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TORSION-FREE ABELIAN GROUPS, VALUATIONS AND TWISTED GROUP RINGS

BY

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Dedicated to the memory of Srinivasa Ramanujan on the occasion of the one hundredth anniversary of his birth.

ABSTRACT. Anderson and Ohm have introduced valuations of monoid rings $k[\Gamma]$ where k is a field and Γ a cancellative torsion-free commutative monoid. We study the residue class fields in question and solve a problem concerning the pure transcendence of the residue fields.

1. **Introduction.** In this paper we give a solution to a problem posed by Anderson and Ohm on valuations of group rings ([1], p. 149). We do this in a general setting by working with quotient fields of group rings and twisted group rings. Throughout k will denote a field, A a torsion-free additive abelian group and $k(A)$ the quotient field of the group ring $k[A]$ or the twisted group ring $k^{\tau}[A]$, which are known to be integral domains. Even though the latter rings have been studied extensively, little is known about the quotient field $k(A)$. The problem in [1] quickly leads to the question as to when the field extension $k(A)/k$ is purely transcendental (Proposition 3.1). Consideration of a natural valuation (Corollary 2.3) provides the answer.

Dealing with monoid rings, one would expect that the order structure of the monoid may have even a stronger influence on the factoriality of monoid rings. This is taken in Corollary 2.4. It is interesting to compare our proof with that of Gilmer and Parker in [5, 4], where a stronger result is presented.

In Section 4 we show how twisted group rings arise in the description of residue class fields of natural valuations defined on group rings.

We denote by $R^{\cdot}$ the set of non-zero elements of R. It is also worthwhile to mention an often used fact on group rings. A torsion-free (abelian) group A carries a total order $\leqq$ (which we can think of being fixed), so we may write an element $y \neq 0$ in the group ring $k[A]$ uniquely as a finite sum

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$$y = \sum_{i=1}^{m} a_i x^{\alpha_i} \ (0 \neq a_i \in k, \alpha_1 < \ldots < \alpha_m \text{ in } A).$$

An element of $k(A)$ is the quotient of two such expressions.

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2. $k(A)$ **as quotient field of an UFD.** We begin with an easy observation.

PROPOSITION 2.0. *Let B be a subgroup of the torsion-free abelian group A and $\{\alpha_i : i \in I\}$ a family of elements of A. Then the set $\{x^{\alpha_i} : i \in I\} \subseteq k(A)$ is algebraically independent over $k(B)$ if and only if the set $\{\alpha_i : i \in I\}$ is linearly independent* (*over* $\mathbf{Z}$) mod B. *Thus if A/B is free abelian, then $k(A)/k(B)$ is purely transcendental.*

PROOF. If $n_1\alpha_1 + \ldots + n_t\alpha_t = \beta \in B(n_i \in \mathbf{Z})$ is a non-trivial relation, say $n_1, \ldots, n_s > 0$, $n_{s+1}, \ldots, n_t < 0$, then

$$(x^{\alpha_1})^{n_1} \ldots (x^{\alpha_s})^{n_s} - x^{\beta}(x^{\alpha_{s+1}})^{-n_{s+1}} \ldots (x^{\alpha_t})^{-n_t} = 0$$

shows that $x^{\alpha_1}, \ldots, x^{\alpha_t}$ are algebraically dependent over $k(B)$. On the other hand, if $\alpha_1, \ldots, \alpha_t$ are linearly independent mod B then no expression of the form $\sum yx^{n_1\alpha_1 + \ldots + n_t\alpha_t}(y \in k[B])$ can vanish because no terms with formally different exponents can cancel. □

The next result, which is as easy, leads to the converse.

THEOREM 2.1. *Let $k(A)$ be the quotient field of the group ring $k[A]$ or twisted group ring $k^{\tau}[A]$. Let R be a unique factorization domain* (UFD) *having $k(A)$ as its quotient field. If $U(R)$ is the group of units of R, then the quotient $A/A \cap U(R)$ is free abelian.*

PROOF. Consider the group of divisibility $(k(A))^{\cdot}/U(R)$ which is free abelian, since R is an UFD ([7], p. 118). If $\theta:(k(A))^{\cdot} \to (k(A))^{\cdot}/U(R)$ is the canonical semi-valuation, then θ restricted to A has kernel $A \cap U(R)$. Thus the image $A/A \cap U(R)$ is free. □

COROLLARY 2.2. *Let $k(A)$ be as in the hypothesis of Theorem* 2.1 *and B a subgroup of A. The field extension $k(A)/k(B)$ is purely transcendental if and only if A/B is free.* (*Then B is a summand of A*).

PROOF. If $k(A)/k(B)$ is purely transcendental, we can take R to be a suitable polynomial ring over $k(B)$ and $A \cap U(R) = B$. By Theorem 2.1, A/B will be free. The converse is Proposition 2.0. □

In the following result, the case when $k[A]$ is an ordinary group ring was also communicated by Warren L. May.

COROLLARY 2.3. *In both the ordinary and twisted group ring cases, the field extension $k(A)/k$ is purely transcendental if and only if A is free abelian.*

If S is a torsion-free cancellative monoid, then S has the structure of a totally ordered monoid. It is not necessary that S be positively ordered; i.e., $0 \leqq S$. Gilmer and Parker [5] have characterized monoid rings $k[S]$ which are factorial. If $k[S]$ is factorial, it is not true that S is a free monoid ([4], Theorem 14.16). The case, when it is indeed so, is influenced by the way S is totally ordered. We first present a weaker version as a corollary to Theorem 2.1.

COROLLARY 2.4. *Let S be a monoid and k a field. Assume that S is positively ordered in some ordering of S. If the monoid ring $k[S]$ is factorial, then the group hull of S is free abelian.*

The authors are thankful to the referee for drawing their attention to the following stronger result ([4], Theorem 14.16). A direct proof in our spirit would use the semi-valuation θ that appears in the proof of Theorem 2.1.

THEOREM 2.5 (Gilmer and Parker). *Let S be a monoid and k a field. Then the following conditions are equivalent for the monoid ring $k[S]$:*
(1) *S is a free monoid isomorphic to the direct sum $\oplus \mathbf{Z}_+$ of copies of the monoid of non-negative integers*
(2) *$k[S]$ is a polynomial ring*
(3) *$k[S]$ is factorial and the monoid S is positively ordered in some ordering of S.*

The converse of Corollary 2.4, of course, is not true, as is seen by taking $S = \{0, 2, 3, \ldots\}$; the monoid ring $k[S]$ in this case is not even integrally closed.

On the other hand, combining Pontryagin's example (Proposition 2.15, p. 101 in [6]) with Corollary 2.4, we get an example of a non-free abelian group G having the following properties:

(i) The group ring $k[G]$ is factorial

(ii) For no ordering P of G, the monoid ring $k[P]$ is factorial.

3. **The problem posed by Anderson and Ohm.** We now turn to the problem posed in [1]. Let Γ be a cancellative torsion-free commutative monoid and $\phi:\Gamma \to C$ a monoid homomorphism into a totally ordered abelian group C. Clearly, ϕ can be extended uniquely to a group homomorphism $\bar{\phi}:A \to C$, where A denotes the group hull of Γ. Observe that, for any field k, the quotient fields $k(\Gamma)$ and $k(A)$ of the semigroup ring $k(\Gamma)$ and the group ring $k[A]$ are identical.

Recall ([1], p. 147) that the infimum valuation $v{:}k[\Gamma]^{\cdot} \rightarrow C$ is defined by

$$v\left(\sum_{i=1}^{m} a_i x^{\gamma_i}\right) = \inf_{i=1,2,\ldots,m} \{\phi(\gamma_i)\}$$

Evidently, k is a subfield of the residue class field k_v of the valuation v. In [1] an example (p. 149) is given, where k_v/k is purely transcendental and the problem is posed to characterize the groups C for which the residue class field k_v is an infinite pure transcendental extension of k. Surprisingly, the pure transcendental character of k_v/k depends only on the kernel of $\bar{\phi}$, as we shall show below. First, a preparatory result, which may be deduced from Propositions 1.3 and 1.4 of [1].

PROPOSITION 3.1. *The residue class field k_v is isomorphic to the field $k(H)$ where H =* ker $\bar{\phi}$.

PROOF. We adopt the notation of [1]. We will show that the valuation ring R_v of v satisfies $R_v = k(H) + M_v$ where M_v is its maximal ideal; as $k(H) \cap M_v = 0$, that $R_v/M_v = k(H)$ will follow at once. Since inclusion in one way is obvious, it suffices to prove $R_v \subseteq k(H) + M_v$.

Let f/g be a unit in R_v, where $f, g \in k[\Gamma]$. We may assume that $v(f) = v(g) = 0$. We may write

$$f = \sum_{i=1}^{m} a_i x^{\alpha_i} + f_1,\ g = \sum_{j=1}^{n} b_j x^{\beta_j} + g_1 \quad (a_i, b_j \in k)$$

where $\alpha_i, \beta_j \in H$ and $f_1 = 0$ or $v(f_1) > 0$, and $g_1 = 0$ or $v(g_1) > 0$. Thus $f_1, g_1 \in M_v$, while $\xi = \sum a_i x^{\alpha_i}$ and $\eta = \sum b_j x^{\beta_j} \in k[H]$. Since

$$f/g - \xi/\eta = (f_1 g - g_1 f)/g(g - g_1) \in M_v$$

and $\xi/\eta \in k(H)$, it follows that $f/g \in k(H) + M_v$. □

We see that if $\bar{\phi}$ is an injective map, then k_v coincides with k. Otherwise, k_v/k is a transcendental extension. Combining Corollary 2.3 and Proposition 3.1, we obtain:

THEOREM 3.2. *Let $\phi{:}A \rightarrow C$ be a group homomorphism of the torsion-free abelian group A into a totally ordered abelian group C, and v the infimum valuation of the field $k(A)$ defined via ϕ. Then the residue class field k_v of v is a purely transcendental extension of k if and only if* ker ϕ *is free.*

Notice that $k(H)/k$ is a finitely generated extension if and only if H is a finitely generated group. With this observation, the question posed by Anderson and Ohm is readily answered:

COROLLARY 3.3. *With the same notation as in the theorem, k_v/k is an infinite pure transcendental extension if and only if* $\ker \phi$ *is free and not finitely generated.*

4. **The twisted group ring.** In [1], Anderson and Ohm defined certain other valuations of monoid rings (Proposition 2.1, p. 150). We will show in this section how twisted group rings naturally appear as subrings of the residue class fields in question.

Let C be a totally ordered abelian group and let $u:k^{\cdot} \to C$ be a valuation of the field k. Let Γ, ϕ, A and $\bar{\phi}$ be as in Section 3. Recall that the Anderson-Ohm valuation v of $k[\Gamma]$ into C is defined by

$$v\left(\sum_{i=1}^{s} a_{\alpha_i} x^{\alpha_i}\right) = \inf_{i=1,2,\ldots,s} \{u(a_{\alpha_i}) + \phi(\alpha_i)\}$$

where $a_{\alpha_i} \in k^{\cdot}$, and $\alpha_i \neq \alpha_j$ for $i \neq j$. We are interested in the residue class field of v, where v is naturally extended to the quotient field $k(\Gamma)$ of $k[\Gamma]$.

We recall next the definition of twisted group rings ([1], p. 154). Let F be a field and H a torsion-free abelian group. An *F-twisting* of H is a map $\tau:H \times H \to F^{\cdot}$ satisfying the following conditions:

(i) $\tau(\alpha, \beta) = \tau(\beta, \alpha)$ and

(ii) $\tau(\alpha, \beta)\tau(\alpha + \beta, \gamma) = \tau(\beta, \gamma)\tau(\alpha, \beta + \gamma)$

for all $\alpha, \beta, \gamma \in H$. The *twisted group ring* $F^{\tau}[H]$ of H over F with respect to the twisting τ is defined as follows: $F^{\tau}[H] = \oplus_{\alpha \in H} F x^{\alpha}$, the F-vector space with basis $\{x^{\alpha}\}_{\alpha \in H}$; multiplication is defined distributively via $x^{\alpha}x^{\beta} = \tau(\alpha, \beta)x^{\alpha+\beta}$. The twisted group ring is indeed a domain. For this and other relevant remarks, we refer to [1], p. 154.

To find the residue class field k_v of the Anderson-Ohm valuation v, we will first define a k_u-twisting τ of the subgroup $H = \phi^{-1}(\phi(A) \cap u(k^{\cdot}))$ of A; here k_u is the residue field of u. We will denote by R_u and R_v the valuation rings of u and v respectively. Let $\theta_u:R_u \to k_u$ be the natural homomorphism and M_u and M_v the maximal ideals of R_u and R_v respectively.

Choose a family $\{y_{\gamma}\}_{\gamma \in H}$ of elements of k as follows: Fix an ordering $\leqq$ of H as an ordered group. If $\gamma = 0$ in H, choose $y_{\gamma} = 1$; if $\gamma > 0$ in H, then $\phi(\gamma) \in u(k^{\cdot})$ so that we may choose $y_{\gamma} \in k^{\cdot}$ with $u(y_{\gamma}) = \phi(\gamma)$; if $\gamma < 0$ in H, choose $y_{\gamma} = y_{(-\gamma)}^{-1}$.

PROPOSITION 4.1. *The map $\tau:H \times H \to k_u^{\cdot}$ defined by $\tau((\alpha, \beta)) = \theta_u(y_{\alpha+\beta}y_{\alpha}^{-1}y_{\beta}^{-1})$ gives a k_u-twisting of H.*

PROOF. Note that τ goes into $k_u^{\cdot}$ as

$$u(y_{\alpha+\beta}y_{\alpha}^{-1}y_{\beta}^{-1}) = \phi(\alpha + \beta) + \phi(-\alpha) + \phi(-\beta) = 0$$

so that $y_{\alpha+\beta}y_\alpha^{-1}\ y_\beta^{-1}$ is a unit of the valuation ring of u. It remains to verify Properties (i) and (ii) of the twisting. Property (i) is clearly valid. As for (ii), if α, β and $\gamma \in H$, then write $y_{\alpha+\beta} = \epsilon y_\alpha y_\beta$ and $y_{(\alpha+\beta)+\gamma} = \eta y_{\alpha+\beta}y_\gamma$ while $y_{\alpha+(\beta+\gamma)} = \mu y_\alpha y_{\beta+\gamma}$ and $y_{\beta+\gamma} = \lambda y_\beta y_\gamma$ where ϵ, η, μ and λ are units of the valuation ring of u. From this we get $\eta\epsilon = \mu\lambda$ so that

$$\tau((\alpha, \beta))\tau((\alpha + \beta, \gamma)) = \tau((\beta, \gamma))\tau((\alpha, \beta + \gamma)),$$

as wanted. □

PROPOSITION 4.2. *The residue class field k_v of the Anderson-Ohm valuation is the quotient field of the twisted group ring $k_u^\tau[H]$, where the twisting τ is given as in Proposition* 4.1.

Proposition 4.2 will be immediate from Propositions 4.4 and 4.5.

LEMMA 4.3. *Let f, $g \in k[\Gamma]^\cdot$ and assume that f/g is a unit of R_v. Then there exist integers s, $t \geqq 1$ and elements a, b, f_1, $g_1 \in k[\Gamma]$ with the following properties*:

(i) $a = \sum_{i=1}^s a_{\alpha_i}x^{\alpha_i}$, $b = \sum_{j=1}^t b_{\beta_j}x^{\beta_j}$ *with* $u(a_{\alpha_i}) + \phi(\alpha_i) = 0 = u(b_{\beta_j}) + \phi(\beta_j)$ *for all* $i = 1, 2, \ldots, s$ *and* $j = 1, 2, \ldots, t$

(ii) $v(f_1) > 0$ *and* $v(g_1) > 0$ *and*

(iii) $f/g = (a + f_1)/(b + g_1)$.

PROOF. Clearly f and g are non-zero elements and we may write $f = \sum_{i=1}^s a_{\alpha_i}x^{\alpha_i} + f_1$ and $g = \sum_{j=1}^t b_{\beta_j}x^{\beta_j} + g_1$, for some s, $t \geqq 1$ with $u(a_{\alpha_i}) + \phi(\alpha_i) = u(b_{\beta_j}) + \phi(\beta_j)$ for all i, j ($1 \leqq i \leqq s$, $1 \leqq j \leqq t$) and $v(f_1) > v(f)$ and $v(g_1) > v(g)$. If we multiply the numerator and denominator by $b_{\beta_1}^{-1}x^{-\beta_1}$ we get the desired representation, after an obvious change of notation.

PROPOSITION 4.4. *Define a mapping $\psi: R_v \to k_u^\tau(H)$ as follows: If $f/g \in M_v$ then $\psi(f/g) = 0$. If f/g is a unit of R_v, choose a representation $f/g = (a + f_1)/(b + g_1)$ as in Lemma* 4.3 *and define*

$$\psi(f/g) = \left\{\sum_{i=1}^s \theta_u(a_{\alpha_i}y_{\alpha_i})x^{\alpha_i}\right\}\Big/\left\{\sum_{j=1}^t \theta_u(b_{\beta_j}y_{\beta_j})x^{\beta_j}\right\}.$$

Then the mapping ψ is well-defined and ψ maps R_v onto $k_u^\tau(H)$.

PROOF. Notice in the first place that $u(a_{\alpha_i}y_{\alpha_i}) = u(a_{\alpha_i}) + \phi(\alpha_i) = 0$ so that $\theta_u(a_{\alpha_i}y_{\alpha_i}) \in k_u$; similarily $\theta_u(b_{\beta_j}y_{\beta_j}) \in k_u$ so that ψ goes into $k_u^\tau(H)$.

Let us write $\bar{a} = \sum_{i=1}^s \theta_u(a_{\alpha_i}y_{\alpha_i})x^{\alpha_i}$ with a similar expression for $\bar{b}$. We have shown above that $\bar{a}/\bar{b} \in k_u^\tau(H)$.

To show ψ is well defined, let $f/g = (c + f_2)/(d + g_2)$ be another representation of f/g as in Lemma 4.3, where

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$$c = \left(\sum_{e=1}^{p} c_{\gamma_e} x^{\gamma_e}\right), \text{ and } d = \left(\sum_{m=1}^{q} d_{\delta_m} x^{\delta_m}\right),$$

with $p, q \geqq 1$, $u(c_{\gamma_e}) + \phi(\gamma_e) = u(d_{\delta_m}) + \phi(\delta_m) = 0$ for all e and m involved and $v(f_2) > 0$ and $v(g_2) > 0$. We want to show $\bar{a}/\bar{b} = \bar{c}/\bar{d}$. We will show $\bar{a}\bar{d} = \bar{b}\bar{c}$ in $k_u^\tau[H]$.

Let $\lambda \in H$ be an exponent that appears with a non-zero coefficient $\bar{z}_\lambda$ in the product $\bar{a}\bar{d}$. Taking into account the twisting τ in question, this coefficient $\bar{z}_\lambda$ is given by

$$\begin{aligned} \bar{z}_\lambda &= \sum_{\alpha_i + \delta_m = \lambda} [\theta_u(a_{\alpha_i} y_{\alpha_i}) \theta_u(d_{\delta_m} y_{\delta_m}) \theta_u(y_\lambda y_{\alpha_i}^{-1} y_{\delta_m}^{-1})] \\ &= \sum_{\alpha_i + \delta_m = \lambda} [\theta_u(a_{\alpha_i} d_{\delta_m} y_\lambda)] \end{aligned} \tag{1}$$

We will show that $\bar{z}_\lambda$ also appears as a non-zero coefficient of $\bar{b}\bar{c}$. To this effect, we consider the equality of the two representations for f/g which gives

$$(a + f_1)(d + g_2) = (b + g_1)(c + f_2) \tag{2}$$

Now λ appears as an exponent with a non-zero coefficient in the expression for the term on the left side of (2), and so also in the expression for the term on the right side of (2). Comparing these coefficients, we get

$$\begin{aligned} &\left(\sum_{\alpha_i + \delta_m = \lambda} a_{\alpha_i} d_{\delta_m}\right) + \left(\text{a sum } \sum_h w_h \text{ of terms } w_h, \text{ where } u(w_h y_\lambda) > 0\right) \\ &= \left(\sum_{\beta_j + \gamma_e = \lambda} b_{\beta_j} c_{\gamma_e}\right) + \left(\text{a sum } \sum_{h'} w'_{h'} \text{ of terms } w'_{h'} \text{ where } u(w'_{h'} y_\lambda) > 0\right) \end{aligned} \tag{3}$$

Moreover $u(a_{\alpha_i} d_{\delta_m} y_\lambda) = 0 = u(b_{\beta_j} c_{\gamma_e} y_\lambda)$. If we multiply both sides of (3) by y_λ and apply the mapping θ_u, we get

$$\sum_{\alpha_i + \delta_m = \lambda} \theta_u(a_{\alpha_i} d_{\delta_m} y_\lambda) = \sum_{\beta_j + \gamma_e = \lambda} \theta_u(b_{\beta_j} c_{\gamma_e} y_\lambda) \tag{4}$$

The term on the right side of (4) is easily checked to appear as coefficient of x^λ in the expression for $\bar{b}\bar{c}$ in the twisted ring $k^\tau[H]$ and it is, of course, $\bar{z}_\lambda$ as we wanted to prove. Thus every non-zero coefficient of $\bar{a}\bar{d}$ appears in the expression for $\bar{b}\bar{c}$ and by symmetry we have $\bar{a}\bar{d} = \bar{b}\bar{c}$.

Finally, it is clear that ψ is onto. This completes the proof of Proposition 4.4.

PROPOSITION 4.5. *The mapping* ψ *defined in Proposition* 4.4 *is a homomorphism of rings.*

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PROOF. This is routine and is facilitated by working with units of R_v: If $a = \sum a_{\alpha_i} x^{\alpha_i}$ and $b = \sum b_{\beta_j} x^{\beta_j}$ with $u(a_{\alpha_i}) + \phi(\alpha_i) = 0 = u(b_{\beta_j}) + \phi(\beta_j)$ then $\psi(a + b) = \psi(a) + \psi(b)$ and $\psi(ab) = \psi(a)\psi(b)$, the latter product being in the twisted group ring.

We now can solve a more general problem than the one posed in [1]:

COROLLARY 4.6. *Let A, k, u and H be as in the introduction of this section. If v is the Anderson-Ohm valuation of $k(A)$ extending the valuation u of k, then the residue class field k_v is a purely transcendental extension of the residue class field k_u if and only if the subgroup H is free abelian.* □

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