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ARONSZAJN'S THEOREM FOR SOME NONLINEAR DIRICHLET PROBLEMS WITH UNBOUNDED NONLINEARITIES

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1. Introduction

We consider the Dirichlet problem

$$u''+u+g(u)=h(x)\quad x\in(0,\pi),\quad u(0)=u(\pi)=0 \tag{1}$$

where g is continuous and $h\in L^2(0,\pi)$.

By integrating (1) we see that a necessary condition for (1) to have a solution is that

$$\omega=\omega(h)=\int_0^\pi h(x)\sin x\,dx\Big/\int_0^\pi \sin x\,dx\in\overline{\text{Range}\,g}. \tag{2}$$

If g is such that

$$g(\infty)\leqq g(u)\leqq g(-\infty)\quad \text{for every } u\in\mathbb{R} \tag{3}$$

then it is well known **[10, 11]** that a sufficient condition for the existence of solutions to (1) is that

$$\omega\in\text{Int}\,(\text{Range}\,g). \tag{4}$$

On the other hand, if g satisfies

$$g(-\infty)\leqq g(u)\leqq g(\infty)\quad \text{for every } u\in\mathbb{R}, \tag{5}$$

then a restriction on g is needed. Indeed, for $g(u)=3u$ and $h(x)=\sin 2x$, problem (1) has no solution. Thus, we shall consider the following hypotheses

there exist constants γ, C such that $C>0$, $\gamma\in[0,3)$, and $|g(u)|\leqq\gamma|u|+C$ for every $u\in\mathbb{R}$, (6)

g is a nondecreasing function. (7)

Ahmad proved in [1] that (1) has at least one solution provided that (6) holds and

$$\overline{g(-\infty)}\int_0^\pi \sin x\,dx<\int_0^\pi h(x)\sin x\,dx<\underline{g(\infty)}\int_0^\pi \sin x\,dx \tag{8}$$

where

$$\overline{g(-\infty)}=\limsup_{u\to-\infty} g(u) \quad \text{and} \quad \underline{g(\infty)}=\liminf_{u\to\infty} g(u).$$

In section 2, we study the case when equality does occur in (8) (Theorem 1(c)). Then, we show that uniqueness does not occur in general even if (4) and (7) are satisfied. However, if g is Lipschitz continuous and the Lipschitz constant "stays away" from the nearest eigenvalue uniqueness occurs (Theorem 2). Thus, we have that if there exists a non-negative constant k with $k<3$ and

$$|g(u)-g(v)|\leqq k|u-v|, \quad \text{for every } u,v,\in\mathbb{R} \tag{9}$$

and g is strictly increasing, then any solution of (1) is unique. The proof of this result is standard, but we give it for the sake of completeness.

In Section 3 we prove our main result (Theorem 4): If (4), (7) and (9) hold, then the set of solutions of (1) in $L^2(0,\pi)$ is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. Following Aronszajn [2] we will call such a set an R_δ. It is known that an R_δ is acyclic and, in particular, it is nonempty, compact and connected. Note that there are compact and connected sets which cannot be continuous images of R_δ's [12, 13]. We shall denote the solution set of (1) by $S(h)$.

Recently, Ballotti proved in [3] that the solution set for a parabolic partial differential equation is an R_δ. Similarly, Górniewicz and Pruszko [6] and De Blasi and Myjak [5] showed that the set of all solutions of a Darboux problem for a partial differential equation of hyperbolic type is an R_δ.

In our last section we present some examples in order to bring out the fact that the set of solutions is not an R_δ in general in the following two cases:

(i) g satisfies (5) instead of (7)

(ii) $\omega\in\operatorname{Bdry}(\operatorname{Range} g)$.

In the sequel and for $p\in[1,\infty]$ we denote by $|\ |_p$ the usual norm in $L^p(0,\pi)$. For $R>0$, $\bar{B}_p(0,R)=\{u\in L^p(0,\pi): |u|_p\leqq R\}$. If $p=2$, $u,v\in L^2(0,\pi)$ we write $\|u\|=|u|_2$, $(u,v)=\int_0^\pi u(x)v(x)\,dx$, and for $R>0$, $\bar{B}(0,R)=\bar{B}_2(0,R)$.

2. Existence of solutions

We have the following existence result.

Theorem 1. *Under assumptions (6) and (7), we have:*

(a) $\omega\in\operatorname{Int}(\operatorname{Range} g)$ *is a sufficient condition for (1) to have a solution.*

(*b*) $\omega \in \overline{\text{Range}\, g}$ *is a necessary condition for (1) to have a solution.*

(*c*) *If* $\omega \in \text{Bdry}(\text{Range}\, g)$, *then (1) has a solution if and only if* $g(0)=0$.

Proof. Part (a) is an immediate consequence of [1]. Part (b) follows by integrating (1), and (c) can be proved as in [9, Th. 2] (see also [10]).

We remark that in the case where g is strictly decreasing, uniqueness is trivial since

$$(u''+u+g(u)-v''-v-g(v), u-v)<0$$

for every $u,v\in E$ with $u\neq v$. However, this is not true for g non-decreasing. Indeed, for $h\equiv 0$ and g strictly increasing with $g(u)=3u$ in a neighbourhood of $u=0$, we have that $a\sin x$, with a sufficiently small, are solutions of (1). We note that the non-uniqueness is due to the presence of an eigenvalue ($\lambda=3$) of the problem

$$u''+u+\lambda u=0, \quad u(0)=u(\pi)=0. \tag{10}$$

Note that the eigenvalues of (10) are $\lambda_i=i^2-1$, $i=1,2,\dots$. Nevertheless, we have the following uniqueness result.

Theorem 2. *If* g *is strictly increasing and (4) and (9) hold with* $k<3$, *then any solution of (1) is unique.*

Proof. Let $E=L^2(0,\pi)$ and define the operator $L\colon D(L)\subset E\to E$ by $Lu=u''+u''$ where $D(L)=\{u\in H^2(0,\pi)\colon u(0)=u(\pi)=0\}$. Let $N\colon E\to E$ be the Nemytskii map associated with the nonlinear part of (1), that is, $Nu=h-g(u)$. Thus, (1) is equivalent to the operator equation $Lu=Nu$. Now, we take $c=(\lambda_1+\lambda_2)/2=3/2$ (see [4, p. 116]). Thus, the operator $L+cI$ is invertible and $\|(L+cI)^{-1}\|\leqq c^{-1}$. On the other hand, for every $x,y\in\mathbb{R}$ we have that $|g(y)-g(x)+c(x-y)|\leqq c|x-y|$, with strict inequality when $x\neq y$ since g is strictly decreasing and (9) holds with $k<3$. Therefore, $\|Nu-Nv+c(u-v)\|\leqq c\|u-v\|$ for every u, $v\in E$. Moreover, if $u\neq v$ in a set of positive measure, then $\|Nu-Nv+c(u-v)\|<c\|u-v\|$. Now, if $u,v\in E$ are two solutions then

$$(L+cI)^{-1}[Nu-Nv+c(u-v)]=u-v \quad \text{and}$$

$$\|u-v\|\leqq c^{-1}\|Nu-Nv+c(u-v)\|$$

which implies that $u=v$. This completes the proof of the theorem.

3. Aronszajn's theorem

We shall use the following result due to Aronszajn [2].

Theorem 3. *Let* K *be a closed, convex and bounded set is a Banach space* E. *Let* $T\colon E\to E$ *be compact such that*

(*a*) $T(K)\subset K$

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(b) *For every* $\varepsilon>0$, *there exists* $T_\varepsilon: E\to E$ *compact with* $\|T_\varepsilon(u)-T(u)\|<\varepsilon$ *for every* $u\in K$

(c) *There exists* $\rho>0$ *such that for every* $\varepsilon>0$, $I-T_\varepsilon$ *maps* K *bijectively onto a set containing* $\bar{B}(0,\rho)=\{u\in E: \|u\|\leqq\rho\}$.

Then the set of fixed points of T, $F(T)=\{u\in E: T(u)=u\}$ *is an* R_σ.

Now we are in a position to prove our main result.

Theorem 4. *Suppose that (4), (7) and (9) hold with* $k<3$. *Then the set of solutions of (1) is an* R_δ.

Proof. Let $\xi(x)=\sqrt{(\pi/2)\sin x}$, $x\in[0,\pi]$ and define the projection $P: E\to E$ by $Pu=(u,\xi)\xi$. Thus, $PE=E_0=\operatorname{Ker} L$ and $E=E_0\oplus E_1$, $E_1=(I-P)E$. The partial inverse of L is $H: E_1\to E_1$ where $Hv=u$ iff $Lu=v, u\in E_1$. It is well known [4] that solutions of $Lu=Nu$ are precisely the fixed points of the operator $R: E\to E, Ru=Pu+H(I-P)Nu+PNu$.

On the other hand, we know [1] that there exists $R>0$ such that $|u|_\infty\leqq R$ for any $u\in S(h)$. Choose $A<-R$, $B>R$ such that $g(A)<\omega(h)<g(B)$.

Consider the following modified problem

$$u''+u+G(u)=h(x),\quad x\in(0,\pi),\quad u(0)=u(\pi)=0 \tag{11}$$

where

$$G(u)=\begin{cases} g(A), & u<A\\ g(u), & A\leqq u\leqq B\\ g(B), & u>B.\end{cases}$$

Now, let $M: E\to E, Mu=h-G(u)$ so that (11) is equivalent to $Lu=Mu$. Any solution of (1) is also a solution of (11) and taking into account Theorem 1(a) we have that (1) is solvable. Hence, if we show that the set of solutions of (11), denoted by $\tilde{S}(h)$, is an R_δ, we can conclude that

$$S(h)=\tilde{S}(h)\cap\bar{B}_\infty(0,R) \tag{12}$$

is an R_δ.

We note that $\tilde{S}(h)$ is bounded since G is bounded and $g(A)<\omega(h)<g(B)$. Let s be such that $\|u\|\leqq s$ for every $u\in\tilde{S}(h)$. Define the retraction $r: E\to\bar{B}(0,s)$ by $ru=u$ for $\|u\|\leqq s$, $ru=(s/\|u\|)u$ for $\|u\|>s$. In order to show that $\tilde{S}(h)$ is an R_δ we consider the operator $T: E\to E$ defined by $Tu=Pru+H(I-P)Mu+PMu$. Note that

$$\tilde{S}(h)=F(T)\cap\bar{B}(0,R). \tag{13}$$

We shall show that $F(T)$ is an R_δ by using Aronszajn's theorem. Let J be such that $\|Mu\|\leqq J$ for $u\in E$ and set $c=\|H(I-P)\|\cdot J$.

Let $K=\bar{B}(0,t)$ where $t=s+c+J+\rho$, $\rho>0$. For $u\in K$, $\|Tu\|\leqq s+c+J<t$ which shows that $T(K)\subset K$.

Now, let $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous and strictly increasing function such that $\phi(\varepsilon)>0$ for $\varepsilon>0$ and $\phi(\varepsilon)<\min\{\pi/(\pi\sqrt{2}\pi(\|H(I-P)\|+1)), 3-K\}$. Define $M_\varepsilon: E\to E$ by $M_\varepsilon(u)=Mu-\phi(\varepsilon)\operatorname{Arctan} u$, and $T_\varepsilon(u)=Pru+H(I-P)M_\varepsilon(u)+PM_\varepsilon(u)$. Thus, for $u\in K$ we get $\|T_\varepsilon(u)-T(u)\|\leqq\|H(I-P)\|\cdot\phi(\varepsilon)\cdot\|\operatorname{Arctan} u\|+\phi(\varepsilon)\cdot\|\operatorname{Arctan} u\|<\varepsilon$. Hence, (b) of Theorem 3 is satisfied.

To show (c) we shall prove that $I-T_\varepsilon$ is one-to-one in K and $\bar{B}(0,\rho)\subset(I-T_\varepsilon)K$. Indeed, let $u,v\in K$ such that $(I-T_\varepsilon)u=(I-T_\varepsilon)v$. Thus, $u-v=T_\varepsilon u-T_\varepsilon v$. Therefore, $u-v\in D(L)$ and $L(u-v)=M_\varepsilon u-M_\varepsilon v=Mu-Mv-\phi(\varepsilon)[\operatorname{Arctan} u-\operatorname{Arctan} v]$. The function $u\to G(u)+\phi(\varepsilon)\operatorname{Arctan} u$ is strictly increasing and Lipschitz continuous with Lipschitz constant $k+\phi(\varepsilon)<3$. By Theorem 2 we can conclude that $u=v$.

Now, for $w\in\bar{B}(0,\rho)$, define the operator $A_\varepsilon: E\to E, A_\varepsilon(u)=T_\varepsilon(u)+w$. For $u\in K$, $\|A_\varepsilon(u)\|\leqq\|T_\varepsilon(u)\|+\rho<t$. In consequence, $A_\varepsilon(K)\subset K$ and by Schauder fixed point theorem A_ε has a fixed point $u\in K$ which is precisely the solution of $(I-T_\varepsilon)u=w$. This shows (c) of Aronszajn's theorem and that the set of fixed points of T is an R_δ.

From (12) and (13) we can conclude that $S(h)$ is an R_δ since, as is well known, any convex subset of a Banach space is an absolute retract.

4. Counterexamples

In this section we show with some examples that Theorem 4 is as sharp as possible in the sense that the solution set is not an R_δ if we have either

(i) (5) instead of (7), or

(ii) $\omega\in\operatorname{Bdry}(\operatorname{Range} g)$ instead of (4)

even if (9) holds with $k<3$.

Example 1. Let $h\equiv 0$ and

$$g(u)=\begin{cases}-1, & u<-1\\ u, & -1<u<0\\ 0, & u>0.\end{cases}$$

Clearly g is non-decreasing, $g(-\infty)=-1$, and $g(\infty)=0$. On the other hand, $\omega(h)=0$ and if u is a solution of (1), then

$$\int_0^\pi g(u(x))\sin x\,dx=0.$$

This implies that $u(x)\geqq 0$ for every $x\in[0,\pi]$ since $g\leqq 0$. Therefore, the set of solutions is given by $\{\alpha\sin x:\alpha\geqq 0\}$ which is not bounded. Hence, the solution set is not an R_δ. Note that the solution set is connected in this case.

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Remark. If $\omega \in \text{Bdry}(\text{Range}\, g)$ and g is strictly increasing, then (1) has no solution by Theorem 1(c). Hence, the solution set would be empty and it cannot be an R_δ.

Example 2. Consider the problem

$$u''+u+g(u)=-\tfrac{1}{2}, \quad u(0)=u(\pi)=0 \tag{14}$$

where

$$g(u)=\begin{cases} u, & u<0 \\ -u, & 0\leqq u\leqq 1 \\ u-2, & u>1. \end{cases}$$

Thus, $\omega=-\frac{1}{2}$ and (4) holds. It is easy to see that g satisfies (5) and (9) with $k<3$. In fact, $k=1$.

We shall show that $S(h)$ is not connected and consequently it is not an R_δ.

If u is a solution of (14) such that $0\leqq u\leqq 1$ in $[0,\pi]$, then u is a solution of the linear problem $u''=-\frac{1}{2}, u(0)=u(\pi)=0$. Thus, $\alpha(t)=-\frac{1}{4}t(t-\pi)$ is a solution of (14) since $0\leqq\alpha\leqq 1$ in $[0,\pi]$. On the other hand, if u is a solution with $u\leqq 0$, then u satisfies the linear problem $u''+2u=-\frac{1}{2}, u(0)=u(\pi)=0$, which has a unique solution given by $\beta(t)=-\frac{1}{4}+\lambda\sin\sqrt{2}t+\mu\cos\sqrt{2}t$ where $\lambda=(1-\cos\sqrt{2}\pi)/4\sin\sqrt{2}\pi, \mu=\frac{1}{4}$. Note that $\lambda<0$ and set $m_0=\beta'(0)=\lambda\sqrt{2}$.

Now, for $m\in\mathbb{R}$, consider the initial value problem

$$u''+2u=-\tfrac{1}{2}, \quad u(0)=0, \quad u'(0)=m$$

which has a unique solution $u_m(t)=-\frac{1}{4}+\lambda_m\sin\sqrt{2}t+\mu_m\cos\sqrt{2}t$ with $\lambda_m=2^{-1/2}\cdot m$ and $\mu_m=\frac{1}{4}$.

We prove below that

$$u_m(t)\leqq 0 \quad \text{for every } m\in[m_0,0], \quad t\in[0,\pi]. \tag{15}$$

Note that $\beta=u_{m_0}$. Thus, β is a solution to (14). If $m\in(m_0,0]$, then $u_m(\pi)\leqq -\frac{1}{4}+\lambda_m\sin\sqrt{2}\pi+\mu_m\cos\sqrt{2}\pi<0$. Hence, u_m is not a solution of (14) for $m\in(m_0,0]$.

On the other hand, the initial value problem $u''+u+g(u)=-\frac{1}{2}, u(0)=0, u'(0)=m$, has a unique solution for any $m\in\mathbb{R}$. For $m\in[m_0,0]$ such a unique solution is precisely u_m. Therefore, (14) has no solution u with $u'(0)\in(m_0,0)$.

Let $c=m_0/2$ and consider the open sets in $X=C^1[0,\pi]$.

$$A=\{u\in X: u'(0)>c\}, \quad B=\{u\in X: u'(0)<c\}.$$

Thus, $\alpha\in A\cap S(h)$, $\beta\in B\cap S(h)$, $A\cup B\supset S(h)$, and $A\cap B=\varnothing$ which means that $S(h)$ is not connected in X. Moreover, $S(h)$ is not connected in E. Indeed, the operator R can be considered as a continuous map form E to X, and $R(S(h))=S(h)$. This implies that $S(h)$ is not connected in E.

Proof of (15). For $t\in[0,2^{-1/2}\cdot\pi]$ we have that $\sin\sqrt{2}t\geqq 0$. Thus, $\lambda_m \sin\sqrt{2}t\leqq 0$ and $u_m(t)\leqq 0$. If $t\in[2^{-1/2}\cdot\pi,\pi]$, then $\cos\sqrt{2}\pi\geqq\cos\sqrt{2}t$ and $\sin\sqrt{2}\pi\leqq\sin\sqrt{2}t\leqq 0$. This implies that $u_m(t)\leqq -\frac{1}{4}+2^{-1/2}\cdot m\sin\sqrt{2}\pi+\frac{1}{4}\cos\sqrt{2}\pi\leqq 0$.

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