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Each Copy of the Real Line in $\mathbb{C}^2$ is Removable

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Abstract. Around 1995, Professors Lupacciolu, Chirka and Stout showed that a closed subset of $\mathbb{C}^N$ ($N \geq 2$) is removable for holomorphic functions, if its topological dimension is less than or equal to $N-2$. Besides, they asked whether closed subsets of $\mathbb{C}^2$ homeomorphic to the real line (the simplest 1-dimensional sets) are removable for holomorphic functions. In this paper we propose a positive answer to that question.

1 Introduction

Around 1995, Professors Lupacciolu, Chirka and Stout characterized the closed subsets X of $\mathbb{C}^N$ ($N \geq 2$) which are removable for holomorphic functions (see [1], [4] and [5]). Each holomorphic function defined on $\mathbb{C}^N - X$ has a unique holomorphic extension over $\mathbb{C}^N$ if and only if both Dolbeault cohomology groups ${}^{\sigma}H_c^{N,N-1}(X)$ and $H_c^{N,N}(X)$ vanish. The set X is then removable when its topological dimension is less than or equal to $N-2$ (see [3]). Besides, Professors Chirka and Stout ask whether closed subsets of $\mathbb{C}^2$ homeomorphic to the real line (the simplest 1-dimensional sets) are removable. We propose answering positively that question by showing the following:

Proposition 1.1 *Let X be a closed subset of $\mathbb{C}^2$ homeomorphic to the real line, then X is removable for holomorphic functions in $\mathbb{C}^2$.*

Proof Let f be a holomorphic function defined on $\mathbb{C}^2 - X$, and let x be an arbitrary point of X. We will show that f extends holomorphically over x by building a relatively compact domain $W \subset \mathbb{C}^2$ such that: $x \in W$, its boundary δW meets X in only two points, $\delta W - X$ is $\mathcal{C}^2$-smooth and $\mathbb{C}^2 - \overline{W}$ is connected; so need we just apply Theorem 3 of [4].

Fix a relatively compact open interval $H \subset X$ which contains to x. Its closure $\overline{H}$ is an arc with two end points, named a and b. Each closure, interior or boundary is calculated with respect to $\mathbb{C}^2$. We can assume, without loss of generality, that $\|a-b\| \geq 3$. Set $E = \{a, b\}$, and for every positive integer k, let $B_k(E)$ be the union of two open balls of radius $1/k$ and centers in a and b respectively. Moreover, consider the compact sets $H_k = \overline{H} - B_k(E)$. Notice that $X - H$ is closed in $\mathbb{C}^2$, so is the distance

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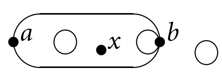
between $X - H$ and each H_k positive. Hence, there exist relatively compact open sets $V_k \subset \mathbb{C}^2$ such that:

(i) $H_k \subset V_k$ and $\overline{V_k} \cap (X - H) = \varnothing$ for every positive integer k.
(ii) The boundary $\delta V_k = \overline{V_k} - V_k$ is a $\mathcal{C}^2$-smooth 3-manifold.

We choose V_k inductively. The sets H_{k+1} and H_k are equal outside $B_k(E)$ because $H_k = H_{k+1} - B_k(E)$, so we can ask $\overline{V_{k+1}}$ and $\overline{V_k}$ to be equal outside $B_k(E)$ as well. Fix $\overline{V_1}$ and demand that $\overline{V_{k+1}} - B_k(E) = \overline{V_k} - B_k(E)$ holds for every positive integer. Chosen in this way, the sequence of sets $\overline{V_k}$ converges to a compact $\widetilde{V} \subset \mathbb{C}^2$ when $k \to \infty$ because the sequence of balls $B_k(E)$ converges to E. Indeed: $\widetilde{V} - B_k(E) = \overline{V_k} - B_k(E)$ for each positive integer k. Now let $V \subset \mathbb{C}^2$ be the interior set of $\widetilde{V}$. The following equalities follow from (ii):

(iii) $\delta V - B_k(E) = \delta V_k - B_k(E) = \delta \widetilde{V} - B_k(E)$.
(iv) $V - B_k(E) = V_k - B_k(E)$.

We have that $x \in V$. Indeed, since $x \notin E$, there exists a positive integer n such that $x \notin B_n(E)$, so $x \in H_n$; the inclusions $x \in V_n - B_n(E) \subset V$ follows then from (i) and (iv). On the other hand, the set $\delta V - E$ is a $\mathcal{C}^2$-smooth 3-manifold because $B_k(E)$ converges to E and facts (ii) and (iii). Besides, we assert that $\delta V \cap X \subset E$. Indeed, suppose there is $w \in \delta V \cap X$ but $w \notin E$. There then exists a positive integer n such that $w \notin B_n(E)$, and so $w \in \overline{V_n} \cap X - V_n$ because (ii) and (iii). We finally deduce that $w \in H - H_n \subset B_n(E)$ from (i), contradiction. Thus, the point x has a neighbourhood V with the desired smooth boundary. However, it is possible that V or $\mathbb{C}^2 - \overline{V}$ may not be connected. A picture of $\overline{V}$ could be the following:

$\overline{V} =$ 

We proceed as follows to solve this problem. Let W_1 be the connected component of V which contains to x, and let W_2 be the only unbounded connected component of $\mathbb{C}^2 - \overline{W_1}$ (notice that $\overline{W_1}$ is compact). Both components W_1 and W_2 are open because $\mathbb{C}^2$ is locally connected. Besides, their boundaries satisfy $\delta W_1 \subset \delta V$ and $\delta W_2 \subset \overline{W_1} - W_1 = \delta W_1$. Finally, consider the open set $W_3 = \mathbb{C}^2 - \overline{W_2}$. It is easy to see that $\overline{W_3} = \mathbb{C}^2 - W_2$ is compact, $W_1 \subset W_3$ and $\delta W_3 = \overline{W_3} \cap \overline{W_2} = \delta W_2$.

From the previous statements, we have that W_3 is a relatively compact open set such that $x \in W_3$ and $\delta W_3 \subset \delta V$, so $\delta W_3 \cap X \subset E$ and $\delta W_3 - E$ is a $\mathcal{C}^2$-smooth 3-manifold. The complement $\mathbb{C}^2 - \overline{W_3} = W_2$ is connected.

We assert that W_3 is connected as well. Let W_4 be an arbitrary connected component of W_3. Its boundary is an infinite set and satisfies $\delta W_4 \subset \delta W_3$. Hence, there exist a point $w \in \delta W_4 - E$ and an open ball B with center in w, such that $B \cap \delta W_3$ is diffeomorphic to $\mathbb{R}^3$ and $B - \delta W_3$ has exactly two connected components: $A = B \cap W_3$ and $B \cap W_2$ (recall that $\delta W_3 - E$ is $\mathcal{C}^2$-smooth). It is easy to see that $A \subset W_4$ because $\delta W_4 \subset \delta W_3$, and $A \cap W_1 \neq \varnothing$ because $\delta W_3 \subset \delta W_1$. That is, the connected set W_1 is contained in W_3 and meets to every connected component W_4; so is W_3 connected.

Finally, notice that E is polynomially convex because it is finite, and the function f is holomorphic in $\delta W_3 - E$ (recall that $\delta W_3 \cap X \subset E$ and f is holomorphic outside

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X). Whence, applying Theorem 3 of [4] or Theorem 3.1.1 of [1], the function f has a unique analytic extension over $x \in W_3$. ■

Actually, the equality $\delta W_3 \cap X = E$ holds, and δW_3 (resp. $\delta W_3 - E$) is connected because $\mathbb{C}^2$ (resp. $\mathbb{C}^2 - E$) is unicoherent (see [2, p. 397]).

We have so far shown that each closed set X homeomorphic to the real line is locally removable in $\mathbb{C}^2$, so can we extend this proof to consider an arbitrary complex manifold $\mathcal{M}$ of dimension two instead of $\mathbb{C}^2$ because $\mathcal{M}$ is locally biholomorphic to an open set of $\mathbb{C}^2$ and we can use exactly the same reasonings. Furthermore, since each finite or countable set is polynomially convex, we can relax the hypotheses over X and just demand that every point $x \in X$ has a relatively compact open neighbourhood V such that $\delta V \cap X$ is at most countable.

However, in order to show that every closed set homeomorphic to $\mathbb{R}^{N-1}$ is removable in $\mathbb{C}^N$ ($N \geq 3$), we firstly need to prove that the topological copies of sphere $\mathcal{S}^{N-2}$ are removable for the boundary in $\mathbb{C}^N$. We have strongly used the fact that $\mathcal{S}^0$ contains just two points.

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