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SOME EXAMPLES OF FINITENESS CONDITIONS IN CENTRE-BY-METABELIAN GROUPS

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Abstract

Centre-by-metabelian groups with the maximal condition for normal subgroups are exhibited which (a) are residually finite but have quotient groups which are not residually finite; and (b) have all quotients residually finite but are not abelian-by-polycyclic.

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Segal, in [1], page 336, observes that the following question is "elusive": *does a finitely generated soluble group which has the maximum condition for normal subgroups and is residually finite have all its quotients residually finite*?

The purpose of this note is to settle that question by exhibiting some examples of centre-extended-by-metabelian groups.

THEOREM. *There exist centre-by-metabelian groups with the maximal condition for normal subgroups which* (a) *are residually-finite but have non-residually-finite quotients*; (b) *have all quotients residually-finite but are not abelian-by-polycyclic. The class is parametrised by sequences of integers and the group theoretical properties are related to properties of the sequences of integers.*

Let G_1 be the group

$$\left\langle t_1, x_1, y_1, z_1^{(i)} \colon \left[z_1^{(i)}, t_1\right] = \left[z_1^{(i)}, x_1\right] = \left[z_1^{(i)}, y_1\right] = 1, \right.$$
$$\left. \left[x_1, y_1^{t_1^i}\right] = z_1^{(i)}, \left[x_1, x_1^{t_1^i}\right] = \left[y_1, y_1^{t_1^i}\right] = 1, (i \in \mathbf{Z}) \right\rangle.$$

Let N_1 denote the normal closure of $\{x_1, y_1\}$ in G_1 and Z_1 denote the normal subgroup generated by the $z_1^{(i)}$. Then G_1/N_1 is cyclic, N_1/Z_1 is abelian and Z_1 is a central free abelian group with basis $\{z_1^{(i)}: i \in \mathbf{Z}\}$.

Let $\Gamma = \{\gamma_i: i \in \mathbf{Z}\}$ be a sequence of integers with $\gamma_0 = 1$ and let $H_\Gamma \leqslant Z_1$ be the subgroup generated by all elements

$$z_1^{(i)}\left(z_1^{(0)}\right)^{-\gamma_i} \qquad (i \in \mathbf{Z}).$$

Then Z_1/H_Γ is infinite cylic, generated by the image of $z_1^{(0)} = [x_1, y_1]$. Let $G(\Gamma) = G_1/H_\Gamma$ (note that H_Γ is central and so normal in G_1). We will denote the images of previously defined elements and subgroups in $G(\Gamma)$ by omitting the subscript.

By a bi-monic linear recurrence relation satisfied by Γ, we mean a relation of the form

$$\gamma_{i+k} + c_{k-1}\gamma_{i+k-1} + \cdots + c_1\gamma_{i+1} + \gamma_i = 0,$$

holding for all $i \in \mathbf{Z}$ and some $k, c_j \in \mathbf{Z}$.

The proof of the theorem involves relating the group-theoretical properties of $G(\Gamma)$ with properties of Γ. We do this first in a simpler case. For each non-negative integer n denote by $G(\Gamma)_n$ the quotient $G(\Gamma)/Z_n$ (so $G(\Gamma)_0 = G(\Gamma)$).

PROPOSITION 1. *For all $n \geqslant 0$ the following two properties are equivalent*:

(a) *$G(\Gamma)_n$ is abelian-by-polycyclic*;

(b) *Γ satisfies a bi-monic linear recurrence relation modulo n*;

Further, if $n \neq 0$ then these are equivalent to

(c) *$G(\Gamma)_n$ is residually-finite*;

(d) *Γ is periodic modulo n.*

PROOF. Let N_n denote the image of N in $G(\Gamma)_n$ and let C_n denote the centre of N_n in $G(\Gamma)_n$. Then (b) holds if and only if

$$z^{(i+k)}z^{(i+k-1)c_{k-1}} \cdots z^{(i)} = 1 \text{ in } G(\Gamma)_n, \quad \text{for all } i \in \mathbf{Z},$$

that is, if and only if,

$$\left[x, y^{t^{i+k}}y^{t^{i+k-1}c_{k-1}} \cdots y^{t^i}\right] = 1.$$

This, in turn, holds if and only if,

$$y^{t^k}y^{t^{k-1}c_{k-1}} \cdots y^t \in C_n.$$

But an expression of this type belongs to C_n if and only if N/C_n is finitely generated. Hence (b) holds if and only if N/C_n is finitely generated. If N/C_n is finitely generated, then $G(\Gamma)_n/C_n$ is polycyclic and so $G(\Gamma_n)$ is abelian-by-polycyclic.

Conversely, if $G(\Gamma)_n$ is abelian-by-polycyclic, then there is an abelian subgroup D_n of N, normal in $G(\Gamma)_n$, with N/D_n finitely generated. Thus for some k, c_j and all i in $\mathbf{Z}$,

$$x^{t^{i+k}}x^{t^{i+k-1}c_{k-1}} \cdots x^{t^i} \in D_n; \; y^{t^{i+k}}y^{t^{i+k-1}c_{k-1}} \cdots y^{t^i} \in D_n.$$

Using the fact that D_n is abelian, it is easily verified that (b) is satisfied.

Suppose now that $n \neq 0$. Then it is clear that (b) and (d) are equivalent. Also, it is well known that (a) implies (c). Assume, then, that $G(\Gamma)_n$ is residually finite. Then there is a normal subgroup T of finite index avoiding the finite normal subgroup Z_n. Thus $G(\Gamma)_n$ is a subdirect product of the metabelian group $G(\Gamma)_n/Z_n$ and the finite group $G(\Gamma)_n/T$; in, particular, it is abelian-by-polycyclic.

The next lemma relates the residual finiteness of $G(\Gamma)$ to that of the $G(\Gamma)_n$.

LEMMA 2. *$G(\Gamma)$ is residually finite if and only if $\{n$: $G(\Gamma)_n$ is residually finite$\}$ is infinite.*

PROOF. Note that, for any infinite set S of positive integers, Z is a subdirect product of $\{Z/Z^n$: $n \in S\}$ and so $G(\Gamma)$ is a subdirect product of $\{G(\Gamma)_n$: $n \in S\}$. Thus, if $\{n$: $G(\Gamma)_n$ is residually finite$\}$ is infinite, then $G(\Gamma)$ is a subdirect product of residually finite groups and so residually finite.

For the converse, suppose that F is a normal subgroup of finite index in $G(\Gamma)$. Then, as in the proof of Proposition 1, $G(\Gamma)/Z \cap F$ is residually finite. If $\{n$: $G(\Gamma)_n$ is residually finite$\}$ is finite, then $T = \{n$: $Z^n = Z \cap F$ for some normal F of finite index$\}$ is finite. Hence the intersection of all the normal subgroups of finite index in $G(\Gamma)$ contains the intersection of all Z^n $(n \in T)$ which is non-trivial. Thus $G(\Gamma)$ is not residually finite.

We can summarise this as follows.

COROLLARY 3. *$G(\Gamma)$ is a finitely generated centre-by-metabelian group with the maximal condition on normal subgroups.*

(a) *$G(\Gamma)$ is residually finite if and only if Γ is periodic modulo n for infinitely many distinct n.*

(b) *Every quotient of $G(\Gamma)$ is residually finite if and only if Γ is periodic modulo n for all n.*

(c) *$G(\Gamma)$ is abelian-by-polycyclic if and only if Γ satisfies a bi-monic linear recurrence relation.*

PROOF. A combination of Proposition 1 and Lemma 2 proves all but the "if" implication of (b). Suppose, then, that Γ is periodic modulo n for all n. By

Proposition 1, $G(\Gamma)_n$, together with all of its quotients, is residually finite for all n. But any monolithic quotient of $G(\Gamma)$ must have a finite centre and so be a quotient of some $G(\Gamma)_n$. Thus each monolithic quotient of $G(\Gamma)$ is residually finite and so $G(\Gamma)$ is residually finite.

Finally, to complete the proof of the Theorem, we must distinguish the three properties of sequences mentioned in the Corollary.

LEMMA 4. (a) *There exists a sequence* Γ *which is periodic modulo n for infinitely many, but not for all*, n; (b) *there exists a sequence* Γ *which is periodic modulo n for all n but which satisfies no bi-monic linear recurrence relation.*

This is surely well-known and elementary but inelegant proofs are not difficult to find. I do not, however, have the short and elegant proof which it seems likely must exist. I therefore leave to the interested reader the (hopefully enjoyable) task of finding a suitable agrument.

Combining Corollary 3 with Proposition 4, the proof of the Theorem is complete.

References

[1] D. Segal, 'On the residual simplicity of certain modules', *Proc. London Math. Soc.* (3) **34** (1977), 327–353.

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