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THE INTERPOLATION PROOF OF GROTHENDIECK'S INEQUALITY

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Introduction

This note is an exposition of the simple and elegant approach to Grothendieck's inequality given in [**2**] and [**4**], with one further simplification. The process of factorizing through L_2 ([**2**], p. 21) introduces a factor of $\sqrt{\pi/2}$ into the final constant. We show that this step can be avoided.

The ingredients are Khinchin's inequality, an interpolation result for p-summing norms and a reformulation of Grothendieck's inequality. No measure theory is used at any stage. We obtain Grothendieck's inequality with constant $2\sqrt{3}$. We present the details for the real case, but the method applies with minor changes to the complex case too (giving the same constant). Some short proofs of known results are included for completeness.

Preliminaries

If X, Y are normed linear spaces, we denote by U_X the unit ball in X and by $L(X,Y)$ the space of continuous linear operators from X to Y.

We denote by l_p^n the space $\mathbb{R}^n$ with l_p-norm. The ith unit vector is denoted by e_i, and the usual inner product on $\mathbb{R}^n$ by $\langle,\rangle$.

Given a finite sequence $(x_1,\ldots,x_k)$ of elements of a normed linear space, define $\mu_p(x_1,\ldots,x_k)$ (for $p \geqq 1$) by

$$[\mu_p(x_1,\ldots,x_k)]^p = \sup\left\{\sum_i |f(x_i)|^p : f \in U_{X^*}\right\}.$$

It is elementary that if K is a norming subset of U_{X^*}, then this is equal to

$$\sup\left\{\sum_i |f(x_i)|^p : f \in K\right\}.$$

In particular, if X is l_∞^n or $C(S)$, then (taking K to be the set of point-evaluations) we have

$$[\mu_p(x_1,\dots,x_k)]^p=\left\|\sum_i |x_i|^p\right\|,$$

where pth powers are defined in the obvious pointwise sense.

The p-summing norm of an operator T is defined by

$$\pi_p(T)=\sup\left\{\left(\sum_i \|Tx_i\|^p\right)^{1/p}:\mu_p(x_1,\dots,x_k)\leqq 1\right\},$$

where finite sequences of any length k are considered.

One of the equivalent forms of Grothendieck's inequality is; there is a constant K_G such that for any m, n and any T in $L(l_1^m, l_2^n)$, we have $\pi_1(T)\leqq K_G\|T\|$. The best possible value of K_G (which is still unknown) is called "Grothendieck's constant".

Let D_k denote the set of all mappings ε from $\{1,\dots,k\}$ to $\{-1,1\}$. Khinchin's inequality states that for $a=(a_1,\dots,a_k)\in\mathbb{R}^k$,

$$2^{-k}\sum_{\varepsilon\in D_k}|\langle\varepsilon,a\rangle|\geqq\frac{1}{\sqrt{2}}\left(\sum_i a_i^2\right)^{1/2}.$$

This has long been known, and is easily proved, with $1/\sqrt{3}$ instead of $1/\sqrt{2}$. For a proof that the best constant is $1/\sqrt{2}$, see [1].

The quantity $\tilde{\pi}_2(T)$

Let Y be $\mathbb{R}^n$ with any norm under which it is a Banach lattice (in particular, any l_p^n). For $y_1,\dots,y_k$ in Y, the element $(y_1^2+\dots+y_k^2)^{1/2}$ is well-defined. For T in $L(l_\infty^m, Y)$, define

$$\tilde{\pi}_2(T)=\sup\left\{\left\|\left[\sum_i (Ta_i)^2\right]^{1/2}\right\|:\mu_2(a_1,\dots,a_k)\leqq 1\right\}.$$

One can verify that $\tilde{\pi}_2$ is a norm, though this is not important for our purposes. Its relevance here is that it provides an equivalent statement of Grothendieck's inequality, as follows.

Proposition 1. $K_G=\sup\{\tilde{\pi}_2(T): T\in L(l_\infty^m, l_1^p),\ \|T\|\leqq 1,\ m,p\in\mathbb{N}\}$.

Proof. Any mapping S in $L(l_1^m, l_2^n)$ can be written as

$$Sx=\sum_{i=1}^n \langle a_i, x\rangle e_i$$

where $a_i\in l_\infty^m$. Then $\|Sx\|^2=\sum_i\langle a_i,x\rangle^2$, so $\|S\|=\mu_2(a_1,\dots,a_n)$. We consider S with $\|S\|=1$.

Any mapping T in $L(l_\infty^m, l_1^p)$ can be written as

$$Ty = \sum_{j=1}^{p} \langle b_j, y\rangle e_j$$

where $b_j \in l_1^m$. Then $\|T\| = \mu_1(b_1, \ldots, b_p)$. We consider T with $\|T\| = 1$.

With S, T as above, we have $(Ta_i)(j) = \langle a_i, b_j\rangle$, so if

$$c = \left[\sum_i (Ta_i)^2\right]^{1/2},$$

then

$$c(j)^2 = \sum_i \langle a_i, b_j\rangle^2 = \|Sb_j\|^2,$$

so $\|c\| = \sum_j \|Sb_j\|$ (where the norms are those of the appropriate spaces). Hence the statement that $\tilde{\pi}_2(T) \leqq K$ for all such T is equivalent to the statement that $\pi_1(S) \leqq K$ for all such S.

Let $a_1, \ldots, a_k \in \mathbb{R}^n$, and define c by: $c \geqq 0$ and $c^2 = \sum_i a_i^2$. In l_2^n we have clearly $\|c\|^2 = \sum \|a_i\|^2$, and hence for T in $L(l_\infty^m, l_2^n)$ we have $\tilde{\pi}_2(T) = \pi_2(T)$. The space l_1^n has the following well-known property ("2-concavity").

Lemma 1. *In l_1^n, we have $\|c\|^2 \geqq \sum_i \|a_i\|^2$.*

Proof. Write $\|a_i\| = \lambda_i$. By Schwarz's inequality,

$$0 \leqq \sum_i \lambda_i |a_i| \leqq \left(\sum_i \lambda_i^2\right)^{1/2} c$$

in the natural ordering of l_1^n. Since the norm of l_1^n is additive for positive elements, it follows that

$$\sum_i \lambda_i^2 \leqq \left(\sum_i \lambda_i^2\right)^{1/2} \|c\|,$$

which gives the result.

Hence for T in $L(l_\infty^m, l_1^n)$, we have $\tilde{\pi}_2(T) \geqq \pi_2(T)$.

The following lemma (cf. [4], Lemma 1.1) is well-known as part of the proof of the weaker version of Khinchin's inequality.

Lemma 2. *Given $a_1, \ldots, a_k \in \mathbb{R}^n$ and $\varepsilon \in D_k$, write $b_\varepsilon = \sum_i \varepsilon_i a_i$. Then*

$$2^{-k} \sum_{\varepsilon \in D_k} b_\varepsilon^4 \leqq 3\left(\sum_i a_i^2\right)^2.$$

Proof. b_ε^4 is the sum of terms of the form $\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l a_i a_j a_k a_l$. When we sum over $\varepsilon \in D_k$, the only terms that do not cancel are

$$\sum_i a_i^4 + 6\sum_{i<j} a_i^2 a_j^2$$

occurring for each ε. The statement follows.

Recall that for non-negative numbers $c_1, \ldots, c_N$,

$$\frac{1}{N}(c_1+\ldots+c_N) \leqq \left[\frac{1}{N}(c_1^4+\ldots+c_N^4)\right]^{1/4}.$$

Proposition 2. *Let Y be $\mathbb{R}^n$ with any norm under which it is a Banach lattice. For T in $L(l_\infty^m, Y)$, we have*

$$\tilde{\pi}_2(T) \leqq 3^{1/4}2^{1/2}\pi_4(T).$$

Proof. Take elements $a_1, \ldots, a_k$ of l_∞^m and define b_ε as in Lemma 2. By Khinchin's inequality,

$$\left(\sum_i (Ta_i)^2\right)^{1/2} \leqq \sqrt{2}\,2^{-k}\sum_{\varepsilon \in D_k} |Tb_\varepsilon|.$$

Since the norm in Y is such that $0 \leqq u \leqq v$ implies $\|u\| \leqq \|v\|$, we have

$$\left\|\left(\sum_i (Ta_i)^2\right)^{1/2}\right\| \leqq \sqrt{2}\,2^{-k}\sum_\varepsilon \|Tb_\varepsilon\|$$

$$\leqq \sqrt{2}\left[2^{-k}\sum_\varepsilon \|Tb_\varepsilon\|^4\right]^{1/4} \qquad \text{by the remark above}$$

$$\leqq \sqrt{2}\,\pi_4(T)\left\|\left(2^{-k}\sum_\varepsilon b_\varepsilon^4\right)^{1/4}\right\|$$

by the definition of π_4

$$\leqq \sqrt{2}\,\pi_4(T)3^{1/4}\left\|\left(\sum_i a_i^2\right)^{1/2}\right\| \qquad \text{by Lemma 2.}$$

Since $\|(\sum_i a_i^2)^{1/2}\| = \mu_2(a_1, \ldots, a_k)$, this completes the proof.

Interpolation

The required interpolation result can be related to the Riesz–Thorin theorem, but it is simpler to prove it directly from Pietsch's theorem, as follows. For T in $L(l_\infty^n, Y)$ (any

Y), Pietsch's theorem ([3], p. 64–5) asserts the equivalence of the following statements:

(i) $\pi_p(T) \leqq A$,

(ii) there is a positive linear functional ϕ on l_∞^n such that $\|\phi\| = A^p$ and $\|Tx\|^p \leqq \phi(|x|^p)$ for all x in l_∞^n.

Statement (ii) can clearly be reformulated as follows:

(ii′) there exist non-negative numbers $\lambda_1, \ldots, \lambda_n$ such that $\sum_j \lambda_j = A^p$ and

$$\left\|\sum_j x_j a_j\right\|^p \leqq \sum_j \lambda_j |x_j|^p$$

for all $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, where $Te_j = a_j$.

Proposition 3. *Let $1 < p < r$. Then for any T in $L(l_\infty^n, Y)$,*

$$\pi_r(T)^r \leqq \pi_p(T)^p \|T\|^{r-p}.$$

In particular,

$$\pi_{2p}(T)^2 \leqq \pi_p(T)\|T\|.$$

Proof. Write $\theta = p/r$. There exist λ_j as in (ii′), with $A = \pi_p(T)$. Take x in l_∞^n and f in U_{Y^*}. Write $|x_j| = y_j$ and $|f(a_j)| = u_j$. Clearly, $\sum_j u_j \leqq \|T\|$. We have

$$f\left(\sum_j x_j a_j\right) \leqq \sum_j y_j u_j$$

$$= \sum_j (y_j u_j^\theta) u_j^{1-\theta}$$

$$\leqq \left(\sum_j y_j^{1/\theta} u_j\right)^\theta \left(\sum_j u_j\right)^{1-\theta} \quad \text{by Hölder's inequality.}$$

Now $\sum_j y_j^{1/\theta} u_j = \sum_j \varepsilon_j y_j^{1/\theta} f(a_j) \leqq \|\sum_j \varepsilon_j y_j^{1/\theta} a_j\|$, where $\varepsilon_j \in \{-1, 1\}$. So by (ii′),

$$\left(\sum_j y_j^{1/\theta} u_j\right)^p \leqq \sum_j \lambda_j y_j^r,$$

and hence

$$\left\|\sum_j x_j a_j\right\|^r \leqq \sum_j \lambda_j y_j^r \|T\|^{r-p}.$$

It follows, by the easy implication in Pietsch's theorem that

$$\pi_r(T)^r \leqq \sum_j \lambda_j \|T\|^{r-p} = \pi_p(T)^p \|T\|^{r-p}.$$

Note. More generally, one can show that

$$\pi_r(T) \leqq \pi_p(T)^{\theta} \pi_q(T)^{1-\theta},$$

where $1/r = \theta/p + (1-\theta)/q$. Simple examples show that these results do not hold for operators whose domain is not an l_∞-space.

Deduction of Grothendieck's inequality

Take T in $L(l_\infty^m, l_1^p)$. We have

$$\begin{aligned} \tilde{\pi}_2(T)^2 &\leqq 2\sqrt{3}\,\pi_4(T)^2 && \text{by Proposition 2} \\ &\leqq 2\sqrt{3}\,\pi_2(T)\|T\| && \text{by Proposition 3} \\ &\leqq 2\sqrt{3}\,\tilde{\pi}_2(T)\|T\| && \text{by Lemma 1.} \end{aligned}$$

Hence $\tilde{\pi}_2(T) \leqq 2\sqrt{3}\|T\|$. By Proposition 1, this is equivalent to Grothendieck's inequality (with $K_G \leqq 2\sqrt{3}$).

Remarks

(1) Clearly, there is a constant $K \leqq K_G$ such that $\pi_2(T) \leqq K\|T\|$ for all T in $L(l_\infty^m, l_1^p)$. (The main part of [4] is concerned with generalizations of this result, rather than Grothendieck's inequality itself.) The exact value of K remains unknown, and is of interest as much as K_G. It is well known that $K_G \leqq K\sqrt{\pi/2}$ in the real case, and $K_G \leqq 2K/\sqrt{\pi}$ in the complex case. It is attractive to conjecture that $K = \sqrt{2}$ in the real case, as it is elementary that this is the value for l_1^2. It is also the value for l_1^3, since $\pi_2(T)^2 \leqq \pi_1(T)\|T\|$ and $\pi_1(l_1^3) = 2$ (however, the estimate given by this reasoning grows with p).

(2) It is easy to show (as in Proposition 2, but without Khinchin's inequality) that $\tilde{\pi}_2(T) \leqq 3^{1/4}\tilde{\pi}_4(T)$. However, Proposition 3 becomes false if $\tilde{\pi}_p$, $\tilde{\pi}_r$ are substituted for π_p, π_r.

(3) Using the exact value of the corresponding Khinchin constant [1], one obtains a slightly better estimate using π_5 instead of π_4: $K_G \leqq 4/\pi^{1/6}$ ($=3.305..$). This is the best estimate afforded by this method using π_p for integral p.

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