Glasgow Math. J. **61** (2019) 349–371.
doi:10.1017/S0017089518000241.

HOMOTOPY TYPES OF GAUGE GROUPS OVER NON-SIMPLYCONNECTED CLOSED 4-MANIFOLDS

TSELEUNG SO

Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom
e-mail: tls1g14@soton.ac.uk

(Received 13 February 2018; revised 11 April 2018; accepted 4 May 2018; first published online 20 June 2018)

Abstract. Let G be a simple, simply connected, compact Lie group, and let M be an orientable, smooth, connected, closed 4-manifold. In this paper, we calculate the homotopy type of the suspension of M and the homotopy types of the gauge groups of principal G-bundles over M when $\pi_1(M)$ is (1) $\mathbb{Z}^{*m}$, (2) $\mathbb{Z}/p^r\mathbb{Z}$, or (3) $\mathbb{Z}^{*m} * (*_{j=1}^{n} \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p and the p_j's are odd primes.

2010 *Mathematics Subject Classification.* Primary 54C35, 55P40; Secondary 55R10.

1. Introduction. Let G be a topological group, and let M be a topological space. Given a principal G-bundle P over M, the associated gauge group $\mathcal{G}(P)$ is the topological group of G-equivariant automorphisms of P, which fixes M. Atiyah, Bott and Gottlieb [**1**, **4**] showed that its classifying space $B\mathcal{G}(P)$ is homotopy equivalent to the connected component $\mathrm{Map}_P(M, BG)$ of the mapping space $\mathrm{Map}(M, BG)$, which contains the map inducing P. When G is a simple, simply connected, compact Lie group and M is an orientable, smooth, connected, closed 4-manifold, it can be shown that the set of isomorphism classes of principal G-bundles over M is in one-to-one correspondence with the homotopy set $[M, BG] \cong \mathbb{Z}$. If a principal G-bundle corresponds to an integer t, then we denote its associated gauge group by $\mathcal{G}_t(M)$. In [**18**], Theriault showed that when M is a simply connected spin 4-manifold, there is a homotopy equivalence

$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{n} \Omega^2 G, \tag{1}$$

where n is the rank of $H^2(M; \mathbb{Z})$. For the non-spin case, this homotopy equivalence still holds when localized away from 2. As a result, the study of $\mathcal{G}_t(M)$ can be reduced to that of $\mathcal{G}_t(S^4)$, which has been being investigated over the last 20 years. Kishimoto, Kono and Tsutaya gave bounds on the numbers of distinct homotopy types of $\mathcal{G}_t(S^4)$ for all G using localization at odd primes [**9**]. Moreover, the homotopy types of $\mathcal{G}_t(S^4)$ are classified for many cases. Let (a, b) be the greatest common divisor of a and b. Then, [**3**, **8**, **10**, **11**, **19**, **21**, **22**]

- when $G = SU(2)$, there is a homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(12, t) = (12, s)$;

- when $G = SU(3)$, there is a homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(24, t) = (24, s)$;
- when $G = SU(5)$, there is a p-homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(120, t) = (120, s)$ for any prime p;
- when $G = Sp(2)$, there is a p-local homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(40, t) = (40, s)$ for any prime p;
- when $G = SU(n)$, there is a p-local homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(n(n^2-1), t) = (n(n^2-1), s)$ for any odd prime p such that $n \leq (p-1)^2 + 1$;
- when $G = Sp(n)$, there is a p-local homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(4n(2n+1), t) = (4n(2n+1), s)$ for any odd prime p such that $2n \leq (p-1)^2 + 1$;
- when $G = G_2$, there is a p-local homotopy equivalence $\mathcal{G}_t(S^4) \simeq \mathcal{G}_s(S^4)$ if and only if $(84, t) = (84, s)$ for any odd prime p.

In addition, a few cases of $\mathcal{G}_t(\mathbb{CP}^2)$ are worked out [**12**, **20**]:

- when $G = SU(2)$, there is a homotopy equivalence $\mathcal{G}_t(\mathbb{CP}^2) \simeq \mathcal{G}_s(\mathbb{CP}^2)$ if and only if $(6, t) = (6, s)$;
- when $G = SU(3)$, there is a p-local homotopy equivalence $\mathcal{G}_t(\mathbb{CP}^2) \simeq \mathcal{G}_s(\mathbb{CP}^2)$ if and only if $(12, t) = (12, s)$ for any prime p.

On the other hand, very little is known about $\mathcal{G}_t(M)$ when M is non-simply connected. The goal of this paper is to study the homotopy types of $\mathcal{G}_t(M)$ for certain non-simply connected 4-manifolds. To achieve this, we need a homotopy decomposition statement.

THEOREM 1.1. *Suppose that G is a simple, simply connected, compact Lie group and Y is a CW-complex of dimension at most 3. Let $\phi : Y \to M$ be a map such that $\Sigma\phi$ has a left homotopy inverse. Then, we have*

$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(C_\phi) \times Map^*(Y, G)$$

where C_ϕ is the cofiber of ϕ.

Using Theorem 1.1, we calculate the homotopy types of $\mathcal{G}_t(M)$ when $\pi_1(M)$ is (1) $\mathbb{Z}^{*m}$, (2) $\mathbb{Z}/p^r\mathbb{Z}$, or (3) $\mathbb{Z}^{*m} * (*_{j=1}^n \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p and the p_j's are odd primes.

THEOREM 1.2. *Suppose that G is a simple, simply connected, compact Lie group and M is an orientable, smooth, connected, closed 4-manifold.*

- *If $\pi_1(M) = \mathbb{Z}^{*m}$ or $\mathbb{Z}/p^r\mathbb{Z}$, then $\mathcal{G}_t(M)$ is homotopy equivalent to a product of $\mathcal{G}_t(S^4)$ or $\mathcal{G}_t(\mathbb{CP}^2)$ and 'loop spaces' on G.*
- *If $\pi_1(M) = \mathbb{Z}^{*m} * (*_{j=1}^n \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, then $\mathcal{G}_t(M) \times \prod^{2d} \Omega^2 G$ is homotopy equivalent to a product of $\mathcal{G}_t(S^4)$ or $\mathcal{G}_t(\mathbb{CP}^2)$ and 'loop spaces' on G for some number d.*

The term 'loop spaces' refers both to iterated based loop spaces ΩG, $\Omega^2 G$ and $\Omega^3 G$ and modular loop spaces $\Omega G\{p^r\}$ and $\Omega^2 G\{p^r\}$, where $G\{p^r\}$ is the homotopy fiber of the p^r-power map on G. Explicit decompositions are stated in Section 3.

Theorem 1.2 shows that the homotopy type of $\mathcal{G}_t(M)$ is related to that of $\mathcal{G}_t(S^4)$ or $\mathcal{G}_t(\mathbb{CP}^2)$ in these three cases. Combining Theorem 1.2 and the known results in [**3**, **10**, **11**, **19**, **22**], we have the following classification.

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

COROLLARY 1.3. *If M is an orientable, smooth, connected, closed 4-manifold with $\pi_1(M) = \mathbb{Z}^{*m}$ or $\mathbb{Z}/p^r\mathbb{Z}$, then the followings hold:*

- *when $G = SU(2)$, there is a homotopy equivalence $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ if and only if $(12, t) = (12, s)$ for M spin, and $(6, t) = (6, s)$ for M non-spin;*
- *when $G = SU(3)$, there is a homotopy equivalence $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ if and only if $(24, t) = (24, s)$ for M spin; there is a p-local homotopy equivalence $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ if and only if $(12, t) = (12, s)$ for any prime p and M non-spin;*
- *when $G = SU(n)$, there is a p-local homotopy equivalence $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ if and only if $(n(n^2-1), t) = (n(n^2-1), s)$ for any odd prime p such that $n \leq (p-1)^2 + 1$;*
- *when $G = Sp(2)$, there is a p-local homotopy equivalence $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ if and only if $(40, t) = (40, s)$ for any odd prime p;*
- *when $G = G_2$, there is a p-local homotopy equivalence $\mathcal{G}_t(M) \simeq \mathcal{G}_s(M)$ if and only if $(84, t) = (84, s)$ for any odd prime p.*

There is an analogous statement to Corollary 1.3.

COROLLARY 1.4. *If M is an orientable, smooth, connected, closed 4-manifold with $\pi_1(M) = \mathbb{Z}^{*m} * (*_{j=1}^{n} \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, then the integral and p-local homotopy equivalences in Corollary 1.3 hold for $\mathcal{G}_t(M) \times \prod^{2d} \Omega^2 G$ for some number d.*

The structure of this paper is as follows. In Section 2, we prove the homotopy decomposition Theorem 1.1 and develop some useful lemmas. In particular, Theorem 1.1 is used to revise homotopy equivalence (1), which is often referred to during the calculations in Section 3. In Section 3, we give the homotopy types of ΣM and $\mathcal{G}_t(M)$ when $\pi_1(M)$ is either $\mathbb{Z}^{*m}$, $\mathbb{Z}/p^r\mathbb{Z}$ or $\mathbb{Z}^{*m} * (*_{j=1}^{n} \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p and the p_j's are odd primes.

2. A homotopy decomposition of gauge groups.

2.1. A homotopy decomposition and some useful lemmas. We are going to extend the homotopy decomposition (1) to a more general situation. Suppose that M is an orientable four-dimensional CW-complex that is constructed by attaching a 4-cell onto a three-dimensional CW-complex M_3 by an attaching map $f : S^3 \to M_3$. Let $u : BG \to K(\mathbb{Z}, 4)$ be a generator of $H^4(BG) \cong \mathbb{Z}$. Since it is a 5-equivalence, $u_* : [M, BG] \to [M, K(\mathbb{Z}, 4)] = \mathbb{Z}$ is a bijection. If a principal G-bundle over M corresponds to some integer t, then we denote its associated gauge group by $\mathcal{G}_t(M)$.

In [**18**], the homotopy decomposition (1) is obtained as a consequence of the attaching map f of the 4-cell in M having the property that Σf is null-homotopic. Observe that Σf is the connecting map of the cofibration sequence

$$M_3 \longrightarrow M \longrightarrow S^4,$$

where M_3 is the 3-skeleton of M. From the point of view of homotopy theory, it can be replaced by a cofibration sequence

$$Y \xrightarrow{\phi} M \xrightarrow{q} C_\phi \tag{2}$$

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

for some space Y and some map $\phi : Y \to M$, with a connecting map that is null-homotopic. Here, q is the quotient map and C_ϕ is the cofiber of ϕ. The nullity condition is equivalent to $\Sigma\phi$ having a left homotopy inverse. If we further restrict the dimension of Y to be at most 3, then by Cellular Approximation theorem ϕ is homotopic to $\iota \circ \varphi$, where $\varphi : Y \to M_3$ is a map on M_3 and $\iota : M_3 \to M$ is the inclusion. The existence of a left homotopy inverse of $\Sigma\phi$ imposes a strong condition on ΣM.

LEMMA 2.1. *Let Y be a CW-complex of dimension at most 3 and let $\phi : Y \to M$ be a map. If ϕ is homotopic to $\iota \circ \varphi$ for some map $\varphi : Y \to M_3$, then the followings are equivalent:*

(1) $\Sigma\phi : \Sigma Y \to \Sigma M$ has a left homotopy inverse ψ;
(2) $\Sigma\varphi$ has a left homotopy inverse ψ' and $\psi' \circ \Sigma f$ is null-homotopic;
(3) ΣM is homotopy equivalent to $\Sigma Y \vee \Sigma C_\phi$ and $\Sigma\phi$ is homotopic to the inclusion;
(4) ΣM_3 is homotopy equivalent to $\Sigma Y \vee \Sigma C_\varphi$, where C_φ is the cofiber of φ, $\Sigma\varphi$ is homotopic to the inclusion and $\psi' \circ \Sigma f$ is null-homotopic.

Proof. First, we show that Conditions (1) and (2) are equivalent. If $\Sigma\phi$ has a left homotopy inverse ψ, then $\psi' = \psi \circ \Sigma\iota$ is a left homotopy inverse of $\Sigma\varphi$ and $\psi' \circ \Sigma f = \psi \circ \Sigma\iota \circ \Sigma f$ is null-homotopic since $S^4 \xrightarrow{\Sigma f} \Sigma M_3 \xrightarrow{\Sigma\iota} \Sigma M$ is a cofibration sequence. Conversely, assume Condition (2). Consider the homotopy commutative diagram

$$\begin{array}{ccccc} S^4 & \xrightarrow{\Sigma f} & \Sigma M_3 & \xrightarrow{\Sigma\iota} & \Sigma M \\ & & \downarrow{\scriptstyle \psi'} & \swarrow{\scriptstyle \psi} & \\ & & \Sigma Y & & \end{array}$$

By hypothesis, $\psi' \circ \Sigma f$ is null-homotopic, so ψ' has an extension ψ. Then, we have

$$\begin{aligned} \psi \circ \Sigma\phi &\simeq \psi \circ \Sigma(\iota \circ \varphi) \\ &\simeq (\psi \circ \Sigma\iota) \circ \Sigma\varphi \\ &\simeq \psi' \circ \Sigma\varphi \\ &\simeq 1\!\!1_{\Sigma Y}, \end{aligned}$$

where $1\!\!1_{\Sigma Y}$ is the identity map on ΣY. Therefore, ψ is a left homotopy inverse of $\Sigma\phi$.

Now, we show that Conditions (1) and (3) are equivalent. If $\Sigma\phi$ has a left homotopy inverse ψ, then let h be the composition

$$h : \Sigma M \xrightarrow{\sigma} \Sigma M \vee \Sigma M \xrightarrow{\psi \vee \Sigma q} \Sigma Y \vee \Sigma C_\phi,$$

where σ is the comultiplication. Observe that h induces an isomorphism $H^*(\Sigma Y \vee \Sigma C_\phi) \to H^*(\Sigma M)$. Since these spaces are suspensions and are simply connected, h is a homotopy equivalence by Whitehead theorem. Therefore, ΣM is homotopy equivalent

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

to $\Sigma Y \vee \Sigma C_\phi$. Moreover, since $\Sigma\phi$ is a co-H-map,

$$\begin{aligned} h \circ \Sigma\phi &\simeq (\psi \vee \Sigma q) \circ \sigma \circ \Sigma\phi \\ &\simeq (\psi \vee \Sigma q) \circ (\Sigma\phi \vee \Sigma\phi) \circ \sigma \\ &\simeq (\psi \circ \Sigma\phi \vee \Sigma q \circ \Sigma\phi) \circ \sigma \\ &\simeq (1\!\!1_{\Sigma Y} \vee *) \circ \sigma \end{aligned}$$

is the inclusion $\Sigma Y \hookrightarrow \Sigma Y \vee \Sigma C_\phi$. Conversely, assume Condition (3). Let h be a homotopy equivalence from ΣM to $\Sigma Y \vee \Sigma C_\phi$ and let ψ be the composition

$$\psi : \Sigma M \xrightarrow{h} \Sigma Y \vee \Sigma C_\phi \xrightarrow{p} \Sigma Y,$$

where p is the pinch map. By hypothesis, $h \circ \Sigma\phi$ is homotopic to the inclusion $\Sigma Y \to \Sigma Y \vee \Sigma C_\phi$, so we have

$$\psi \circ \Sigma\phi \simeq p \circ h \circ \Sigma\phi \simeq 1\!\!1_{\Sigma Y}$$

and ψ is a left homotopy inverse of $\Sigma\phi$.

The equivalence between (2) and (4) can be shown similarly. □

We can extend the cofibration (2) to the homotopy commutative diagram whose rows and columns are cofibration sequences

$$\begin{array}{ccccccc} * & \longrightarrow & Y & = & Y & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \phi} & & \downarrow \\ S^3 & \xrightarrow{f} & M_3 & \xrightarrow{\iota} & M & \xrightarrow{p} & S^4 \\ \| & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \| \\ S^3 & \xrightarrow{f'} & C_\varphi & \longrightarrow & C_\phi & \xrightarrow{p'} & S^4 \end{array} \tag{3}$$

where q', p and p' are the quotient maps and $f' = q' \circ f$. The bottom row implies that C_ϕ is constructed by attaching a 4-cell onto C_φ via f'. The generator u of $H^4(BG)$ induces a bijection between $[C_\phi, BG]$ and $[C_\phi, K(\mathbb{Z}, 4)] \cong \mathbb{Z}$, so any principal G-bundle over C_ϕ corresponds to some integer t. Denote the associated gauge group by $\mathcal{G}_t(C_\phi)$. We want to compare $\mathcal{G}_t(M)$ and $\mathcal{G}_t(C_\phi)$ via the pullback q^*.

LEMMA 2.2. *Assume the conditions in Lemma 2.1. Then, $q^* : [C_\phi, BG] \to [M, BG]$ is a group isomorphism.*

Proof. The naturality of u^* implies the commutative diagram

$$\begin{array}{ccc} [C_\phi, BG] & \xrightarrow{q^*} & [M, BG] \\ \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle u^*} \\ H^4(C_\phi) & \xrightarrow{q^*} & H^4(M). \end{array}$$

Since the group structures on $[M, BG]$ and $[C_\phi, BG]$ are induced by bijections

$$u^* : [M, BG] \to H^4(M) \cong \mathbb{Z} \text{ and } u^* : [C_\phi, BG] \to H^4(C_\phi) \cong \mathbb{Z},$$

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

it suffices to show that $q^* : H^4(C_\phi) \to H^4(M)$ is an isomorphism.

The cofibration (2) induces an exact sequence

$$H^4(\Sigma M) \xrightarrow{(\Sigma\phi)^*} H^4(\Sigma Y) \xrightarrow{\epsilon^*} H^4(C_\phi) \xrightarrow{q^*} H^4(M) \xrightarrow{\phi^*} H^4(Y),$$

where ϵ^* is induced by the connecting map $\epsilon : C_\phi \to \Sigma Y$. Since $\Sigma\phi$ has a left homotopy inverse, ϵ is null-homotopic and ϵ^* is trivial. Also, the dimension of Y is at most 3, so $H^4(Y)$ is trivial. By exactness $q^* : H^4(C_\phi) \to H^4(M)$ is an isomorphism. □

LEMMA 2.3. *For any integer t, let $Map_t^*(M, BG)$ and $Map_t^*(C_\phi, BG)$ be the connected components of $Map^*(M, BG)$ and $Map^*(C_\phi, BG)$ containing $t \in [M, BG] \cong [C_\phi, BG]$, respectively. Then,*

$$Map^*(\Sigma Y, BG) \xrightarrow{\epsilon^*} Map_t^*(C_\phi, BG) \xrightarrow{q^*} Map_t^*(M, BG)$$

is a homotopy fibration sequence.

Proof. First, $q^* : \mathrm{Map}_t^*(C_\phi, BG) \to \mathrm{Map}_t^*(M, BG)$ is well defined by Lemma 2.2. Second, the cofibration (2) induces a homotopy fibration sequence

$$\mathrm{Map}^*(\Sigma Y, BG) \xrightarrow{\epsilon^*} \mathrm{Map}^*(M, BG) \xrightarrow{q^*} \mathrm{Map}^*(C_\phi, BG).$$

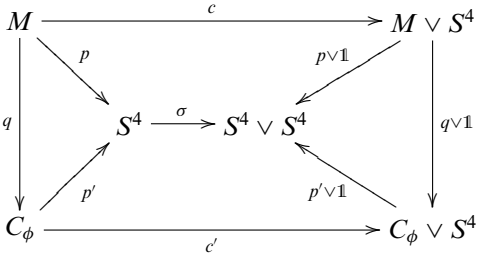
Since ϵ^* is null-homotopic, $\mathrm{Map}^*(\Sigma Y, BG)$ is mapped into $\mathrm{Map}_0^*(M, BG)$ and hence is the homotopy fiber of $q^* : \mathrm{Map}_0^*(M, BG) \to \mathrm{Map}_0^*(C_\phi, BG)$. To show that it is true for any $t \in \mathbb{Z}$, let $\alpha_t : \mathrm{Map}_0^*(M, BG) \to \mathrm{Map}_t^*(M, BG)$ be a map sending a pointed map $g : M \to BG$ to the composition

$$\alpha_t(g) : M \xrightarrow{c} M \vee S^4 \xrightarrow{g \vee t} BG \vee BG \xrightarrow{\nabla} BG,$$

where c is the coaction map of M and ∇ is the folding map, and let $\beta_t : \mathrm{Map}_0^*(C_\phi, BG) \to \mathrm{Map}_t^*(C_\phi, BG)$ be a map defined similarly. Since α_t and β_t are homotopy equivalences, it suffices to prove the commutate diagram

$$\begin{array}{ccc} \mathrm{Map}_0^*(C_\phi, BG) & \xrightarrow{q^*} & \mathrm{Map}_0^*(M, BG) \\ \downarrow{\scriptstyle \alpha_t} & & \downarrow{\scriptstyle \beta_t} \\ \mathrm{Map}_t^*(C_\phi, BG) & \xrightarrow{q^*} & \mathrm{Map}_t^*(M, BG). \end{array}$$

Consider the homotopy commutative diagram



https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

where c' is the coaction map, and p and p' are the quotient maps, and σ is the comultiplication of S^4. The left and the right triangles are due to the bottom-right square in diagram (3), and the top and the bottom quadrangles are due to the property of coaction maps. Extend it to get the cofibration diagram

$$\begin{array}{ccccccc} M & \xrightarrow{c} & M \vee S^4 & \xrightarrow{q^*g \vee t} & BG \vee BG & \xrightarrow{\nabla} & BG \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q \vee 1\!\!1} & & \| & & \| \\ C_\phi & \xrightarrow{c'} & C_\phi \vee S^4 & \xrightarrow{g \vee t} & BG \vee BG & \xrightarrow{\nabla} & BG \end{array}$$

The upper row around the diagram is $\beta_t(q^*g)$, while the lower row around the diagram is $q^*\alpha_t(g)$. Therefore, q^* commutes with α_t and β_t and the asserted statement follows. □

THEOREM 2.4. *Let Y be a CW-complex of dimension at most 3 and let $\phi : Y \to M$ be a map. If ϕ satisfies one of the four conditions in Lemma 2.1, then there are homotopy equivalences*

$$\Sigma M \simeq \Sigma C_\phi \vee \Sigma Y \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(C_\phi) \times Map^*(Y, G).$$

Proof. Let $\mathrm{Map}_t(M, BG)$ and $\mathrm{Map}_t(C_\phi, BG)$ be the connected components of $\mathrm{Map}(M, BG)$ and $\mathrm{Map}(C_\phi, BG)$ containing $t \in [M, BG] \cong [C_\phi, BG]$. There are evaluation fibrations

$$\mathrm{Map}_t^*(M, BG) \to \mathrm{Map}_t(M, BG) \to BG \text{ and } \mathrm{Map}_t^*(C_\phi, BG) \to \mathrm{Map}_t(C_\phi, BG) \to BG.$$

By Lemma 2.2, $q^* : [C_\phi, BG] \to [M, BG]$ is an isomorphism, so there is a homotopy commutative diagram whose rows are fibration sequences

$$\begin{array}{ccccccc} G & \longrightarrow & \mathrm{Map}_t^*(C_\phi, BG) & \longrightarrow & \mathrm{Map}_t(C_\phi, BG) & \longrightarrow & BG \\ \| & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \| \\ G & \longrightarrow & \mathrm{Map}_t^*(M, BG) & \longrightarrow & \mathrm{Map}_t(M, BG) & \longrightarrow & BG \end{array} \tag{4}$$

As in [**18**], from the left square in (4) we obtain a homotopy fibration diagram

$$\begin{array}{ccccc} * & \longrightarrow & \Omega\mathrm{Map}_t^*(M, BG) & = & \Omega\mathrm{Map}_t^*(M, BG) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G}_t(C_\phi) & \xrightarrow{q^*} & \mathcal{G}_t(M) & \xrightarrow{h} & \mathrm{Map}^*(\Sigma Y, BG) \\ \| & & \downarrow & & \downarrow{\scriptstyle \epsilon^*} \\ \mathcal{G}_t(C_\phi) & \longrightarrow & G & \longrightarrow & \mathrm{Map}_t^*(C_\phi, BG) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Map}_t^*(M, BG) & = & \mathrm{Map}_t^*(M, BG) \end{array} \tag{5}$$

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

The right column is due to Lemma 2.3. The nullity of ϵ^* implies that h has a right homotopy inverse. The group multiplication in $\mathcal{G}_t(M)$ then gives a homotopy equivalence

$$\Phi : \mathcal{G}_t(C_\phi) \times \mathrm{Map}^*(\Sigma Y, BG) \longrightarrow \mathcal{G}_t(M) \times \mathcal{G}_t(M) \longrightarrow \mathcal{G}_t(M),$$

so $\mathcal{G}_t(M)$ is homotopy equivalent to $\mathcal{G}_t(C_\phi) \times \mathrm{Map}^*(Y, G)$. □

When calculating $\mathcal{G}_t(M)$, we will use Theorem 2.4 several times and apply it to M and C_ϕ in some cases. So, here, we establish some facts about C_ϕ.

LEMMA 2.5. *Let $q' : M \to C_\phi$ be the quotient map and let f' be the composition*

$$S^3 \xrightarrow{f} M_3 \xrightarrow{q'} C_\varphi.$$

Then, $\Sigma f'$ is null-homotopic if and only if Σf is null-homotopic.

Proof. The necessity is obvious. Assume that $\Sigma f'$ is null-homotopic. Since the composition

$$h : \Sigma M_3 \xrightarrow{\sigma} \Sigma M_3 \vee \Sigma M_3 \xrightarrow{\psi \vee \Sigma q'} \Sigma Y \vee \Sigma C_\varphi$$

is a homotopy equivalence, it suffices to show that $h \circ \Sigma f$ is null-homotopic. Consider the homotopy commutative diagram

$$\begin{array}{ccccc} S^4 & \xrightarrow{\sigma} & S^4 \vee S^4 & & \\ \downarrow{\scriptstyle \Sigma f} & & \downarrow{\scriptstyle \Sigma f \vee \Sigma f} & & \\ \Sigma M_3 & \xrightarrow{\sigma} & \Sigma M_3 \vee \Sigma M_3 & \xrightarrow{\psi \vee \Sigma q'} & \Sigma Y \vee \Sigma C_\varphi \end{array}$$

where the two σ's are the comultiplications of S^4 and ΣM_3. The lower direction around the diagram is $h \circ \Sigma f$, and the upper direction around the diagram is

$$(\psi \circ \Sigma f \vee \Sigma q' \circ \Sigma f) \circ \sigma \simeq (\psi \circ \Sigma f \vee \Sigma f') \circ \sigma.$$

By hypothesis $\Sigma f'$ is null-homotopic, and by Lemma 2.1(4) $\psi \circ \Sigma f$ is null-homotopic. Therefore, $h \circ \Sigma f$ is null-homotopic and hence so is Σf. □

LEMMA 2.6. *Let $H^2_{free}(X)$ be the free part of $H^2(X)$ for any space X. If $H^2(Y)$ is torsion, then $q^* : H^2_{free}(C_\phi) \to H^2_{free}(M)$ is an isomorphism. Moreover, if for any $\alpha' \in H^2_{free}(M)$ there exists $\beta' \in H^2_{free}(M)$ such that $\alpha' \cup \beta' \in H^4(M)$ is a generator, then for any $\alpha \in H^2_{free}(C_\phi)$ there exists $\beta \in H^2_{free}(C_\phi)$ such that $\alpha \cup \beta \in H^4(C_\phi)$ is a generator.*

Proof. Cofibration (2) induces the long exact sequence of cohomology groups

$$\cdots \longrightarrow H^k(C_\phi) \xrightarrow{(\Sigma q)^*} H^k(M) \xrightarrow{(\Sigma \phi)^*} H^k(Y) \longrightarrow \cdots$$

Since $\Sigma\phi$ has a left homotopy inverse, the sequence splits for $k \geq 1$, and we have

$$H^k(M) \cong H^k(Y) \oplus H^k(C_\phi).$$

By hypothesis $H^2(Y)$ is torsion, so $H^2_{\text{free}}(C_\phi)$ is isomorphic to $H^2_{\text{free}}(M)$.

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

For any $\alpha \in H^2_{\text{free}}(C_\phi)$, $q^*(\alpha)$ is in $H^2_{\text{free}}(M)$. By hypothesis there exists $\beta' \in H^2_{\text{free}}(M)$ such that $q^*(\alpha) \cup \beta' \in H^4(M)$ is a generator. Since $q^* : H^2_{\text{free}}(C_\phi) \to H^2_{\text{free}}(M)$ is an isomorphism, there exists $\beta \in H^2_{\text{free}}(C_\phi)$ such that $q^*(\beta) = \beta'$. Therefore, we have $q^*(\alpha) \cup \beta' = q^*(\alpha \cup \beta)$. Observe that $q^* : H^4(C_\phi) \to H^4(M)$ is an isomorphism since Y has dimension at most 3. It follows that $\alpha \cup \beta$ is a generator of $H^4(C_\phi)$. □

The second part of Lemma 2.6 says that the cup product on $H^2_{\text{free}}(C_\phi)$ is unimodular if the cup product on $H^2_{\text{free}(M)}$ is unimodular, which follows from Poincaré Duality when M is an orientable compact manifold. Furthermore, if C_ϕ has a subcomplex Y' satisfying Theorem 2.4, then the cup product on $H^2_{\text{free}}(C_\phi / Y')$ is still unimodular.

Next, we consider two variations of Theorem 2.4 when M has a special structure.

LEMMA 2.7. *Suppose that M_3 is homotopy equivalent to $Z \vee Z'$. Let Y and Y' be CW-complexes of dimension at most 3, and let $\varphi : Y \to Z$ and $\varphi' : Y' \to Z'$ be maps. If $\Sigma\varphi$ and $\Sigma\varphi'$ have left homotopy inverses ψ and ψ', respectively, and the compositions*

$$S^4 \xrightarrow{\Sigma f} \Sigma M_3 \xrightarrow{pinch} \Sigma Z \xrightarrow{\psi} \Sigma Y \text{ and } S^4 \xrightarrow{\Sigma f} \Sigma M_3 \xrightarrow{pinch} \Sigma Z' \xrightarrow{\psi'} \Sigma Y'$$

are null-homotopic, then we have

$$\Sigma M \simeq \Sigma M' \vee \Sigma Y \vee \Sigma Y' \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(M') \times Map^*(\Sigma Y, BG) \times Map^*(\Sigma Y', BG),$$

where M' is the cofiber of the map $Y \vee Y' \xrightarrow{\varphi \vee \varphi'} Z \vee Z' \hookrightarrow M$.

Proof. Let Φ be the composition

$$\Phi : Y \vee Y' \xrightarrow{\varphi \vee \varphi'} Z \vee Z' \simeq M_3.$$

The map $\psi \vee \psi' : \Sigma Z \vee \Sigma Z' \to \Sigma Y \vee \Sigma Y'$ is a left homotopy inverse of $\Sigma\Phi$. We show that $(\psi \vee \psi') \circ \Sigma f$ is null-homotopic, implying that Φ satisfies the hypothesis of Theorem 2.4.

Notice that the composition

$$h : \Sigma M_3 \xrightarrow{\sigma} \Sigma M_3 \vee \Sigma M_3 \xrightarrow{p_1 \vee p_2} \Sigma Z \vee \Sigma Z'$$

is a homotopy equivalence, where $p_1 : \Sigma M_3 \to \Sigma Z$ and $p_2 : \Sigma M_3 \to \Sigma Z'$ are the pinch maps. Since Σf is a co-H-map, we have

$$\begin{aligned}(\psi \vee \psi') \circ h \circ \Sigma f &\simeq (\psi \vee \psi') \circ (p_1 \vee p_2) \circ \sigma \circ \Sigma f \\ &\simeq (\psi \vee \psi') \circ (p_1 \vee p_2) \circ (\Sigma f \vee \Sigma f) \circ \sigma \\ &\simeq (\psi \circ p_1 \circ \Sigma f \vee \psi' \circ p_2 \circ \Sigma f) \circ \sigma,\end{aligned}$$

which is null-homotopic by assumption. Therefore, $(\psi \vee \psi') \circ \Sigma f$ is null-homotopic and Theorem 2.4 implies the asserted statement. □

LEMMA 2.8. *Suppose $M \cong X \# X'$, where X and X' are orientable, smooth, connected, closed 4-manifolds. Let Y and Y' be CW-complexes of dimensions at most 3 and let $\varphi : Y \to X_3$ and $\varphi' : Y' \to X'_3$ be maps satisfying the hypothesis of Theorem 2.4. Then, we have*

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

$$\Sigma M \simeq \Sigma M' \vee \Sigma Y \vee \Sigma Y' \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(M') \times Map^*(\Sigma Y, BG) \times Map^*(\Sigma Y', BG),$$

where M' is the cofiber of the inclusion $Y \vee Y' \xrightarrow{\varphi\vee\varphi'} X_3 \vee X_3' \hookrightarrow M$.

Proof. Let $f : S^3 \to X_3$ and $f' : S^3 \to X_3'$ be the attaching maps of the 4-cells in X and X', respectively. By Lemma 2.1, $\Sigma\varphi$ and $\Sigma\varphi'$ have left homotopy inverses ψ and ψ' and $\psi \circ \Sigma f$ and $\psi' \circ \Sigma f'$ are null-homotopic. We show that φ and φ' satisfy the hypothesis in Lemma 2.7.

The 3-skeleton of M is $X_3 \vee X_3'$ and the attaching map of the 4-cell is

$$f_\# : S^3 \xrightarrow{\sigma} S^3 \vee S^3 \xrightarrow{f\vee f'} X_3 \vee X_3'.$$

Let $p_1 : \Sigma X_3 \vee \Sigma X_3' \to \Sigma X_3$ and $p_2 : \Sigma X_3 \vee \Sigma X_3' \to \Sigma X_3'$ be the pinch maps. Consider the homotopy commutative diagram

$$\begin{array}{ccccc} S^4 & \xrightarrow{\Sigma f_\#} & \Sigma X_3 \vee \Sigma X_3' & & \\ \| & & \downarrow{\scriptstyle p_1} & & \\ S^4 & \xrightarrow{\Sigma f} & \Sigma X_3 & \xrightarrow{\psi} & \Sigma Y \end{array}$$

Since $\psi \circ \Sigma f$ is null-homotopic, the composition $\psi \circ p_1 \circ \Sigma f_\#$ is null-homotopic. Similarly $\psi' \circ p_2 \circ \Sigma f_\#$ is null-homotopic. Then, Lemma 2.7 implies the statement. □

2.2. Gauge groups over simply connected 4-manifolds. Now, we revise homotopy equivalence (1) using Theorem 2.4. When M is simply connected, its 3-skeleton M_3 is homotopy equivalent to $\bigvee_{i=1}^n S^2$. If Σf is null-homotopic, then we can apply Theorem 2.4 by taking Y to be the whole of M_3 and $\varphi : Y \to M_3$ to be the identity map and get the homotopy equivalence (1). In the following, we assume the homotopy class of Σf is not trivial.

To distinguish the 2-spheres, denote the ith copy of S^2 in M_3 by S_i^2. Let $\eta : S^3 \to S^2$ be the Hopf map and let η_i be the composition

$$\eta_i : S^3 \xrightarrow{\eta} S_i^2 \hookrightarrow \bigvee_{i=1}^n S_i^2.$$

We also denote the suspensions $\Sigma\eta$ and $\Sigma\eta_i$ by $\bar\eta$ and $\bar\eta_i$ for short.

LEMMA 2.9. *If M_3 is homotopy equivalent to $\bigvee_{i=1}^n S_i^2$, then Σf is homotopic to $\sum_{i=1}^n a_i\bar\eta_i$, where $a_i \in \mathbb{Z}/2\mathbb{Z}$.*

Proof. The Hilton–Milnor theorem implies

$$\pi_3(\bigvee_{i=n}^n S_i^2) \cong \bigoplus_{i=1}^n \pi_3(S_i^2) \oplus \bigoplus_{i\neq j} \pi_3(\Sigma S_i^1 \wedge S_j^1),$$

where $\pi_3(S_i^2)$ is generated by η_i and $\pi_3(\Sigma S_i^1 \wedge S_j^1)$ is generated by Whitehead products of the identity maps on S_i^2 and S_j^2. As an element in $\pi_3(\bigvee_{i=1}^n S_i^2)$, f is homotopic

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

to $\sum_{i=1}^{n} \tilde{a}_i \eta_i + w$, where $\tilde{a}_i$ is an integer and w is a sum of Whitehead products in $\bigoplus_{i \neq j} \pi_3(S_i^2 \wedge S_j^1)$. After suspension Σf is homotopic to $\sum_{i=1}^{n} a_i \bar{\eta}_i$, where $a_i \equiv \tilde{a}_i \pmod 2$, since suspensions of Whitehead products are null-homotopic and $\bar{\eta}_i$ has order 2. □

If the homotopy class of Σf is not trivial, then at least one of the a_i's is not zero. Relabelling the spheres if necessary, we may assume that Σf is homotopic to $\sum_{i=1}^{m} \bar{\eta}_i$ for some integer m such that $1 \leq m \leq n$. Then, we can simplify this expression with the following lemma.

LEMMA 2.10. *If M_3 is homotopy equivalent to $\bigvee_{i=1}^{n} S_i^2$ and Σf is homotopic to $\sum_{i=1}^{m} \bar{\eta}_i$, then there is a map $\tilde{f} : S^3 \to M_3$ such that its cofiber $C_{\tilde{f}}$ is homotopy equivalent to M and $\Sigma \tilde{f}$ is homotopic to $\bar{\eta}_1$. Moreover, $p_1 \circ \tilde{f}$ is homotopic to $p_1 \circ f$, where $p_1 : \bigvee_{i=1}^{n} S_i^2 \to S_1^2$ is the pinch map.*

Proof. For each $1 < j \leq m$, define a map $\xi_j : M_3 \to M_3$ as follows. On $S_1^2 \vee S_j^2$, ξ_j is the composition

$$S_1^2 \vee S_j^2 \xrightarrow{\sigma \vee \mathbb{1}} S_1^2 \vee S_1^2 \vee S_j^2 \xrightarrow{\mathbb{1} \vee \nabla} S_1^2 \vee S_j^2,$$

where σ is a comultiplication of S_1^2 and ∇ is the folding map of S_1^2 and S_j^2. On the remaining spheres, ξ_j is the identity. Consider the homotopy cofibration diagram

$$\begin{array}{ccccc} S^3 & \xrightarrow{f} & M_3 & \longrightarrow & M \\ \| & & \downarrow{\scriptstyle \xi_j} & & \downarrow{\scriptstyle \tilde{\xi}_j} \\ S^3 & \xrightarrow{\xi_j \circ f} & M_3 & \longrightarrow & C_{\xi_j \circ f} \end{array}$$

where $\tilde{\xi}_j$ is an induced map and $C_{\xi_j \circ f}$ is the cofiber of $\xi_j \circ f$. Since ξ_j is a homology isomorphism, so is $\tilde{\xi}_j$ by the 5-lemma. Therefore, $\tilde{\xi}_j$ is a homotopy equivalence and M is homotopy equivalent to $C_{\xi_j \circ f}$.

By Lemma 2.9, $\Sigma(\xi_j \circ f)$ is homotopic to $\sum_{i=1}^{n} a_i \bar{\eta}_i$, where $a_i \in \mathbb{Z}/2\mathbb{Z}$. For $1 \leq i \leq n$, let $p_i : \bigvee_{i=1}^{n} S_i^3 \to S_i^3$ be the pinch map. Then, a_i is $p_i \circ \Sigma(\xi_j \circ f)$. By the definition of ξ_j, $a_i = 1$ for $i \neq j$ and $1 \leq i \leq m$, and $a_i = 0$ otherwise, that is

$$\Sigma(\xi_j \circ f) \simeq \bar{\eta}_1 + \cdots + \bar{\eta}_{j-1} + \bar{\eta}_{j+1} + \cdots + \bar{\eta}_m.$$

Let $\tilde{f}$ be $\xi_m \circ \ldots \circ \xi_2 \circ f$ and let $C_{\tilde{f}}$ be its cofiber. Then, $\Sigma \tilde{f}$ is homotopic to $\bar{\eta}_1$ and $C_{\tilde{f}}$ is homotopy equivalent to M.

Last, observe that $p_1 \circ \xi_j$ is homotopic to p_1. It follows that $p_1 \circ \xi_j \circ f$ is homotopic to $p_1 \circ f$ and so is $p_1 \circ \tilde{f}$. □

LEMMA 2.11. *Let M be a 4-dimensional simply connected CW-complex. If Σf is non-trivial, then there are homotopy equivalences*

$$\Sigma M \simeq \Sigma \mathbb{CP}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(C_{a\eta}) \times \prod_{i=1}^{n-1} \Omega^2 G,$$

where a is an odd integer and n is the rank of $H^2(M)$ and $C_{a\eta}$ is the cofiber of $a\eta$.

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

Proof. By Lemma 2.9, Σf is homotopic to $\sum_{i=1}^{m} \bar{\eta}_i$ for some m such that $1 \leq m \leq n$. By Lemma 2.10, there exists a map $\tilde{f}$ such that $\Sigma\tilde{f}$ is homotopic to $\bar{\eta}_1$ and its cofiber $\tilde{M}$ is homotopy equivalent to M. Replacing f by $\tilde{f}$ and M by $\tilde{M}$, we can assume $\Sigma f \simeq \bar{\eta}_i$. Use Theorem 2.4 by taking Y to be $\bigvee_{i=2}^{n} S_i^2$ and $\phi : \bigvee_{i=2}^{n} S_i^2 \to M$ to be the inclusion and get

$$\Sigma M \simeq \Sigma C_\phi \vee \left(\bigvee_{i=1}^{n-1} S^3\right) \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(C_\phi) \times \prod_{i=1}^{n-1} \Omega^2 G,$$

where C_ϕ is the cofiber of ϕ. We need to show that C_ϕ is homotopy equivalent to $C_{a\eta}$ for some odd integer a and ΣC_ϕ is homotopy equivalent to $\Sigma\mathbb{CP}^2$.

Consider the cofibration diagram

$$\begin{array}{ccccc}
* & \longrightarrow & S^3 & = & S^3 \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p_1\circ f} \\
\bigvee_{i=2}^{n} S_i^2 & \xrightarrow{\iota} & \bigvee_{i=1}^{n} S_i^2 & \xrightarrow{p_1} & S_1^2 \\
\| & & \downarrow & & \downarrow \\
\bigvee_{i=2}^{n} S_i^2 & \xrightarrow{\phi} & M & \longrightarrow & C_\phi
\end{array}$$

where $\iota : \bigvee_{i=2}^{n} S_i^2 \to \bigvee_{i=1}^{n} S_i^2$ is the inclusion, $p_1 : \bigvee_{i=2}^{n} S_i^3 \to S_1^2$ is the pinch map. Since $p_1 \circ f$ is in $\pi_3(S^2) \cong \mathbb{Z}$, it is homotopic to $a\eta$ for some integer a. The right column implies that C_ϕ is homotopy equivalent to the cofiber $C_{a\eta}$ of $a\eta$. Moreover, Σf is homotopic to $\bar{\eta}_1$, so $\Sigma(p_1 \circ f)$ is homotopic to $\Sigma\eta$ and a is an odd number. It follows that ΣC_ϕ is homotopy equivalent to the cofiber of $\Sigma\eta$, which is $\Sigma\mathbb{CP}^2$. □

We can modify the result of Lemma 2.11 a bit better.

LEMMA 2.12. *Let a be an odd number. Then, we have*

$$\mathcal{G}_t(C_{a\eta}) \times \Omega^2 G \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \Omega^2 G.$$

Proof. Let g be the composition

$$g : S^3 \xrightarrow{\sigma} S^3 \vee S^3 \xrightarrow{\eta\vee a\eta} S^2 \vee S^2$$

and let C_g be its cofiber. To distinguish the 2-spheres in the range, denote the ith copy by S_i^2. Now, we calculate $\mathcal{G}_t(C_g)$. By Lemma 2.10, we can assume that Σg is homotopic to $\bar{\eta}_1$ and $p_1 \circ g$ is homotopic to η. Use Theorem 2.4 by taking Y to be S_2^2 and $\varphi : S_2^2 \to S_1^2 \vee S_2^2$ to be the inclusion and obtain $\mathcal{G}_t(C_g) \simeq \mathcal{G}_t(C_g/S_2^2) \times \Omega^2 G$.

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

Consider the homotopy cofibration diagram

$$\begin{array}{ccccc} * & \longrightarrow & S_2^2 & = & S_2^2 \\ \downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \varphi} \\ S^3 & \xrightarrow{g} & S_1^2 \vee S_2^2 & \longrightarrow & C_g \\ \| & & \downarrow{\scriptstyle p_1} & & \downarrow \\ S^3 & \xrightarrow{\eta} & S_1^2 & \longrightarrow & \mathbb{CP}^2, \end{array}$$

where j is the inclusion and p_1 is the pinch map. The right column implies that C_g/S_2^2 is homotopy equivalent to $\mathbb{CP}^2$, so we have

$$\mathcal{G}_t(C_g) \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \Omega^2 G.$$

Similarly, let g' be composition

$$g' : S^3 \xrightarrow{\sigma} S^3 \vee S^3 \xrightarrow{a\eta \vee \eta} S_1^2 \vee S_2^2$$

and let $C_{g'}$ be its cofiber. By Lemma 2.10, $\Sigma g'$ is $\bar{\eta}_1$ and $p_1 \circ g'$ is homotopic to $a\eta$ and $p_1 \circ g'$ is homotopic to $a\eta$. Use Theorem 2.4 by taking Y to be S_2^2 and $\varphi : S_2^2 \to S_1^2 \vee S_2^2$ and obtain $\mathcal{G}_t(M) \simeq \mathcal{G}_t(C_{g'}/S_2^2) \times \Omega^2 G$. Since $C_{g'}/S_2^2$ is homotopy equivalent to $C_{a\eta}$, we have

$$\mathcal{G}_t(C_{g'}) \simeq \mathcal{G}_t(C_{a\eta}) \times \Omega^2 G.$$

Observe that $g = T \circ g'$, where $T : S_1^2 \vee S_2^2 \to S_2^2 \vee S_1^2$ is the swapping map. Since T is a homotopy equivalence, $C_{g'}$ and C_g are homotopy equivalent. Combining the two homotopy equivalences gives the asserted lemma. □

PROPOSITION 2.13. *Suppose that M is a 4-dimensional simply connected Poincaré-complex. Let the rank of $H^2(M)$ be n. If Σf is null-homotopic, then there are homotopy equivalences*

$$\Sigma M \simeq S^5 \vee (\bigvee_{i=1}^{n} S^3) \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{n} \Omega^2 G.$$

If Σf is non-trivial, then there are homotopy equivalences

$$\Sigma M \simeq \Sigma \mathbb{CP}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{i=1}^{n-1} \Omega^2 G.$$

Proof. If Σf is null-homotopic, then use Theorem 2.4 by taking Y to be M_3 and $\varphi : Y \to M_3$ to be the identity map and get the first two homotopy equivalences.

If Σf is non-trivial, by Lemma 2.11 there are homotopy equivalences

$$\Sigma M \simeq \Sigma \mathbb{CP}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(C_{a\eta}) \times \prod_{i=1}^{n-1} \Omega^2 G,$$

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

where a is an odd integer. It suffices to show that we can replace $\mathcal{G}_t(C_{a\eta})$ by $\mathcal{G}_t(\mathbb{CP}^2)$.

When $n = 1$, the Poincaré complex condition implies that f has Hopf invariant equal to 1 or -1. Therefore, M is homotopy equivalent to $\mathbb{CP}^2$ and the statement holds. When $n \geq 2$, there is at least one copy of $\Omega^2 G$ on the right-hand side. By Lemma 2.12, we can replace $\mathcal{G}_t(C_{a\eta})$ by $\mathcal{G}_t(\mathbb{CP}^2)$ to obtain the proposition. □

It would be interesting to know if Lemma 2.12 can be improved to $\mathcal{G}_t(C_{a\eta}) \simeq \mathcal{G}_t(\mathbb{CP}^2)$. Observe that Proposition 2.13 is an improvement on the homotopy equivalence (1) for the non-spin case in [**18**], which gives a decomposition only after localization away from 2.

3. Gauge groups over non-simply connected 4-manifolds. From now on we assume that M is an orientable, smooth, connected, closed 4 manifold. By Morse theory, M admits a CW-structure with one 4-cell [**14**, Theorem 3.35]. In this section, we calculate the homotopy types of ΣM and $\mathcal{G}_t(M)$ when $\pi_1(M)$ is (1) a free group $\mathbb{Z}^{*m}$, (2) a cyclic group $\mathbb{Z}/p^r\mathbb{Z}$, or (3) a free product of types $(\mathbb{Z}^{*m}) * (*_{j=1}^{n} \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p and the p_j's are odd primes. Our strategy is to apply Theorem 2.4 and its variations to decompose $\mathcal{G}_t(M)$ into a product of a gauge group of a simply connected space, whose homotopy type is worked out in Proposition 2.13, and some complementary factors that do not depend on t.

3.1. The case when $\pi_1(M) = \mathbb{Z}^{*m}$. When $\pi_1(M)$ is a free group, M_3 is homotopy equivalent to a wedge sum of spheres [**7**]

$$M_3 \simeq (\bigvee_{i=1}^{m} S^3) \vee (\bigvee_{j=1}^{n} S^2) \vee (\bigvee_{k=1}^{m} S^1).$$

Using Theorem 2.4, we can calculate the homotopy types of ΣM and $\mathcal{G}_t(M)$.

THEOREM 3.1. *Suppose $\pi_1(M) \cong \mathbb{Z}^{*m}$. Let the rank of $H^2(M)$ be n. If Σf is null-homotopic, then there are homotopy equivalences*

$$\Sigma M \simeq S^5 \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{j=1}^{n} S^3) \vee (\bigvee_{k=1}^{m} S^2)$$
$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{j=1}^{n} \Omega^2 G \times \prod_{k=1}^{m} \Omega G.$$

If Σf is non-trivial, then there are homotopy equivalences

$$\Sigma M \simeq \Sigma\mathbb{CP}^2 \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{j=1}^{n-1} S^3) \vee (\bigvee_{k=1}^{m} S^2)$$
$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{j=1}^{n-1} \Omega^2 G \times \prod_{k=1}^{m} \Omega G.$$

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

Proof. Denote the ith copy of S^3 in M_3 by S^3_i and the kth copy of S^1 by S^1_k. We show that the inclusions $\varphi^3_i : S^3_i \to M_3$ and $\varphi^1_k : S^1_k \to M_3$ satisfy the hypothesis of Lemma 2.7 for all i's and k's.

Let $p^3_i : M_3 \to S^3_i$ and $p^1_k : M_3 \to S^1_k$ be the pinch maps. Then, Σp^3_i and Σp^1_k are left homotopy inverses of $\Sigma\varphi^3_i$ and $\Sigma\varphi^1_k$. Moreover, $\Sigma p^3_i \circ \Sigma f \simeq \Sigma(p^3_i \circ f)$ is null-homotopic since f induces a trivial homomorphism $f^* : H^3(M_3) \to H^3(S^3)$, and $\Sigma p^1_k \circ \Sigma f \simeq \Sigma(p^1_k \circ f)$ is null-homotopic since $p^1_k \circ f$ is null-homotopic by $\pi_3(S^1) = 0$. Apply Lemma 2.7 and get

$$\Sigma M \simeq \Sigma M' \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{k=1}^{m} S^2) \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(M') \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{k=1}^{m} \Omega G,$$

where M' is the cofiber of the inclusion $(\bigvee^m_{i=1} S^1) \vee (\bigvee^m_{k=1} S^3) \hookrightarrow M$. Observe that the 3-skeleton of M' is homotopy equivalent to $\bigvee^n_{i=1} S^2$. Since $H^2(S^1_i)$ and $H^2(S^3_k)$ are zero, Lemma 2.6 implies that M' satisfies Poincaré Duality. Let f' be the attaching map of the 4-cell in M'_3. By Proposition 2.13, if $\Sigma f'$ is null-homotopic, then we have

$$\Sigma M' \simeq S^5 \vee (\bigvee_{i=1}^{n} S^3) \text{ and } \mathcal{G}_t(M') \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{n} \Omega^2 G.$$

If $\Sigma f'$ is non-trivial, then we have

$$\Sigma M' \simeq \Sigma\mathbb{CP}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \text{ and } \mathcal{G}_t(M') \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{i=1}^{n-1} \Omega^2 G.$$

By Lemma 2.5, $\Sigma f'$ is null-homotopic if and only if Σf is null-homotopic. Combining these homotopy equivalences gives the theorem. □

3.2. The case when $\pi_1(M) = \mathbb{Z}/p^r\mathbb{Z}$. Recall that an n-dimensional Moore space $P^n(k)$ is the cofiber of the degree-k map $S^{n-1} \xrightarrow{k} S^{n-1}$ for $n \geq 2$. With integral coefficients, $\tilde{H}_i(P^n(k))$ is $\mathbb{Z}/k\mathbb{Z}$ for $i = n-1$ and is zero otherwise. With mod-k coefficients, $\tilde{H}_i(P^n(k); \mathbb{Z}/k\mathbb{Z})$ is $\mathbb{Z}/k\mathbb{Z}$ for $i = n-1$ and n and is zero otherwise. Let u be a generator of $\tilde{H}_{n-1}(P^n(k))$. By the Universal Coefficient theorem, $\tilde{H}_{n-1}(P^n(k); \mathbb{Z}/k\mathbb{Z})$ is generated by the mod-k reduction $\bar{u}$ of u, and $\tilde{H}_n(P^n(k); \mathbb{Z}/k\mathbb{Z})$ is generated by $\bar{v} = \beta\bar{u}$, where β is the Bockstein homomorphism.

For any space X, the mod-k homotopy group $\pi_n(X; \mathbb{Z}/k\mathbb{Z})$ is defined to be $[P^n(k), X]$. When $n \geq 3$, $\pi_n(X; \mathbb{Z}/k\mathbb{Z})$ has a group structure induced by the comultiplication of $P^n(k)$ and when $n \geq 4$, $\pi_n(X; \mathbb{Z}/k\mathbb{Z})$ is abelian. There are two associated homomorphisms: the mod-k Hurewicz homomorphism

$$\bar{h} : \pi_n(X; \mathbb{Z}/k\mathbb{Z}) \to H_n(X; \mathbb{Z}/k\mathbb{Z}),$$

which is defined to be $\bar{h}(f) = f_*(\bar{v})$ for $f \in \pi_n(X; \mathbb{Z}/k\mathbb{Z})$, and the homotopy Bockstein homomorphism

$$\bar{\beta}_\pi : \pi_n(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_{n-1}(X),$$

which is defined to be $\bar{\beta}_\pi(f) = \iota^* \circ f$ and $\iota : S^{n-1} \to P^n(k)$ is the inclusion. They are compatible with the standard Hurewicz homomorphism h, and Bockstein

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

homomorphisms β in the commutative diagram [**15**]

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_{n+1}(X;\mathbb{Z}/k\mathbb{Z}) & \xrightarrow{\beta_\pi} & \pi_n(X) & \xrightarrow{k} & \pi_n(X) & \longrightarrow & \pi_n(X;\mathbb{Z}/k\mathbb{Z}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \bar{h}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \bar{h}} & \\
\cdots \longrightarrow & H_{n+1}(X;\mathbb{Z}/k\mathbb{Z}) & \xrightarrow{\beta} & H_n(X) & \xrightarrow{k} & H_n(X) & \longrightarrow & H_n(X;\mathbb{Z}/k\mathbb{Z}) & \longrightarrow \cdots
\end{array} \tag{6}$$

For any map $g : P^3(k) \to P^3(k)$, let C_g be its cofiber, and let a and b be generators of $H^2(C_g;\mathbb{Z}/k\mathbb{Z})$ and $H^4(C_g;\mathbb{Z}/k\mathbb{Z})$. Then, the mod-$k$ Hopf invariant $\bar{H}(g) \in \mathbb{Z}/k\mathbb{Z}$ is defined by the formula $a \cup a \equiv \bar{H}(g)b \pmod{k}$.

LEMMA 3.2. *[**16**, Corollary 11.12] Let p be an odd prime and let $g : P^3(p^r) \to P^3(p^r)$ be a map in the kernel of $\bar{h}$. Then, g is null-homotopic if and only if its mod-p^r Hopf invariant $\bar{H}(g)$ is zero.*

Back to the calculation of $\mathcal{G}_t(M)$. When $\pi_1(M) = \mathbb{Z}/p^r\mathbb{Z}$, Poincaré Duality and the Universal Coefficient theorem imply the homology groups of M are as follows:

$$H_i(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 4 \\ \mathbb{Z}/p^r\mathbb{Z} & i = 1 \\ \mathbb{Z}^{\oplus n} \oplus \mathbb{Z}/p^r\mathbb{Z} & i = 2 \\ 0 & \text{else.} \end{cases}$$

Now, we calculate the homotopy type of ΣM to find a possible subcomplex Y satisfying the hypothesis of Theorem 2.4.

LEMMA 3.3. *Let p be an odd prime. Then, $\pi_4(P^4(p^r))$ and $\pi_4(P^3(p^r))$ are trivial.*

Proof. After localizing away from p, any $P^n(p^r)$ is contractible so $\pi_4(P^4(p^r))$ and $\pi_4(P^3(p^r))$ are p-torsion. Localize at p and consider the long exact sequence of homotopy groups for the pair $(P^4(p^r), S^3)$:

$$\cdots \longrightarrow \pi_4(S^3) \longrightarrow \pi_4(P^4(p^r)) \xrightarrow{j_*} \pi_4(P^4(p^r), S^3) \longrightarrow \cdots$$

The pair $(P^4(p^r), S^3)$ is 3-connected, so $\pi_4(P^4(p^r), S^3)$ is $\mathbb{Z}$ by Hurewicz Theorem. Since $\pi_4(S^3)$ is trivial at odd primes, j_* is an injection. But $\pi_4(P^4(p^r))$ is torsion, so this injection only makes sense if $\pi_4(P^4(p^r))$ is trivial.

Now, we calculate $\pi_4(P^3(p^r))$. Let $F^3\{p^r\}$ be the homotopy fiber of the pinch map $P^3(p^r) \longrightarrow S^3$. Then, $\pi_4(P^3(p^r))$ equals $\pi_4(F^3\{p^r\})$ since $\pi_4(\Omega S^3)$ and $\pi_4(S^3)$ are trivial at odd primes. By [**15**, Proposition 11.7.1], there is a p-local homotopy equivalence

$$\Omega F^3\{p^r\} \simeq S^1 \times \Omega\Sigma(\bigvee_\alpha P^{n_\alpha}(p^r)) \times \prod_j S^{2p^j-1}\{p^{r+1}\},$$

where n_α is either 3 or greater than 4, and $S^n\{p^{r+1}\}$ is the homotopy fiber of the degree map $p^{r+1} : S^n \longrightarrow S^n$. Since $\pi_4(F^3\{p^r\})$ is $\pi_3(\Omega F^3\{p^r\})$, we need to calculate the third homotopy group of each factor on the right-hand side. The first factor $\pi_3(S^1)$ and the last factor is trivial since $S^{2p^j-1}\{p^{r+1}\}$ is 3-connected for $j \geq 1$. For the remaining factor

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

of $\Omega F^3\{p^r\}$, consider the string of isomorphisms

$$\begin{aligned}\pi_3(\Omega\Sigma(\bigvee_\alpha P^{n_\alpha}(p^r))) &\cong \pi_3(\prod_\alpha \Omega\Sigma P^{n_\alpha}(p^r))\\ &\cong \pi_3(\Omega\Sigma P^3(p^r))\\ &\cong \pi_4(P^4(p^r))\\ &\cong 0.\end{aligned}$$

The first isomorphism is obtained from the Hilton–Milnor theorem using dimension and connectivity considerations. The second isomorphism holds since only one n_α equals 3, while the rest are strictly larger than 4. The third isomorphism holds by adjunction, and the fourth isomorphism holds as we have already seen that $\pi_4(P^4(p^r)) = 0$. Therefore, $\pi_4(F^3\{p^r\})$ is trivial and so is $\pi_4(P^3(p^r))$. □

LEMMA 3.4. *If $\pi_1(M) \cong \mathbb{Z}/p^r\mathbb{Z}$, then there is a homotopy equivalence*

$$\Sigma M_3 \simeq P^4(p^r) \vee (\bigvee_{i=1}^n S^3) \vee P^3(p^r).$$

Proof. Since ΣM_3 is simply connected, it has a minimal cell structure. By [**6**, Proposition 4H.3] ΣM_3 is homotopy equivalent to the cofiber of a map

$$g : P^3(p^r) \vee (\bigvee_{i=1}^n S^2) \to P^3(p^r)$$

such that g induces a trivial homomorphism $g_* : H_2(P^3(p^r) \vee (\bigvee_{i=1}^n S^2)) \to H_2(P^3(p^r))$. We claim that g is null-homotopic. To distinguish the 2-spheres in the wedge sum, denote the ith copy by S_i^2. Since

$$[P^3(p^r) \vee (\bigvee_{i=1}^n S_i^2), P^3(p^r)] \cong [P^3(p^r), P^3(p^r)] \oplus \left(\bigoplus_{i=1}^n [S_i^2, P^3(p^r)]\right),$$

we write $g = g' \oplus (\bigoplus_{i=1}^n g_i'')$, where $g' \in [P^3(p^r), P^3(p^r)]$ and $g_i'' \in [S_i^2, P^3(p^r)]$.

Consider the commutative diagram

$$\begin{array}{ccc}\pi_2(S_i^2) & \xrightarrow{g_i''} & \pi_2(P^3(p^r))\\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h}\\ H_2(S_i^2) & \xrightarrow{(g_i'')_*} & H_2(P^3(p^r)).\end{array}$$

Both Hurewicz homomorphisms h are isomorphisms by Hurewicz theorem. Since g_* is trivial, so is $(g_i'')_*$. The diagram implies that g_i'' is null-homotopic. Therefore, ΣM_3 is homotopy equivalent to $C_{g'} \vee (\bigvee_{i=1}^n S^3)$, where $C_{g'}$ is the cofiber of g'. It suffices to show that g' is null-homotopic.

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

Consider the commutative diagram

$$\begin{array}{ccc} \pi_3(P^3(p^r);\mathbb{Z}/p^r\mathbb{Z}) & \xrightarrow{\bar{h}} & H_3(P^3(p^r);\mathbb{Z}/p^r\mathbb{Z}) \\ \downarrow{\scriptstyle \beta_\pi} & & \downarrow{\scriptstyle \beta} \\ \pi_2(P^3(p^r)) & \xrightarrow{h} & H_2(P^3(p^r)) \end{array}$$

from (6). The induced homomorphism g'_* is trivial, and we have

$$h \circ \beta_\pi(g') = \beta \circ \bar{h}(g') = \beta \circ ((g')_* v) = 0.$$

Observe that $\beta : H_3(P^3(p^r);\mathbb{Z}/p^r\mathbb{Z}) \to H_2(P^3(p^r))$ is an isomorphism in this case, so g' is in the kernel of $\bar{h}$. Since $C_{g'}$ retracts off the suspension ΣM_3, it is a co-H-space and $H^*(C_{g'};\mathbb{Z}/p^r\mathbb{Z})$ has trivial cup products. Therefore, the mod-p^r Hopf invariant $\bar{H}(g')$ is zero and g' is null-homotopic by Lemma 3.2. □

Lemma 3.4 says that ΣM_3 contains $P^3(p^r) \vee P^4(p^r)$ as its wedge summands. This, however, does not necessarily imply that M_3 contains $P^2(p^r) \vee P^3(p^r)$ since M is not simply connected.

LEMMA 3.5. *If $\pi_1(M) \cong \mathbb{Z}/p^r\mathbb{Z}$, then there exists a map $\epsilon : P^2(p^r) \to M_3$ satisfying the hypothesis of Theorem 2.4 and its cofiber C_ϵ is simply connected.*

Proof. By the Cellular Approximation theorem, $\pi_1(M_3)$ equals $\pi_1(M) \cong \mathbb{Z}/p^r\mathbb{Z}$. Let $j : S^1 \to M_3$ represent a generator of $\pi_1(M_3)$. It has order p^r, so there exists an extension $\epsilon : P^2(p^r) \to M_3$.

$$\begin{array}{ccccc} S^1 & \xrightarrow{p^r} & S^1 & \longrightarrow & P^2(p^r) \\ & & \downarrow{\scriptstyle j} & \swarrow{\scriptstyle \epsilon} & \\ & & M_3 & & \end{array}$$

Since $\pi_1(M_3)$ is abelian, $H_1(M_3)$ is $\pi_1(M_3) \cong \mathbb{Z}/p^r\mathbb{Z}$ by Hurewicz theorem and the induced map $\epsilon_* : H_1(P^2(p^r)) \to H_1(M_3)$ is an isomorphism. Therefore, the cofiber C_ϵ of ϵ has $H_1(C_\epsilon) = 0$, implying that C_ϵ is simply connected.

Now, we show that $\Sigma\epsilon$ has a left homotopy inverse. Let ψ be the composition

$$\psi : \Sigma M_3 \simeq P^4(p^r) \vee (\bigvee_{i=1}^{n} S_i^3) \vee P^3(p^r) \xrightarrow{pinch} P^3(p^r).$$

Observe that

$$\psi_* : H_2(\Sigma M_3;\mathbb{Z}/p^r\mathbb{Z}) \to H_2(P^3(p^r);\mathbb{Z}/p^r\mathbb{Z}) \text{ and } (\Sigma\epsilon)_* : H_2(P^3(p^r);\mathbb{Z}/p^r\mathbb{Z}) \to H_2(\Sigma M_3;\mathbb{Z}/p^r\mathbb{Z})$$

are isomorphism, so $(\psi \circ \Sigma\epsilon)_* : H_2(P^3(p^r);\mathbb{Z}/p^r\mathbb{Z}) \to H_2(P^3(p^r);\mathbb{Z}/p^r\mathbb{Z})$ is an isomorphism. Then, Bockstein homomorphism implies that $(\psi \circ \Sigma\epsilon)_* : H_3(P^3(p^r);\mathbb{Z}/p^r\mathbb{Z}) \to H_3(P^3(p^r);\mathbb{Z}/p^r\mathbb{Z})$ is an isomorphism. Therefore, $\psi \circ \Sigma\epsilon$ is a homotopy equivalence and ψ is a left homotopy inverse of $\Sigma\epsilon$.

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

Moreover, the composition $\psi \circ \Sigma f$ is null-homotopic since $\pi_4(P^3(p^r))$ is trivial by Lemma 3.3. Therefore, ϵ satisfies the hypothesis of Theorem 2.4. □

Denote the pointed mapping space $Map^*(P^n(p^r), G)$ by $\Omega^n G\{p^r\}$. This notation is justified since $Map^*(P^n(p^r), G)$ is the homotopy fiber of the power map $p^r : \Omega^n G \to \Omega^n G$. Now, we calculate the homotopy type of $\mathcal{G}_t(M)$.

THEOREM 3.6. *Suppose $\pi_1(M) \cong \mathbb{Z}/p^r\mathbb{Z}$, where p is an odd prime. Let the rank of $H^2(M)$ be n. If Σf is null-homotopic, then there are homotopy equivalences*

$$\Sigma M \simeq S^5 \vee P^4(p^r) \vee (\bigvee_{i=1}^{n} S^3) \vee P^3(p^r)$$
$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(S^4) \times \Omega^3 G\{p^r\} \times \prod_{i=1}^{n} \Omega^2 G \times \Omega^2 G\{p^r\}.$$

If Σf is non-trivial, then there are homotopy equivalences

$$\Sigma M \simeq \Sigma\mathbb{CP}^2 \vee P^4(p^r) \vee (\bigvee_{i=1}^{n-1} S^3) \vee P^3(p^r)$$
$$\mathcal{G}_t(M) \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \Omega^3 G\{p^r\} \times \prod_{i=1}^{n-1} \Omega^2 G \times \Omega^2 G\{p^r\}.$$

Proof. We decompose $\mathcal{G}_t(M)$ in the following steps.

Step 1: By Lemma 3.5, there exists a map $\epsilon : P^2(p^r) \to M_3$ satisfying the hypothesis of Theorem 2.4. Apply the theorem and get

$$\Sigma M \simeq \Sigma(M') \vee P^3(p^r) \text{ and } \mathcal{G}_t(M) \simeq \mathcal{G}_t(M') \times \Omega^2 G\{p^r\},$$

where M' is the cofiber of $P^2(p^r) \xrightarrow{\epsilon} M_3 \hookrightarrow M$.

Step 2: By Lemma 3.5 M' is simply connected, so its 3-skeleton M'_3 is $(\bigvee_{i=1}^{n} S^2) \vee P^3(p^r)$. We show that the inclusion $\varphi : P^3(p^r) \hookrightarrow M'_3$ satisfies the condition of Theorem 2.4. The pinch map $\Sigma M'_3 \to P^4(p^r)$ is a left homotopy inverse of $\Sigma\varphi$. Let f' be the attaching map of the 4-cell in M'. Then, the composition

$$S^4 \xrightarrow{\Sigma f'} \Sigma M'_3 \xrightarrow{pinch} P^4(p^r)$$

is null-homotopic since $\pi_4(P^4(p^r))$ is trivial by Lemma 3.3. Apply Theorem 2.4 and get

$$\Sigma M' \simeq \Sigma M'' \vee P^4(p^r) \text{ and } \mathcal{G}_t(M') \simeq \mathcal{G}_t(M'') \times \Omega^3 G\{p^r\},$$

where M'' is $M'/P^3(p^r)$.

Step 3: By Lemma 3.4, the 3-skeleton of M'' is homotopy equivalent to $\bigvee_{i=1}^{n} S^2$. Since $H^2(P^2(p^r))$ and $H^2(P^3(p^r))$ are torsion, Lemma 2.6 implies that M'' satisfies Poincaré Duality. Let f'' be the attaching map of the 4-cell in M''. By Proposition 2.13, if $\Sigma f''$ is null-homotopic, then

$$\Sigma M'' \simeq S^5 \vee (\bigvee_{i=1}^{n} S^3) \text{ and } \mathcal{G}_t(M'') \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{n} \Omega^2 G$$

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

If $\Sigma f''$ is non-trivial, then

$$\Sigma M'' \simeq \Sigma\mathbb{CP}^2 \vee (\bigvee_{i=1}^{n-1} S^3) \text{ and } \mathcal{G}_t(M'') \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{i=1}^{n-1} \Omega^2 G$$

Step 4: Combining all the homotopy equivalences from Steps 1 to 3 gives the theorem. □

3.3. The case when $\pi_1 = (\mathbb{Z}^{*m}) * (*_{j=1}^n \mathbb{Z}/p_j^{r_j}\mathbb{Z})$. Suppose that $\pi_1(M)$ is $(\mathbb{Z}^{*m}) * (*_{j=1}^n \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p_j is an odd prime. The Stable Decomposition theorem [**13**, Theorem 1.3] implies that for some number d there is a diffeomorphism

$$M\#_d(S^2 \times S^2) \cong N\#(\#_{j=1}^n L_j), \tag{7}$$

where N and L_j's are orientable, smooth, connected, closed 4-manifolds with $\pi_1(N) = \mathbb{Z}^{*m}$ and $\pi_1(L_j) = \mathbb{Z}/p_j^{r_j}\mathbb{Z}$. We will calculate the suspensions and gauge groups for both sides of isomorphism (7).

LEMMA 3.7. *There are homotopy equivalences*

$$\Sigma(M\#_d(S^2 \times S^2)) \simeq (\Sigma M) \vee (\bigvee_{s=1}^{2d} S^3) \text{ and } \mathcal{G}_t(M\#_d(S^2 \times S^2)) \simeq \mathcal{G}_t(M) \times \prod_{s=1}^{2d} \Omega^2 G.$$

Proof. By induction it suffices to show the lemma for the $d = 1$ case. We use Lemma 2.8 to prove it. For M, the inclusion $\varphi : \{m_0\} \hookrightarrow M$ of the basepoint m_0 obviously satisfies the hypothesis of Theorem 2.4. For $S^2 \times S^2$, take the inclusion $\varphi' : S^2 \vee S^2 \hookrightarrow S^2 \times S^2$ to be the inclusion of the 3-skeleton. Since $\Sigma(S^2 \times S^2)$ is homotopy equivalent to $S^5 \vee S^3 \vee S^3$, φ' satisfies the hypothesis as well. Apply Lemma 2.8 to get

$$\Sigma(M\#(S^2 \times S^2)) \simeq (\Sigma C_j) \vee (\bigvee_{s=1}^{2} S^3) \text{ and } \mathcal{G}_t(M\#(S^2 \times S^2)) \simeq \mathcal{G}_t(C_j) \times \prod_{s=1}^{2} \Omega^2 G,$$

where C_j is the cofiber of the inclusion $j : \{m_0\} \vee (S^2 \vee S^2) \hookrightarrow M\#(S^2 \times S^2)$. Consider the cofibration diagram

$$\begin{array}{ccccc}
* & \longrightarrow & S^3 & = & S^3 \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_M} \\
S^2 \vee S^2 & \longrightarrow & M_3 \vee S^2 \vee S^2 & \longrightarrow & M_3 \\
\| & & \downarrow & & \downarrow \\
S^2 \vee S^2 & \xrightarrow{j} & M\#(S^2 \times S^2) & \longrightarrow & M
\end{array}$$

where f_M is the attaching map of the 4-cell in M. The bottom row implies that C_j is homotopy equivalent to M and the asserted homotopy equivalences follow. □

THEOREM 3.8. *Let $\pi_1(M) \cong (\mathbb{Z}^{*m}) * (*_{j=1}^n \mathbb{Z}/p_j^{r_j}\mathbb{Z})$, where p_j is an odd prime and let l be the rank of $H^2(M)$. Then, there exists a number d such that $\Sigma(M\#_d(S^2 \times S^2))$ is*

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

homotopy equivalent to either

$$S^5 \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{j=1}^{n} P^4(p_j^{r_j})) \vee (\bigvee_{k=1}^{l+2d} S^3) \vee (\bigvee_{j'=1}^{n} P^3(p_{j'}^{r_{j'}})) \vee (\bigvee_{i'=1}^{m} S^2) \quad \text{or}$$
$$\Sigma\mathbb{CP}^2 \vee (\bigvee_{i=1}^{m} S^4) \vee (\bigvee_{j=1}^{n} P^4(p_j^{r_j})) \vee (\bigvee_{k=1}^{l+2d-1} S^3) \vee (\bigvee_{j'=1}^{n} P^3(p_{j'}^{r_{j'}})) \vee (\bigvee_{i'=1}^{m} S^2).$$

In the first case, we have

$$\mathcal{G}_t(M) \times \prod_{s=1}^{2d} \Omega^2 G \simeq \mathcal{G}_t(S^4) \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{j=1}^{n} \Omega^3 G\{p_j^{r_j}\} \times \prod_{k=1}^{l+2d} \Omega^2 G \times \prod_{j'=1}^{n} \Omega^2 G\{p_{j'}^{r_{j'}}\} \times \prod_{i'=1}^{m} \Omega G.$$

In the second case, we have

$$\mathcal{G}_t(M) \times \prod_{s=1}^{2d} \Omega^2 G \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{j=1}^{n} \Omega^3 G\{p_j^{r_j}\} \times \prod_{k=1}^{l+2d-1} \Omega^2 G \times \prod_{j'=1}^{n} \Omega^2 G\{p_{j'}^{r_{j'}}\} \times \prod_{i'=1}^{m} \Omega G.$$

Proof. Denote $N\#(\#_{j=1}^{n} L_j)$ by X. Using the stable decomposition (7) and Lemma 3.7, we only need to calculate ΣX and $\mathcal{G}_t(X)$. Let N_3 and $(L_j)_3$ be the 3-skeletons of N and L_j. Then, the 3-skeleton X_3 of X is the wedge sum $N_3 \vee (\bigvee_{j=1}^{n}(L_j)_3)$.

Step 1: The 3-skeleton N_3 is homotopy equivalent to $(\bigvee_{i=1}^{m} S^3) \vee (\bigvee_{j'=1}^{l'} S^2) \vee (\bigvee_{k=1}^{m} S^1)$. Denote the i^{th} copy of S^3 in N_3 by S_i^3 and the k^{th} copy of S^1 by S_k^1. In the proofs of Theorem 3.1, we show that inclusions $\varphi_i^3 : S_i^3 \to N_3$ and $\varphi_k^1 : S_k^1 \to N_3$ satisfy the hypothesis of Theorem 2.4. For each j, by Lemma 3.5 there exists a map $\epsilon_j : P^2(p_j^{r_j}) \to (L_j)_3$ satisfying the hypothesis as well. Apply Lemma 2.8 and get

$$\Sigma X \simeq \Sigma X' \vee \left(\bigvee_{i=1}^{m} S^4\right) \vee \left(\bigvee_{k=1}^{m} S^2\right) \vee \left(\bigvee_{j=1}^{n} P^3(p_j^{r_j})\right)$$
$$\mathcal{G}_t(X) \simeq \mathcal{G}_t(X') \times \prod_{i=1}^{m} \Omega^3 G \times \prod_{k=1}^{m} \Omega G \times \prod_{j=1}^{n} \Omega G\{p_j^{r_j}\},$$

where X' is the cofiber of the inclusion $\left(\bigvee_{i=1}^{m} S^3\right) \vee \left(\bigvee_{k=1}^{m} S^1\right) \vee \left(\bigvee_{j=1}^{n} P^2(p_j^{r_j})\right) \hookrightarrow X$.

Step 2: Since X' is simply connected, its 3-skeleton X_3' has a minimal cell structure

$$X_3 \simeq \left(\bigvee_{j'=1}^{l''} S^2\right) \vee \left(\bigvee_{j=1}^{n} P^3(p_j^{r_j})\right).$$

https://doi.org/10.1017/S0017089518000241 Published online by Cambridge University Press

We show that inclusions $\varphi_j : P^3(p_j^{r_j}) \to X'_3$ satisfy the hypothesis of Theorem 2.4. For each j, the pinch maps $\Sigma X'_3 \to P^4(p_j^{r_j})$ is a left homotopy inverse of $\Sigma\varphi_j$. Let f' be the attaching map of the 4-cell in X'. Then, the composition

$$S^4 \xrightarrow{\Sigma f'} \Sigma X'_3 \xrightarrow{pinch} P^4(p_j^{r_j})$$

is null-homotopic since $\pi_4(P^3(p_j^{r_j}))$ is trivial by Lemma 3.3. Apply Lemma 2.7 and get

$$\Sigma X' \simeq \Sigma X'' \vee \left(\bigvee_{j=1}^{n} P^3(p_j^{r_j})\right) \text{ and } \mathcal{G}_t(X') \simeq \mathcal{G}_t(X'') \times \prod_{j=1}^{n} \Omega^2 G\{p_j^{r_j}\},$$

where X'' is $X'/(\bigvee_{j=1}^{n} P^3(p_j^{r_j}))$.

Step 3: The 3-skeleton of X'' is homotopy equivalent to $\bigvee_{j'=1}^{l''} S^2$. Since the subcomplexes S^1_i, S^3_k, $P^2(p_j^{r_j})$ and $P^3(p_j^{r_j})$ in Steps 1 and 2 have either zero or torsion second cohomology groups, Lemma 2.6 implies that X'' satisfies Poincaré Duality. By Proposition 2.13, we have

$$\Sigma X'' \simeq S^5 \vee \left(\bigvee_{j'=1}^{l''} S^3\right) \text{ and } \mathcal{G}_t(X'') \simeq \mathcal{G}_t(S^4) \times \prod_{j'=1}^{l''} \Omega^2 G$$

or

$$\Sigma X'' \simeq \Sigma\mathbb{CP}^2 \vee \left(\bigvee_{j'=1}^{l''-1} S^3\right) \text{ and } \mathcal{G}_t(X'') \simeq \mathcal{G}_t(\mathbb{CP}^2) \times \prod_{j'=1}^{l''-1} \Omega^2 G.$$

Step 4: Combining all homotopy equivalences from Steps 1 to 3, the stable decomposition (7) and Lemma 3.7 together imply the theorem. Furthermore, $H_2(M\#_d(S^2 \times S^2))$ has rank $l = 2d$, so $l'' = l + 2d$. □

Remark. The proofs of Theorems 3.1 and 3.6 are also valid for any orientable 4-dimensional CW-complex with one 4-cell, while Theorem 3.8 requires the smoothness of M for the stable diffeomorphism splitting in [**13**]. One can ask under what condition the stabilizing factor $\#_d(S^2 \times S^2)$ can be cancelled, so that the factors $\prod_{s=1}^{2d} \Omega^2 G$ can be removed from the equation. If this can be achieved, Theorems 3.1 and 3.6 will be corollaries of Theorem 3.8 when M is smooth.

ACKNOWLEDGEMENTS. The author would like to thank the reviewer for carefully reading the paper and giving many constructive comments.

REFERENCES

1. M. Atiyah and R. Bott, The Yang–Mills equations over Riemann surfaces, *Philos. Trans. R. Soc. Lond. Ser. A: Math. Phys. Sci.* **308**(1505) (1983), 523–615.

2. F. Cohen, J. Moore, and J. Neisendorfer, Torsion in homotopy goups, *Ann. Math., Second Ser.*, **109** (1979), 121–168.

3. H. Hamanaka and A. Kono, Unstable K-group and homotopy type of certain gauge groups, *Proc. R. Soc. Edinburgh Sect. A: Math.* **136**(2006), 149–155.

4. D. Gottlieb, Applications of bundle map theory, *Trans. Amer. Math. Soc.* **171**(1972), 23–50.

5. M. Greenberg and J. Harper, *Algebraic topology: A first course* (Benjamin/Cummings Publishing Company, Massachusetts, US, 1981).

6. A. Hatcher, *Algebraic topology* (Cambridge University Press, Cambridge, UK, 2002).

7. T. Matumoto and A. Katanaga, On 4-dimensional closed manifolds with free fundamental groups, *Hiroshima Math. J.* **25**(1995), 367–370.

8. D. Kishimoto and A. Kono, On the homotopy types of $Sp(n)$ gauge groups, arXiv:1803.06477.

9. D. Kishimoto, A. Kono and M. Tsutaya, On p-local homotopy types of gauge groups, *Proc. R. Soc. Edinburgh Sect. A: Math.* **144**(2014), 149–160.

10. D. Kishimoto, S. Theriault, and M. Tsutaya, The homotopy types of G_2-gauge groups, *Topol. its Appl.* **228** (2017), 92–107.

11. A. Kono, A note on the homotopy type of certain gauge groups, *Proc. R. Soc. Edinburgh Section A: Math.* **117** (1991), 295–297.

12. A. Kono and S. Tsukuda, A remark on the homotopy type of certain gauge groups, *J. Math. Kyoto Univ.* **36** (1996), 115–121.

13. M. Kreck, W. Lück, and P. Teichner, Stable prime decompositions of 4-manifolds, in *Prospects in topology* (Quinn F., Editor), Annals of mathematics studies, 138 (Princeton University Press, Princeton, 1995), 251–269.

14. Y. Matsumoto, *An Introduction to Morse Theory*, Translations of mathematical monographs, (American Mathematical Society, Rhode Island, US, 2002).

15. J. Neisendorfer, *Algebraic methods in unstable homotopy theory* (Cambridge University Press, Cambridge, UK, 2010).

16. J. Neisendorfer, Homotopy groups with coefficients, *J. Fixed Point Theory Appl.* **8** (2010), 247–338.

17. W. Sutherland, Function spaces related to gauge groups, *Proc. R. Soc. Edinburgh Section A: Math.* **121** (1992), 185–190.

18. S. Theriault, Odd primary homotopy decompositions of gauge groups, *Algebraic Geom. Topol.* **10** (2010), 535–564.

19. S. Theriault, The homotopy types of $Sp(2)$-gauge groups, *Kyoto J. Math.* **50** (2010), 591–605.

20. S. Theriault, *Homotopy types of $SU(3)$-gauge groups*, *Publ. RIMS Kyoto Univ.* **48** (2012), 543–563.

21. S. Theriault, The homotopy types of $SU(5)$-gauge groups, *Osaka J. Math.* **52** (2015), 15–31.

22. S. Theriault, Odd primary homotopy types of $SU(n)$-gauge groups, *Algebraic Geom. Topol.* **17** (2017), 1131 – 1150.