s/(1+\eta))|F_{n_{i,j}+k}|}.$$

Thus we have

$$\begin{aligned}
\sum_{i=1}^{\infty} P^{(s/(1+\eta))}(E_i) &> \sum_{i=1}^{\infty} P_{e^{-|F_{N_i+k}|}}^{(s/(1+\eta))}(E_i) - 1 \geq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{-(s/(1+\eta))|F_{n_{i,j}+k}|} - 1 \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{-(s/(1+\eta))(|F_{n_{i,j}+k}|/|F_{n_{i,j}}|)|F_{n_{i,j}}|} - 1 \geq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{-s|F_{n_{i,j}}|} - 1 \\
&> \sum_{i=1}^{\infty} \left(P(E_i, N_i, \varepsilon, s, \{F_n\}) - \frac{1}{2^i} \right) - 1 \\
&\geq \sum_{i=1}^{\infty} P(E_i, \varepsilon, s, \{F_n\}) - 2 = \infty,
\end{aligned}$$

which implies that $\mathcal{P}^{(s/(1+\eta))}(E) = \infty$ and then $\dim_P E \geq (s/(1+\eta))$.

Letting $\eta \to 0$, we have $\dim_P E \geq s$. Since $0 < s < h_{\text{top}}^P(E, \{F_n\})$ is chosen arbitrarily, we obtain that $h_{\text{top}}^P(E, \{F_n\}) \leq \dim_P E$.

A.2. *Proof of Proposition 5.8.* Recall that for Proposition 5.8 we let $0 \leq \alpha < \beta \leq 1$, $A = \{0, 1\}$ and let $H \subset G$ such that

$$\liminf_{n\to+\infty} \frac{|H \cap F_n|}{|F_n|} = \alpha \quad \text{and} \quad \limsup_{n\to+\infty} \frac{|H \cap F_n|}{|F_n|} = \beta.$$

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$X_{\alpha,\beta} \subset \{0,1\}^G$ is defined by

$$X_{\alpha,\beta} = \{(x_g)_{g\in G} : x_g = 0 \text{ if } g \notin H\}.$$

Let $\mu \in M(X_{\alpha,\beta})$ such that $\mu([x|_{F_n}]) = (1/2^{|H\cap F_n|})$ for every $x \in X_{\alpha,\beta}$ and $n \in \mathbb{N}$. Noticing that $B_{F_n}(x,1) = [x|_{F_n}]$ (see (A.1)), we have

$$\begin{aligned}\underline{h}_\mu^{\mathrm{loc}}(X_{\alpha,\beta},\{F_n\}) &= \int_{X_{\alpha,\beta}} \lim_{\varepsilon\to 0}\liminf_{n\to+\infty} -\frac{1}{|F_n|}\log \mu(B_{F_n}(x,\varepsilon))\,d\mu\\ &\geq \int_{X_{\alpha,\beta}} \liminf_{n\to+\infty} -\frac{1}{|F_n|}\log \mu([x|_{F_n}])\,d\mu\\ &= \liminf_{n\to+\infty} -\frac{1}{|F_n|}\log\frac{1}{2^{|H\cap F_n|}} = \alpha\log 2,\end{aligned}$$

and similarly,

$$\overline{h}_\mu^{\mathrm{loc}}(X_{\alpha,\beta},\{F_n\}) \geq \beta\log 2.$$

Applying [**25**, Theorem 3.1] (the variational principle for amenable Bowen entropy) and Theorem 1.3 (the variational principle for amenable packing entropy) respectively, we have

$$h_{\mathrm{top}}^B(X_{\alpha,\beta},\{F_n\}) \geq \alpha\log 2 \quad \text{and} \quad h_{\mathrm{top}}^P(X_{\alpha,\beta},\{F_n\}) \geq \beta\log 2.$$

To prove $h_{\mathrm{top}}^B(X_{\alpha,\beta},\{F_n\}) \leq \alpha\log 2$, by Proposition 5.4, we only need to prove that $\dim_H(X_{\alpha,\beta}) \leq \alpha\log 2$.

Let

$$E_n = \{(x_g)_{g\in G} : x_g = 0 \text{ if } g \notin H\cap F_n\}, \tag{A.4}$$

which is a subset of $X_{\alpha,\beta}$ with cardinality $\#E_n = 2^{|H\cap F_n|}$. Note that $\bigcup_{x\in E_n}[x|_{F_n}] \supseteq X_{\alpha,\beta}$ and $\mathrm{diam}[x|_{F_n}] = e^{-|F_n|}$ for each $x \in E_n$. Let $\delta > 0$ and $N \in \mathbb{N}$ be fixed. When $n > N$,

$$\mathcal{H}^{(\alpha+\delta)\log 2}_{e^{-|F_N|}}(X_{\alpha,\beta}) \leq \sum_{x\in E_n} e^{-(\alpha+\delta)\log 2|F_n|} = e^{\log 2|F_n|((|H\cap F_n|/|F_n|)-\alpha-\delta)}.$$

Since $\liminf_{n\to+\infty}(|H\cap F_n|/|F_n|) = \alpha$, we have

$$\mathcal{H}^{(\alpha+\delta)\log 2}_{e^{-|F_N|}}(X_{\alpha,\beta}) = 0 \quad \text{and} \quad \mathcal{H}^{(\alpha+\delta)\log 2}(X_{\alpha,\beta}) = \lim_{N\to\infty}\mathcal{H}^{(\alpha+\delta)\log 2}_{e^{-|F_N|}}(X_{\alpha,\beta}) = 0.$$

Thus

$$\dim_H(X_{\alpha,\beta}) \leq (\alpha+\delta)\log 2,$$

from which we deduce that

$$\dim_H(X_{\alpha,\beta}) \leq \alpha\log 2.$$

Finally, we will prove that $h_{\mathrm{top}}^P(X_{\alpha,\beta},\{F_n\}) \leq \beta\log 2$.

Let $\varepsilon > 0$ such that $\varepsilon \in (e^{-|F_k|}, e^{-|F_{k-1}|}]$ for some $k \in \mathbb{N}$. Noticing that $[x|_{F_{n+k}}] \subseteq B_{F_n}(x,\varepsilon)$ (see (A.3); here condition (1) of Proposition 5.4 is used), one can check that the set E_{n+k}, defined by (A.4), is an (F_n,ε)-spanning set of $X_{\alpha,\beta}$. Hence $r_{F_n}(X_{\alpha,\beta},\varepsilon) \leq$

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$2^{|H\cap F_{n+k}|}$. Then

$$\limsup_{n\to+\infty}\frac{1}{|F_n|}\log r_{F_n}(X_{\alpha,\beta},\varepsilon)\le\limsup_{n\to+\infty}\frac{|H\cap F_{n+k}|}{|F_n|}\log 2=\beta\log 2,$$

where for the last equality we use condition (2) of Proposition 5.4. Thus

$$h^P_{\mathrm{top}}(X_{\alpha,\beta},\{F_n\})\le h^{UC}_{\mathrm{top}}(X_{\alpha,\beta},\{F_n\})\le\beta\log 2.$$

A.3. *Proof of Proposition 5.10.* (1) For any $\varepsilon>0$ and $x\in A^{\mathbb{Z}}$, in the system $(A^{\mathbb{Z}}\times A^{\mathbb{Z}},S)$, we have

$$\begin{aligned}B_n((x,y),\varepsilon,\rho)\cap E_y&=\{(x',y):\rho(S^i(x',y),S^i(x,y))<\varepsilon,\ \text{for } 0\le i\le n-1\}\\&=\{(x',y):d(T^{\omega(y,i)}(x'),T^{\omega(y,i)}x)<\varepsilon,\ \text{for } 0\le i\le n-1\}\\&=B_{[m(y,n-1),M(y,n-1)]}(x,\varepsilon,d)\times\{y\}.\end{aligned}\tag{A.5}$$

Let μ be the $\{\frac12,\frac12\}$ Bernoulli measure on $(A^{\mathbb{Z}},T)$ and δ_y be the Dirac probability measure at the point y. Then

$$\begin{aligned}\overline{h}^{\mathrm{loc}}_{\mu\times\delta_y}(E_y,S)&=\int_{E_y}\lim_{\varepsilon\to0}\limsup_{n\to+\infty}-\frac1n\log(\mu\times\delta_y)(B_n((x,y),\varepsilon,\rho))\,d(\mu\times\delta_y)\\&=\int_{A^{\mathbb{Z}}}\lim_{\varepsilon\to0}\limsup_{n\to+\infty}-\frac1n\log\mu(B_{[m(y,n-1),M(y,n-1)]}(x,\varepsilon,d))\,d\mu\\&=\limsup_{n\to+\infty}\frac{M(y,n)-m(y,n)}{n}h_{\mathrm{top}}(A^{\mathbb{Z}},T),\end{aligned}\tag{A.6}$$

where for the last inequality we use the simple facts that $M(y,n-1)\le M(y,n)\le M(y,n-1)+1$ and $m(y,n-1)-1\le m(y,n)\le m(y,n-1)$. Hence by Theorem 1.3, we have

$$h^P_{\mathrm{top}}(E_y,S)\ge\overline{h}^{\mathrm{loc}}_{\mu\times\delta_y}(E_y,S)=\limsup_{n\to\infty}\frac{M(y,n)-m(y,n)}{n}h_{\mathrm{top}}(A^{\mathbb{Z}},T).$$

Let $E\times\{y\}$ be any (n,ε)-separated set for E_y. Note that, from (A.5), E must be an $([m(y,n-1),M(y,n-1)],\varepsilon)$-separated set for $A^{\mathbb{Z}}$. We have

$$\begin{aligned}h^{UC}_{\mathrm{top}}(E_y,S)&\le\limsup_{n\to+\infty}\frac{M(y,n)-m(y,n)}{n}h^{UC}_{\mathrm{top}}(A^{\mathbb{Z}},T)\\&=\limsup_{n\to+\infty}\frac{M(y,n)-m(y,n)}{n}h_{\mathrm{top}}(A^{\mathbb{Z}},T).\end{aligned}$$

By Proposition 2.4,

$$h^P_{\mathrm{top}}(E_y,S)\le h^{UC}_{\mathrm{top}}(E_y,S)\le\limsup_{n\to\infty}\frac{M(y,n)-m(y,n)}{n}h_{\mathrm{top}}(A^{\mathbb{Z}},T).$$

This finishes the proof of (1).

(2) Similarly to (A.6), we have

$$\underline{h}^{\mathrm{loc}}_{\mu\times\delta_y}(E_y,S)=\int_{E_y}\lim_{\varepsilon\to0}\liminf_{n\to+\infty}-\frac1n\log(\mu\times\delta_y)(B_n((x,y),\varepsilon,\rho))\,d(\mu\times\delta_y)$$

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$$= \int_{A^{\mathbb{Z}}} \lim_{\varepsilon\to 0} \liminf_{n\to+\infty} -\frac{1}{n} \log \mu(B_{[m(y,n-1),M(y,n-1)]}(x,\varepsilon,d))\, d\mu$$
$$= \liminf_{n\to+\infty} \frac{M(y,n)-m(y,n)}{n} h_{\mathrm{top}}(A^{\mathbb{Z}},T).$$

Hence by [25], the variational principle for amenable Bowen entropy, we have

$$h^B_{\mathrm{top}}(E_y,S) \geq \underline{h}^{\mathrm{loc}}_{\mu\times\delta_y}(E_y,S) = \liminf_{n\to\infty} \frac{M(y,n)-m(y,n)}{n} h_{\mathrm{top}}(A^{\mathbb{Z}},T).$$

We now prove the upper bound for $h^B_{\mathrm{top}}(E_y)$, that is,

$$h^B_{\mathrm{top}}(E_y,S) \leq \liminf_{n\to\infty} \frac{M(y,n)-m(y,n)}{n} h_{\mathrm{top}}(A^{\mathbb{Z}},T).$$

Let $\delta>0$ be fixed. For any $\varepsilon>0$, there exists $k\in\mathbb{N}$ such that for any $x,z\in A^{\mathbb{Z}}$, whenever $x_i=z_i$ for every $|i|\leq k$, we have that $d(x,y)<\varepsilon$. Hence for any interval $[m.n]$ of integers, we have

$$B_{[m,n]}(x,\varepsilon,d) \supseteq [x|_{[-k+m,k+n]}],$$

where $[x|_{[-k+m,k+n]}] := \{z\in A^{\mathbb{Z}} : z_i = x_i, \text{ for every } -k+m\leq i\leq k+n\}$ is the cylinder in $A^{\mathbb{Z}}$.

Let

$$E_n = \{x\in A^{\mathbb{Z}} : x_i = 1 \text{ if } i\notin [-k+m(y,n-1), k+M(y,n-1)]\}.$$

Consider the family

$$\{B_n((x,y),\varepsilon,\rho)\cap E_y\}_{x\in E_n}.$$

This evidently covers E_y, since $B_n((x,y),\varepsilon,\rho)\cap E_y = B_{[m(y,n-1),M(y,n-1)]}(x,\varepsilon,d)\times\{y\}$ (by (A.5)). Hence for any $N\in\mathbb{N}$, we have for any $n\geq N$,

$$\begin{aligned}\mathcal{M}(E_y,N,\varepsilon,s,\{F_n\}) &\leq (\#E_n)e^{-sn}\\ &= 2^{M(y,n-1)-m(y,n-1)+2k+1}e^{-sn}\\ &= e^{n(((M(y,n-1)-m(y,n-1)+2k+1)/n)\log 2-s)}.\end{aligned}$$

Note that there exist infinitely many $n\in\mathbb{N}$ such that

$$\frac{M(y,n-1)-m(y,n-1)+2k+1}{n} < \liminf_{n\to+\infty}\frac{M(y,n)-m(y,n)}{n}+\delta.$$

Then for any $s>(\liminf_{n\to+\infty}((M(y,n)-m(y,n))/n)+\delta)\log 2$, we can deduce that

$$\mathcal{M}(E_y,N,\varepsilon,s,\{F_n\}) = 0.$$

Letting $N\to+\infty$, $\varepsilon\to 0$ and then $\delta\to 0$, from the definition of the Bowen entropy, we can conclude that

$$h^B_{\mathrm{top}}(E_y,S) \leq \liminf_{n\to\infty}\frac{M(y,n)-m(y,n)}{n} h_{\mathrm{top}}(A^{\mathbb{Z}},T).$$

This finishes the proof of (2) of Proposition 5.10.

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