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Higher Moments of Fourier Coefficients of Cusp Forms

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Abstract. Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight k for the full modular group $SL(2,\mathbb{Z})$. Let $\lambda_f(n)$, $\lambda_g(n)$, $\lambda_h(n)$ be the n-th normalized Fourier coefficients of three distinct holomorphic primitive cusp forms $f(z) \in S_{k_1}(\Gamma)$, $g(z) \in S_{k_2}(\Gamma)$, and $h(z) \in S_{k_3}(\Gamma)$, respectively. In this paper we study the cancellations of sums related to arithmetic functions, such as $\lambda_f(n)^4\lambda_g(n)^2$, $\lambda_g(n)^6$, $\lambda_g(n)^2\lambda_h(n)^4$, and $\lambda_g(n^3)^2$ twisted by the arithmetic function $\lambda_f(n)$.

1 Introduction

Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Suppose that $f(z) \in S_{k_1}(\Gamma)$, $g(z) \in S_{k_2}(\Gamma)$ and $h(z) \in S_{k_3}(\Gamma)$ are primitive cusp forms. We shall denote their corresponding normalized Fourier coefficients by $\lambda_f(n)$, $\lambda_g(n)$, and $\lambda_h(n)$.

Fourier coefficients of cusp forms are mysterious objects, and it is of interest to study their distribution. In 1927, Hecke [7] proved that

$$S(x) = \sum_{n\le x} \lambda_f(n) \ll_f x^{\frac{1}{2}}.$$

Subsequent improvements on $S(x)$ were made by Wilton [34], Walfisz [33], and implied by the work of Kloosterman [13], Davenport [1], Salié [27], and Weil [35]. As a corollary of the Ramanujan–Petersson conjecture proved by Deligne [2], it is known that for any $\varepsilon > 0$,

$$S(x) = \sum_{n\le x} \lambda_f(n) \ll_{f,\varepsilon} x^{\frac{1}{3}+\varepsilon}.$$

Further improvements are due to Hafner and Ivić [6], Rankin [24], and Wu [36].

In the 1930's, Rankin [23] and Selberg [28] introduced a method (the Rankin–Selberg method) and showed that

$$\sum_{n\le x} \lambda_f^2(n) = c_0 x + O_f(x^{\frac{3}{5}}), \qquad \sum_{n\le x} \lambda_f(n)\lambda_g(n) = O_{f,g}(x^{\frac{3}{5}}) \quad (f \ne g).$$

Nearly half a century later, the work of Moreno and Shahidi [22] implied that:

$$\sum_{n\le x} \lambda_f^4(n) \sim c_1 x \log x, \qquad x \to \infty.$$

In 2001, Fomenko [3] strengthened and then generalized these results by showing that

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(a) For any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^3(n) \ll_{f,\varepsilon} x^{\frac{5}{6}+\varepsilon}.$$

(b) For any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^2(n)\lambda_g(n) \ll_{f,g,\varepsilon} x^{\frac{5}{6}+\varepsilon}.$$

(c) Let F_1 be the Gelbart–Jacquet lift on $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$ associated with f, and let F_2 be the Gelbart–Jacquet lift on $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$ associated with g. If F_1 and F_2 are distinct, then for any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^2(n)\lambda_g^2(n) = c_2x + O_{f,g,\varepsilon}(x^{\frac{9}{10}+\varepsilon}).$$

(d) For any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^4(n) = c_1x\log x + c_3x + O_{f,\varepsilon}(x^{\frac{9}{10}+\varepsilon}).$$

In a series of papers [18–20], the first author further improved Fomenko's results and was able to consider more general higher moments of Fourier coefficients of cusp forms. For instance, the following results were established:

(a) for any $\varepsilon > 0$,

$$\sum_{n\le x}\lambda_f^6(n) = xP_1(\log x) + O_{f,\varepsilon}(x^{\frac{31}{32}+\varepsilon}),$$

where $P_1(x)$ is a polynomial of degree 4;

(b) for any $\varepsilon > 0$,

$$\sum_{n\le x}\lambda_f^8(n) = xP_2(\log x) + O_{f,\varepsilon}(x^{\frac{127}{128}+\varepsilon}),$$

where $P_2(x)$ is a polynomial of degree 13.

In fact, it is clear that for two distinct primitive cusp forms f and g, the earlier existing methods are able to establish asymptotic formulae with acceptable error terms (or nontrivial estimates) for sums of the type

$$\sum_{n\le x}\lambda_f(n)^i\lambda_g(n)^j,$$

for any $1\le i, j\le 4$; and for one primitive cusp form f, for sums of the type

$$\sum_{n\le x}\lambda_f(n)^j,$$

for any $1\le j\le 8$.

More recently, in [21] it was shown that the changes of sign in $\lambda_f(n)\lambda_g(n)$ cause cancellations on the twisted sums related to the positive-valued functions $\lambda_f(n)^4$ and $\lambda_h(n)^4$, namely,

$$\sum_{n\le x}\lambda_f(n)^5\lambda_g(n) \ll x^{\frac{31}{32}+\varepsilon}, \quad \sum_{n\le x}\lambda_f(n)\lambda_g(n)\lambda_h(n)^4 \ll x^{\frac{31}{32}+\varepsilon}.$$

This means that sequences $\{\lambda_f(n)\lambda_g(n)\}$ and $\{\lambda_f(n)^4\}$ (or $\{\lambda_h(n)^4\}$) are asymptotically orthogonal as $x\to\infty$.

In this short note, we study cancellations on sums related to $\lambda_f(n)^4\lambda_g(n)^2$, $\lambda_g(n)^6$, $\lambda_g(n)^2\lambda_h(n)^j$, $\lambda_g(n^2)^3$, and $\lambda_g(n^3)^2$ twisted by $\lambda_f(n)$. More precisely, we will prove the following theorems.

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Theorem 1.1 *For any* $\varepsilon > 0$, *we have*

$$\sum_{n\le x}\lambda_f^5(n)\lambda_g(n)^2 \ll_{f,g,\varepsilon} x^{\frac{184}{187}+\varepsilon}, \quad \sum_{n\le x}\lambda_f(n)\lambda_g(n)^6 \ll_{f,g,\varepsilon} x^{\frac{63}{64}+\varepsilon}.$$

Theorem 1.2 *For any* $\varepsilon > 0$, *we have*

$$\sum_{n\le x}\lambda_f(n)\lambda_g(n)^2\lambda_h(n)^j \ll_{f,g,h,\varepsilon} x^{1-\frac{1}{2^{j+2}}+\varepsilon},$$

where $2 \le j \le 4$.

Theorem 1.3 *For any* $\varepsilon > 0$, *we have*

$$\sum_{n\le x}\lambda_f(n)\lambda_g(n^2)^3 \ll_{f,g,\varepsilon} x^{\frac{26}{27}+\varepsilon}, \quad \sum_{n\le x}\lambda_f(n)\lambda_g(n^3)^2 \ll_{f,g,\varepsilon} x^{\frac{15}{16}+\varepsilon}.$$

Remark 1.4 The proofs of Theorems 1.1, 1.2, and 1.3 make use of the j-th symmetric power lifts with $j \le 4$ and the Rankin–Selberg convolution theory. In addition, we shall also exploit the important progress on the Langlands program, namely the functorial product for $GL_2 \times GL_3$ (see [15]).

2 Some Lemmas

Suppose that $f(z) \in S_{k_1}(\Gamma)$, $g(z) \in S_{k_2}(\Gamma)$, and $h(z) \in S_{k_3}(\Gamma)$ are primitive cusp forms. According to Deligne [2], for any prime number p there are $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p) \quad \text{and} \quad |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1.$$

We shall also use the notations $\alpha_g(p)$, $\alpha_h(p)$, $\beta_g(p)$, and $\beta_h(p)$ with the same meanings. The Hecke L-function $L(f,s)$ is defined by

$$L(f,s) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s} = \prod_p\left(1-\alpha_f(p)p^{-s}\right)^{-1}\left(1-\beta_f(p)p^{-s}\right)^{-1}.$$

The j-th symmetric power L-function attached to f is defined by

$$L(\mathrm{sym}^j f, s) = \prod_p\prod_{m=0}^{j}\left(1-\alpha_f(p)^{j-m}\beta_f(p)^m p^{-s}\right)^{-1} := \prod_p L_p(\mathrm{sym}^j f, s) \tag{2.1}$$

for $\Re s > 1$.

It is well known that for every $f \in S_{k_1}(\Gamma)$, there is associated an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$. This representation factors as a restricted tensor product of local GL_2 representations $\pi_f = \otimes_v{}'\pi_{f,v}$, where v runs over all places of $\mathbb{Q}$. If $v = p$ is finite, $\pi_{f,p}$ is an unramified principal series representation, and one associates with it a semi-simple $SL_2(\mathbb{C})$-conjugacy class

$$g_f^\sharp(p) = \begin{pmatrix}\alpha_f(p) & 0\\ 0 & \beta_f(p)\end{pmatrix}^\sharp.$$

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It should be stressed that this $g_f^\sharp(p)$ and $g_f(p)$ below are related to π_f, and the reader should not confuse them with $g(z)$ and π_g from the context. The automorphic L-function associated with π_f is defined by

$$L(\pi_f, s) = \prod_p \det\big(I - p^{-s} g_f(p)\big)^{-1},$$

which coincides with $L(f,s)$. It is well known that

$$\lambda_f(p^j) = \sum_{m=0}^{j} \alpha_f(p)^{j-m} \beta_f(p)^m = \operatorname{tr}\big(\operatorname{sym}^j(g_f(p))\big),$$

where sym^j denotes the symmetric j-th power representation of the standard representation of GL_2. Thus the local L-function of the j-th symmetric power L-function is given by

$$L_p(\operatorname{sym}^j f, s) = \prod_p \det\big(I - p^{-s} \operatorname{sym}^j(g_f(p))\big)^{-1}.$$

As a part of the far-reaching Langlands program, there exists an automorphic cuspidal self-dual representation, denoted by $\operatorname{sym}^j \pi_f = \otimes_v{}' \operatorname{sym}^j \pi_{f,v}$ of $\mathrm{GL}_{j+1}(\mathbb{A}_\mathbb{Q})$ whose local L-factors $L(\operatorname{sym}^j \pi_{f,p}, s)$ agree with the local L-factors $L_p(\operatorname{sym}^j f, s)$.

Thanks to the works of Gelbart and Jacquet [4], Kim and Shahidi [15,16], and Kim [14] in which it is established that the automorphy of the j-th symmetric power lifts (up to 4), the predicted analytic properties and functional equations of the symmetric power L-functions $L(\operatorname{sym}^j f, s)$ $(j = 2,3,4)$ actually hold, we have the following lemma.

Lemma 2.1 *Let $f(z) \in S_k(\Gamma)$ be a primitive cusp form. The j-th symmetric power L-function $L(\operatorname{sym}^j f, s)$ is defined in (2.1). For $j = 1,2,3,4$, there exists an automorphic cuspidal self-dual representation, denoted by $\operatorname{sym}^j \pi_f = \otimes' \operatorname{sym}^j \pi_{f,v}$ of $\mathrm{GL}_{j+1}(\mathbb{A}_\mathbb{Q})$ whose local L-factors $L(\operatorname{sym}^j \pi_{f,p}, s)$ agree with the local L-factors $L_p(\operatorname{sym}^j f, s)$ in (2.1). In particular, for $j = 1,2,3,4$, $L(\operatorname{sym}^j f, s)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$, and it satisfies a certain functional equation of Riemann-type.*

Proof This lemma follows from Gelbart and Jacquet [4] for $k = 2$, and from the recent works of Kim and Shahidi [15,16] and Kim [14] when $k = 3,4$. ■

Besides Lemma 2.1, we need the result of Kim and Shadihi [15] on the automorphy of the tensor product transfer from automorphic representations on $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6.

Lemma 2.2 *Let π and π' be cuspidal automorphic representations of $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, respectively. Then there exists an isobaric automorphic representation $\pi \boxtimes \pi'$ of $\mathrm{GL}_6(\mathbb{A}_\mathbb{Q})$ such that*

$$L(\pi \boxtimes \pi', s) = L(\pi \otimes \pi', s),$$

where $L(\pi \otimes \pi', s)$ is the Rankin–Selberg L-function associated with π and π'.

We shall also use the special case of the cuspidality criterion for the functorial product $\mathrm{GL}_2 \times \mathrm{GL}_3$ proved by Ramakrishnan and Wang [26] which is embodied in :

Lemma 2.3 *Suppose that $f(z) \in S_{k_1}(\Gamma)$ and $g(z) \in S_{k_2}(\Gamma)$ are distinct primitive cusp forms, and $L(\pi_f \otimes \mathrm{sym}^2 \pi_g, s)$ is the Rankin-Selberg L-function associated to π_f on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ and $\mathrm{sym}^2 \pi_g$ on $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$. Then there exists a cuspidal representation $\pi_f \boxtimes \mathrm{sym}^2 \pi_g := \Pi_{f,g}$ on $\mathrm{GL}_6(\mathbb{A}_\mathbb{Q})$ such that*

$$L(\pi_f \otimes \mathrm{sym}^2 \pi_g, s) = L(\Pi_{f,g}, s).$$

Proof Theorem 3.1 in Ramakrishnan and Wang [26] implies that $\pi' \boxtimes \pi$ is cuspidal if (i) π' is not dihedral and (ii) π is not a twist of the adjoint square $\mathrm{Ad}(\pi')$.

Since dihedral forms are not present in our case (holomorphic primitive cusp forms for the full modular group), π_f is not dihedral. It is also clear that the equivalence $\mathrm{Ad}(\pi_f) \cong \mathrm{Ad}(\pi_g)$ (namely $\mathrm{sym}^2 \pi_f \cong \mathrm{sym}^2 \pi_g$ in our case) implies that $\pi_f \cong \pi_g$. This shows that $\pi_f \boxtimes \mathrm{sym}^2 \pi_g := \Pi_{f,g}$ is cuspidal. ∎

Lemma 2.4 *For $\Re s > 1$, define*

$$L_{j-1}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\lambda_g(n)^2\lambda_h(n)^j}{n^s},$$

where $j = 2, 3, 4$. Then we have

$$\begin{aligned}
L_1(s) &= L(\pi_f, s)L(\pi_f \otimes \mathrm{sym}^2 \pi_g, s)L(\pi_f \otimes \mathrm{sym}^2 \pi_h, s)\\
&\qquad \times L((\pi_f \otimes \mathrm{sym}^2 \pi_g) \otimes \mathrm{sym}^2 \pi_h, s)U_1(s),\\
L_2(s) &= L(\pi_f \otimes \pi_h, s)^2 L(\pi_f \otimes \mathrm{sym}^3 \pi_h, s)L((\pi_f \otimes \mathrm{sym}^2 \pi_g) \otimes \pi_h, s)^2\\
&\qquad \times L((\pi_f \otimes \mathrm{sym}^2 \pi_g) \otimes \mathrm{sym}^3 \pi_h, s)U_2(s),\\
L_3(s) &= L(\pi_f, s)^2 L(\pi_f \otimes \mathrm{sym}^2 \pi_h, s)^3 L(\pi_f \otimes \mathrm{sym}^4 \pi_h, s)L(\pi_f \otimes \mathrm{sym}^2 \pi_g, s)^2\\
&\qquad \times L((\pi_f \otimes \mathrm{sym}^2 \pi_g) \otimes \mathrm{sym}^2 \pi_h, s)^3 L((\pi_f \otimes \mathrm{sym}^2 \pi_g) \otimes \mathrm{sym}^4 \pi_h, s)U_3(s),
\end{aligned}$$

where $U_j(s)$ are Dirichlet series, which converge uniformly and absolutely in the half plane $\Re s \geq 1/2 + \varepsilon$ for any $\varepsilon > 0$.

Here $L(\pi_f \otimes \mathrm{sym}^j \pi_g, s)$, $L((\pi_f \otimes \mathrm{sym}^2 \pi_g) \otimes \mathrm{sym}^j \pi_h, s)$ (with $j \leq 4$) are the Rankin-Selberg L-functions associated with corresponding automorphic cuspidal representations.

Proof The Rankin-Selberg L-function $L(\pi_f \otimes \mathrm{sym}^j \pi_g, s)$ is initially defined by

$$L(\pi_f \otimes \mathrm{sym}^j \pi_g, s) =$$
$$\prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)\alpha_g(p)^{j-m}\beta_g(p)^m p^{-s})^{-1}(1 - \beta_f(p)\alpha_g(p)^{j-m}\beta_g(p)^m p^{-s})^{-1}.$$

The product over primes also gives a Dirichlet series representation for $L(\pi_f \otimes \mathrm{sym}^j \pi_g, s)$: for $\Re s > 1$,

$$L(\pi_f \otimes \mathrm{sym}^j \pi_g, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi_f \otimes \mathrm{sym}^j \pi_g}(n)}{n^s},$$

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where $\lambda_{\pi_f\otimes \mathrm{sym}^j \pi_g}(n)$ is a multiplicative function that satisfies

$$\lambda_{\pi_f\otimes \mathrm{sym}^j \pi_g}(p) = \lambda_f(p)\lambda_g(p^j),$$

when p is a prime.

Similarly, the Rankin–Selberg L-function $L((\pi_f\otimes \mathrm{sym}^2 \pi_g)\otimes \mathrm{sym}^j \pi_h, s)$ is defined by

$$\begin{aligned}
&L\big((\pi_f \otimes \mathrm{sym}^2 \pi_g) \otimes \mathrm{sym}^j \pi_h, s\big)\\
&\quad = \prod_p \prod_{m=0}^{j} (1-\alpha_f(p)\alpha_g(p)^2\alpha_h(p)^{j-m}\beta_h(p)^m p^{-s})^{-1}\\
&\qquad \times (1-\beta_f(p)\beta_g(p)^2\alpha_h(p)^{j-m}\beta_h(p)^m p^{-s})^{-1}\\
&\qquad \times (1-\alpha_f(p)\alpha_h(p)^{j-m}\beta_h(p)^m p^{-s})^{-1}(1-\beta_f(p)\alpha_h(p)^{j-m}\beta_h(p)^m p^{-s})^{-1}\\
&\qquad \times (1-\alpha_f(p)\beta_g(p)^2\alpha_h(p)^{j-m}\beta_h(p)^m p^{-s})^{-1}\\
&\qquad \times (1-\beta_f(p)\beta_g(p)^2\alpha_h(p)^{j-m}\beta_h(p)^m p^{-s})^{-1}.
\end{aligned}$$

By using similar notations, one can easily check that $\lambda_{(\pi_f\otimes \mathrm{sym}^2 \pi_g)\otimes \mathrm{sym}^j \pi_h}(n)$ is a multiplicative function, which satisfies

$$\lambda_{(\pi_f\otimes \mathrm{sym}^2 \pi_g)\otimes \mathrm{sym}^j \pi_h}(p) = \lambda_f(p)\lambda_g(p^2)\lambda_h(p^j),$$

when p is a prime and $j = 2, 3, 4$.

On the other hand, from the recursive relations (coming from the Hecke Theory), we have

$$\lambda_h(p)^2 = 1+\lambda_h(p^2), \quad \lambda_h(p)^3 = 2\lambda_h(p)+\lambda_h(p^3), \quad \lambda_h(p)^4 = 2+3\lambda_h(p^2)+\lambda_h(p^4).$$

Then

$$\begin{aligned}
\lambda_f(p)\lambda_g(p)^2\lambda_h(p)^2 &= \lambda_f(p) + \lambda_f(p)\lambda_h(p^2) + \lambda_f(p)\lambda_g(p^2)\\
&\quad + \lambda_f(p)\lambda_g(p^2)\lambda_h(p^2);\\
\lambda_f(p)\lambda_g(p)^2\lambda_h(p)^3 &= 2\lambda_f(p)\lambda_h(p) + \lambda_f(p)\lambda_h(p^3) + 2\lambda_f(p)\lambda_g(p^2)\lambda_h(p)\\
&\quad + \lambda_f(p)\lambda_g(p^2)\lambda_h(p^3);\\
\lambda_f(p)\lambda_g(p)^2\lambda_h(p)^4 &= 2\lambda_f(p) + 3\lambda_f(p)\lambda_h(p^2) + \lambda_f(p)\lambda_h(p^4) + 2\lambda_f(p)\lambda_g(p^2)\\
&\quad + 3\lambda_f(p)\lambda_g(p^2)\lambda_h(p^2) + \lambda_f(p)\lambda_g(p^2)\lambda_h(p^4).
\end{aligned}$$

These identities essentially determine Lemma 2.4.

To illustrate this, we consider $L_3(s)$. Since $\lambda_f(n)\lambda_g(n)^2\lambda_h(n)^4$ is a multiplicative function, we have

$$L_3(s) = \prod_p\Big(1+\frac{\lambda_f(p)\lambda_g(p)^2\lambda_h(p)^4}{p^s}+\frac{\lambda_f(p^2)\lambda_g(p^2)^2\lambda_h(p^2)^4}{p^{2s}}+\cdots\Big). \tag{2.2}$$

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On the other hand, we have

(2.3)

$$
\begin{aligned}
&L(\pi_f,s)^2L(\pi_f\otimes\operatorname{sym}^2\pi_h,s)^3L(\pi_f\otimes\operatorname{sym}^4\pi_h,s)L(\pi_f\otimes\operatorname{sym}^2\pi_g,s)^2\\
&\qquad\times L((\pi_f\otimes\operatorname{sym}^2\pi_g)\otimes\operatorname{sym}^2\pi_h,s)^3L((\pi_f\otimes\operatorname{sym}^2\pi_g)\otimes\operatorname{sym}^4\pi_h,s)\\
&\quad=\prod_p\Big(1+\frac{\lambda_f(p)}{p^s}+\cdots\Big)^2\Big(1+\frac{\lambda_{\pi_f\otimes\operatorname{sym}^2\pi_h}(p)}{p^s}+\cdots\Big)^3\\
&\qquad\times\Big(1+\frac{\lambda_{\pi_f\otimes\operatorname{sym}^4\pi_h}(p)}{p^s}+\cdots\Big)\Big(1+\frac{\lambda_{\pi_f\otimes\operatorname{sym}^2\pi_g}(p)}{p^s}+\cdots\Big)^2\\
&\qquad\times\Big(1+\frac{\lambda_{(\pi_f\otimes\operatorname{sym}^2\pi_g)\otimes\operatorname{sym}^2\pi_h}(p)}{p^s}+\cdots\Big)^3\Big(1+\frac{\lambda_{(\pi_f\otimes\operatorname{sym}^2\pi_g)\otimes\operatorname{sym}^4\pi_h}(p)}{p^s}+\cdots\Big)\\
&\quad=\prod_p\Big(1+\frac{\lambda_f(p)\lambda_g(p)^2\lambda_h(p)^4}{p^s}+\cdots\Big).
\end{aligned}
$$

Here we have used

$$
\begin{aligned}
&2\lambda_f(p)+3\lambda_{\pi_f\otimes\operatorname{sym}^2\pi_h}(p)+\lambda_{\pi_f\otimes\operatorname{sym}^4\pi_h}(p)+2\lambda_{\pi_f\otimes\operatorname{sym}^2\pi_g}(p)\\
&\qquad+3\lambda_{(\pi_f\otimes\operatorname{sym}^2\pi_g)\otimes\operatorname{sym}^2\pi_h}(p)+\lambda_{(\pi_f\otimes\operatorname{sym}^2\pi_g)\otimes\operatorname{sym}^4\pi_h}(p)\\
&\quad=2\lambda_f(p)+3\lambda_f(p)\lambda_h(p^2)+\lambda_f(p)\lambda_h(p^4)+2\lambda_f(p)\lambda_g(p^2)\\
&\qquad+3\lambda_f(p)\lambda_g(p^2)\lambda_h(p^2)+\lambda_f(p)\lambda_g(p^2)\lambda_h(p^4)\\
&\quad=\lambda_f(p)\lambda_g(p)^2\lambda_h(p)^4.
\end{aligned}
$$

By comparing (2.2) with (2.3), we find that

$$
\begin{aligned}
L_3(s)&=L(\pi_f,s)^2L(\pi_f\otimes\operatorname{sym}^2\pi_h,s)^3L(\pi_f\otimes\operatorname{sym}^4\pi_h,s)L(\pi_f\otimes\operatorname{sym}^2\pi_g,s)^2\\
&\quad\times L((\pi_f\otimes\operatorname{sym}^2\pi_g)\otimes\operatorname{sym}^2\pi_h,s)^3L((\pi_f\otimes\operatorname{sym}^2\pi_g)\otimes\operatorname{sym}^4\pi_h,s)U_3(s),
\end{aligned}
$$

where $U_3(s)$ is a Dirichlet series, which converge uniformly and absolutely in the half plane $\Re s\geq 1/2+\varepsilon$ for any $\varepsilon>0$. Here we have used the Ramanujan–Petterson bound $|\lambda_f(n)|\leq d(n)$ (established by Deligne). ■

Lemma 2.5 *For $\Re s>1$, define*

$$
A(s)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)^5\lambda_g(n)^2}{n^s},\qquad B(s)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)\lambda_g(n)^6}{n^s}.
$$

Then we have

$$
\begin{aligned}
A(s)&=L(\pi_f,s)^2L(\operatorname{sym}^2\pi_f\otimes\pi_f,s)^3L(\operatorname{sym}^4\pi_f\otimes\pi_f,s)L(\pi_f\otimes\operatorname{sym}^2\pi_g,s)^2\\
&\quad\times L(\operatorname{sym}^2\pi_f\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s)^3L(\operatorname{sym}^4\pi_f\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s)U(s),
\end{aligned}
$$

and

$$
\begin{aligned}
B(s)&=L(\pi_f,s)^2L(\operatorname{sym}^2\pi_g\otimes\pi_f,s)^3L(\operatorname{sym}^4\pi_g\otimes\pi_f,s)L(\pi_f\otimes\operatorname{sym}^2\pi_g,s)^2\\
&\quad\times L(\operatorname{sym}^2\pi_g\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s)^3L(\operatorname{sym}^4\pi_g\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s)V(s),
\end{aligned}
$$

where $U(s)$ and $V(s)$ are Dirichlet series, which converge uniformly and absolutely in the half plane $\Re s\geq 1/2+\varepsilon$ for any $\varepsilon>0$.

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Proof The proof of this lemma is similar to that of Lemma 2.4. Recall that

$$\lambda_f(p)^4 = 2 + 3\lambda_f(p^2) + \lambda_f(p^4),$$

where p is prime. Then this lemma is based on the following two identities:

$$\begin{aligned}
\lambda_f(p)^5\lambda_g(p)^2 &= \lambda_f(p)^4(\lambda_f(p) + \lambda_f(p)\lambda_g(p^2)) \\
&= 2\lambda_f(p) + 3\lambda_f(p^2)\lambda_f(p) + \lambda_f(p^4)\lambda_f(p) \\
&\quad + 2\lambda_f(p)\lambda_g(p^2) + 3\lambda_f(p^2)\lambda_f(p)\lambda_g(p^2) + \lambda_f(p^4)\lambda_f(p)\lambda_g(p^2) \\
&= 2\lambda_f(p) + 3\lambda_{\mathrm{sym}^2\pi_f\otimes\pi_f}(p) + \lambda_{\mathrm{sym}^4\pi_f\otimes\pi_f}(p) \\
&\quad + 2\lambda_{\pi_f\otimes\mathrm{sym}^2\pi_g}(p) + 3\lambda_{\mathrm{sym}^2\pi_f\otimes(\pi_f\otimes\mathrm{sym}^2\pi_g)}(p) \\
&\quad + \lambda_{\mathrm{sym}^4\pi_f\otimes(\pi_f\otimes\mathrm{sym}^2\pi_g)}(p),
\end{aligned}$$

and

$$\begin{aligned}
\lambda_f(p)\lambda_g(p)^6 &= (\lambda_f(p) + \lambda_f(p)\lambda_g(p^2))\lambda_g(p)^4 \\
&= 2\lambda_f(p) + 3\lambda_g(p^2)\lambda_f(p) + \lambda_g(p^4)\lambda_f(p) \\
&\quad + 2\lambda_f(p)\lambda_g(p^2) + 3\lambda_g(p^2)\lambda_f(p)\lambda_g(p^2) + \lambda_g(p^4)\lambda_f(p)\lambda_g(p^2) \\
&= 2\lambda_f(p) + 3\lambda_{\mathrm{sym}^2\pi_g\otimes\pi_f}(p) + \lambda_{\mathrm{sym}^4\pi_g\otimes\pi_f}(p) \\
&\quad + 2\lambda_{\pi_f\otimes\mathrm{sym}^2\pi_g}(p) + 3\lambda_{\mathrm{sym}^2\pi_g\otimes(\pi_f\otimes\mathrm{sym}^2\pi_g)}(p) \\
&\quad + \lambda_{\mathrm{sym}^4\pi_g\otimes(\pi_f\otimes\mathrm{sym}^2\pi_g)}(p).
\end{aligned}$$

■

Lemma 2.6 *For $\Re s > 1$, define*

$$C(s) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)\lambda_g(n^2)^3}{n^s}, \quad D(s) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)\lambda_g(n^3)^2}{n^s}.$$

Then we have

$$\begin{aligned}
C(s) &= L(\pi_f\otimes\mathrm{sym}^2\pi_g, s)L(\mathrm{sym}^2\pi_g\otimes(\pi_f\otimes\mathrm{sym}^2\pi_g), s) \\
&\quad\times L(\mathrm{sym}^4\pi_g\otimes(\pi_f\otimes\mathrm{sym}^2\pi_g), s)U'(s),
\end{aligned}$$

and

$$D(s) = L(\pi_f, s)L(\mathrm{sym}^4\pi_g\otimes(\pi_f\otimes\mathrm{sym}^2\pi_g), s)V'(s),$$

where $U'(s)$ and $V'(s)$ are Dirichlet series, which converge uniformly and absolutely in the half plane $\Re s \ge 1/2 + \varepsilon$ for any $\varepsilon > 0$.

Proof Recall that $g_f^\sharp(p)$ is the semi-simple $\mathrm{SL}_2(\mathbb{C})$-conjugacy class associated with π_f. Then we have that for $a \ge b$,

$$\mathrm{sym}^a g_f(p)\otimes\mathrm{sym}^b g_f(p) = \bigoplus_{0\le r\le b}\mathrm{sym}^{a+b-2r} g_f(p). \tag{2.4}$$

In particular, (2.4) with $(a,b) = (2,2)$, $(4,2)$, and $(3,3)$ implies that

$$\begin{aligned}
\lambda_g(p^2)^2 &= \lambda_{\mathrm{sym}^2\pi_g}(p)^2 = 1 + \lambda_g(p^2) + \lambda_g(p^4), \\
\lambda_g(p^2)\lambda_g(p^4) &= \lambda_{\mathrm{sym}^2\pi_g}(p)\lambda_{\mathrm{sym}^4\pi_g}(p) = \lambda_g(p^2) + \lambda_g(p^4) + \lambda_g(p^6),
\end{aligned}$$

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and

$$\lambda_g(p^3)^2 = \lambda_{\mathrm{sym}^3\,\pi_g}(p)^2 = 1 + \lambda_g(p^2) + \lambda_g(p^4) + \lambda_g(p^6).$$

Hence, it is easy to see that

$$\begin{aligned}(2.5)\ \lambda_f(p)\lambda_g(p^2)^3 & \\ &= \lambda_f(p)\lambda_g(p^2)(1+\lambda_g(p^2)+\lambda_g(p^4))\\ &= \lambda_f(p)\lambda_g(p^2) + \lambda_f(p)\lambda_g(p^2)\lambda_g(p^2) + \lambda_f(p)\lambda_g(p^2)\lambda_g(p^4)\\ &= \lambda_{\pi_f\otimes\mathrm{sym}^2\,\pi_g}(p) + \lambda_{\mathrm{sym}^2\,\pi_g\otimes(\pi_f\otimes\mathrm{sym}^2\,\pi_g)}(p) + \lambda_{\mathrm{sym}^4\,\pi_g\otimes(\pi_f\otimes\mathrm{sym}^2\,\pi_g)}(p),\\ (2.6)\ \lambda_f(p)\lambda_g(p^3)^2 & \\ &= \lambda_f(p)(1+\lambda_g(p^2)+\lambda_g(p^4)+\lambda_g(p^6))\\ &= \lambda_f(p)(1+\lambda_g(p^2)\lambda_g(p^4)) = \lambda_f(p) + \lambda_f(p)\lambda_g(p^2)\lambda_g(p^4)\\ &= \lambda_f(p) + \lambda_{\mathrm{sym}^4\,\pi_g\otimes(\pi_f\otimes\mathrm{sym}^2\,\pi_g)}(p).\end{aligned}$$

The identities (2.5) and (2.6) determine Lemma 2.6. ■

3 Proof of the Theorems

Suppose that $f(z) \in S_{k_1}(\Gamma)$, $g(z) \in S_{k_2}(\Gamma)$, and $h(z) \in S_{k_3}(\Gamma)$ are primitive cusp forms. The L-function $L((\pi_f \otimes \mathrm{sym}^2\,\pi_g) \otimes \mathrm{sym}^j\,\pi_h, s)$ $(1 \le j \le 4)$ has an analytic continuation to be an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann-type.

In fact, from Lemma 2.1, for $1 \le j \le 4$, $\mathrm{sym}^j\,\pi_h$ is an automorphic cuspidal self-dual representation on $\mathrm{GL}_{j+1}(\mathbb{A}_\mathbb{Q})$. From Lemma 2.3, $\pi_f \otimes \mathrm{sym}^2 \pi_g$ is an automorphic cuspidal self-dual representation on $\mathrm{GL}_6(\mathbb{A}_\mathbb{Q})$. Then from the works about the Rankin–Selberg theory associated with two automorphic cuspidal representations developed by Jacquet, Piatetski-Shapiro, and Shalika [10], Jacquet and Shalika [11, 12], Shahidi [29–32], and the reformulation of Rudnick and Sarnak [25], we know the analytic properties for the Rankin–Selberg L-functions $L((\pi_f \otimes \mathrm{sym}^2\,\pi_g) \otimes \mathrm{sym}^j\,\pi_h, s)$ with $j = 1, 2, 3, 4$.

Remark 3.1 In the sense of Iwaniec and Kowalski [9, Chapter 5], L-functions appearing in Lemmas 2.4, 2.5, and 2.6:

$$L(\pi_f, s)L(\pi_f \otimes \mathrm{sym}^2\,\pi_g, s)L(\pi_f \otimes \mathrm{sym}^2\,\pi_h, s)L((\pi_f \otimes \mathrm{sym}^2\,\pi_g) \otimes \mathrm{sym}^2\,\pi_h, s),$$

$$\begin{aligned}&L(\pi_f \otimes \pi_h, s)^2 L(\pi_f \otimes \mathrm{sym}^3\,\pi_h, s)L((\pi_f \otimes \mathrm{sym}^2\,\pi_g) \otimes \pi_h, s)^2\\ &\qquad\qquad \times L((\pi_f \otimes \mathrm{sym}^2\,\pi_g) \otimes \mathrm{sym}^3\,\pi_h, s),\\ &L(\pi_f, s)^2 L(\pi_f \otimes \mathrm{sym}^2\,\pi_h, s)^3 L(\pi_f \otimes \mathrm{sym}^4\,\pi_h, s)L(\pi_f \otimes \mathrm{sym}^2\,\pi_g, s)^2\\ &\qquad\qquad \times L((\pi_f \otimes \mathrm{sym}^2\,\pi_g) \otimes \mathrm{sym}^2\,\pi_h, s)^3 L((\pi_f \otimes \mathrm{sym}^2\,\pi_g) \otimes \mathrm{sym}^4\,\pi_h, s),\\ &L(\pi_f, s)^2 L(\mathrm{sym}^2\,\pi_f \otimes \pi_f, s)^3 L(\mathrm{sym}^4\,\pi_f \otimes \pi_f, s)L(\pi_f \otimes \mathrm{sym}^2\,\pi_g, s)^2\\ &\qquad\qquad \times L(\mathrm{sym}^2\,\pi_f \otimes (\pi_f \otimes \mathrm{sym}^2\,\pi_g), s)^3 L(\mathrm{sym}^4\,\pi_f \otimes (\pi_f \otimes \mathrm{sym}^2\,\pi_g), s),\end{aligned}$$

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$$L(\pi_f,s)^2L(\operatorname{sym}^2\pi_g\otimes\pi_f,s)^3L(\operatorname{sym}^4\pi_g\otimes\pi_f,s)L(\pi_f\otimes\operatorname{sym}^2\pi_g,s)^2 \times L(\operatorname{sym}^2\pi_g\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s)^3L(\operatorname{sym}^4\pi_g\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s),$$

$$L(\pi_f\otimes\operatorname{sym}^2\pi_g,s)L(\operatorname{sym}^2\pi_g\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s)L(\operatorname{sym}^4\pi_g\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s),$$

and

$$L(\pi_f,s)L(\operatorname{sym}^4\pi_g\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s)$$

are general L-functions of degree 32, 64, 128, 128, 128, 54, and 32, respectively. In particular, they satisfy the conditions of Lemma 2.4 in Lau and Lü [17], which states that if we suppose that $L(f,s)$ is a product of two general L-functions L_1, L_2 with both $\deg L_i \ge 2$, and $L(f,s)$ satisfies the Ramanujan conjecture, then for any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f(n) = M(x) + O(x^{1-\frac{2}{m}+\varepsilon}),$$

where $M(x) = \operatorname{res}_{s=1}L(f,s)x^s/s$ and $m = \deg L$. This proves Theorems 1.1, 1.2, and 1.3 in general.

The results stated in the theorems need not be the best, and it might be possible to improve them slightly. For instance, the above arguments lead only to

$$\sum_{n\le x}\lambda_f^5(n)\lambda_g(n)^2 \ll_{f,g,\varepsilon} x^{\frac{63}{64}+\varepsilon}.$$

The aim here is to go further. We make use of the identity

$$L(\operatorname{sym}^2\pi_f\otimes\pi_f,s) = L(\pi_f,s)L(\operatorname{sym}^3\pi_f,s)$$

so that we have

$$A(s) = L(\pi_f,s)^5L(\operatorname{sym}^3\pi_f,s)^3L(\operatorname{sym}^4\pi_f\otimes\pi_f,s)L(\pi_f\otimes\operatorname{sym}^2\pi_g,s)^2 \times L(\operatorname{sym}^2\pi_f\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s)^3L(\operatorname{sym}^4\pi_f\otimes(\pi_f\otimes\operatorname{sym}^2\pi_g),s)U(s).$$

We also need the following two lemmas (to be used in the sequel).

Lemma 3.2 *Suppose that $\mathfrak{L}(s)$ is a general L-function of degree m. Then for any $\varepsilon > 0$, we have*

$$\mathfrak{L}(\sigma+it) \ll (|t|+1)^{\frac{m}{2}(1-\sigma)+\varepsilon} \tag{3.1}$$

uniformly for $1/2 \le \sigma \le 1+\varepsilon$ and $|t| \ge 1$.

Proof This is the convexity bound for $\mathfrak{L}(s)$, which can be proved by the functional equation, the asymptotic properties of the Γ-function, and the Phragmén–Lindelöf theorem. ∎

Lemma 3.3 *Let f is a primitive holomorphic cusp form and $\varepsilon > 0$. Then we have*

$$\int_0^T \left|L(f,\tfrac{5}{8}+i\tau)\right|^4 \mathrm{d}\tau \ll_\varepsilon T^{1+\varepsilon}$$

uniformly for $T \geqslant 1$, and

$$L(f,\sigma+i\tau) \ll_{f,\varepsilon} (|\tau|+1)^{\max\{(2/3)(1-\sigma),\,0\}+\varepsilon} \tag{3.2}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$ and $|\tau| \geqslant 1$.

Proof See, *e.g.*, [8, Theorem 2, (1.8)] and [5, Corollary].

Recall that

$$
\begin{aligned}
A(s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)^5 \lambda_g(n)^2}{n^s} \\
&= L(\pi_f, s)^5 L(\operatorname{sym}^3 \pi_f, s)^3 L(\operatorname{sym}^4 \pi_f \otimes \pi_f, s) L(\pi_f \otimes \operatorname{sym}^2 \pi_g, s)^2 \\
&\quad \times L(\operatorname{sym}^2 \pi_f \otimes (\pi_f \otimes \operatorname{sym}^2 \pi_g), s)^3 L(\operatorname{sym}^4 \pi_f \otimes (\pi_f \otimes \operatorname{sym}^2 \pi_g), s) U(s)
\end{aligned}
$$

can be analytically continued to be an entire function in the half-plane $\Re s > 1/2$.

By the Perron formula (see [9, Proposition 5.54]), we have

$$
\sum_{n \le x} \lambda_f^5(n) \lambda_g(n)^2 = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} A(s) \frac{x^s}{s} \, ds + O\Big(\frac{x^{1+\varepsilon}}{T} \Big),
$$

where $b = 1 + \varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later.

Then we move the line of integration to $\Re s = \frac{5}{8}$. By Cauchy's theorem, we have

$$
\begin{aligned}
\sum_{n \le x} \lambda_f^5(n) \lambda_g(n)^2 &= \frac{1}{2\pi i} \Big\{ \int_{\frac{5}{8}-iT}^{\frac{5}{8}+iT} + \int_{\frac{5}{8}+iT}^{b+iT} + \int_{b-iT}^{\frac{5}{8}-iT} \Big\} A(s) \frac{x^s}{s} \, ds + O\Big(\frac{x^{1+\varepsilon}}{T} \Big) \\
&:= J_1 + J_2 + J_3 + O\Big(\frac{x^{1+\varepsilon}}{T} \Big).
\end{aligned} \tag{3.3}
$$

For J_1, we have

$$
\begin{aligned}
J_1 \ll x^{\frac{5}{8}} \int_1^T \Big| & L\Big(\pi_f, \frac{5}{8} + it\Big)^5 L\Big(\operatorname{sym}^3 \pi_f, \frac{5}{8} + it\Big)^3 L\Big(\operatorname{sym}^4 \pi_f \otimes \pi_f, \frac{5}{8} + it\Big) \\
& \times L\Big(\pi_f \otimes \operatorname{sym}^2 \pi_g, \frac{5}{8} + it\Big)^2 L\Big(\operatorname{sym}^2 \pi_f \otimes (\pi_f \otimes \operatorname{sym}^2 \pi_g), \frac{5}{8} + it\Big)^3 \\
& \times L\Big(\operatorname{sym}^4 \pi_f \otimes (\pi_f \otimes \operatorname{sym}^2 \pi_g), \frac{5}{8} + it\Big) \Big| t^{-1} dt + x^{\frac{5}{8}+\varepsilon}.
\end{aligned}
$$

By Lemmas 3.2 and 3.3, we have

$$
\begin{aligned}
J_1 &\ll x^{\frac{5}{8}} T^{\frac{2}{3} \times \frac{3}{8} + (2 \times \frac{3}{8}) \times 3 + 5 \times \frac{3}{8} + (3 \times \frac{3}{8}) \times 2 + (9 \times \frac{3}{8}) \times 3 + 15 \times \frac{3}{8} + \varepsilon} \int_1^T \Big| L\Big(\pi_f, \frac{5}{8} + it\Big) \Big|^4 t^{-1} dt \\
&\quad + x^{\frac{5}{8}+\varepsilon} \\
&\ll x^{\frac{5}{8}+\varepsilon} T^{\frac{179}{8}+\varepsilon}.
\end{aligned} \tag{3.4}
$$

For the integrals over the horizontal segments, we use (3.2) (for the factor $L(\pi_f, s)^5$ whose degree is 10) and (3.1) (for rest of the factors) with $m = 118$ to bound

$$
\begin{aligned}
J_2 + J_3 &\ll \max_{\frac{5}{8} \le \sigma \le b} x^{\sigma} T^{(59 + \frac{10}{3})(1-\sigma)+\varepsilon} T^{-1} \ll \max_{\frac{5}{8} \le \sigma \le b} x^{\sigma} T^{\frac{187}{3}(1-\sigma)+\varepsilon} T^{-1} \\
&= \max_{\frac{5}{8} \le \sigma \le b} \Big(\frac{x}{T^{\frac{187}{3}}} \Big)^{\sigma} T^{\frac{184}{3}+\varepsilon} \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{5}{8}+\varepsilon} T^{\frac{179}{8}+\varepsilon}.
\end{aligned} \tag{3.5}
$$

From (3.3), (3.4), and (3.5), we have

$$
\sum_{n \le x} \lambda_f^5(n) \lambda_g(n)^2 \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{5}{8}+\varepsilon} T^{\frac{179}{8}+\varepsilon}. \tag{3.6}
$$

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On taking $T = x^{\frac{3}{187}}$ in (3.6), we have

$$\sum_{n\leq x} \lambda_f^5(n)\lambda_g(n)^2 \ll x^{\frac{184}{187}+\varepsilon}.$$

This completes the proof. ∎

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