BULL. AUSTRAL. MATH. SOC. 08B10, 03G25
VOL. 22 (1980), 411-430.

VARIETIES GENERATED BY FINITE BCK-ALGEBRAS

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Iséki's BCK-algebras form a quasivariety of groupoids and a finite BCK-algebra must satisfy the identity (E_n) : $xy^n = xy^{n+1}$, for a suitable positive integer n . The class of BCK-algebras which satisfy (E_n) is a variety which has strongly equationally definable principal congruences, congruence-3-distributivity, and congruence-3-permutability. Thus, a finite BCK-algebra generates a 3-based variety of BCK-algebras. The variety of bounded commutative BCK-algebras which satisfy (E_n) is generated by n finite algebras, each of which is semiprimal.

Introduction

BCK-algebras were introduced as an algebraic formulation of certain implicational fragments of the propositional calculus by Iséki in [11]. They form a quasivariety of algebras amongst whose subclasses can be found the earlier implicational models of Henkin [10], algebras of sets closed under set-subtraction, and dual relatively pseudocomplemented upper semi-lattices. Many of the articles in the Mathematics Seminar Notes of Kobe University, Volume 3 (1975) onwards, are devoted to these algebras; the papers [14] and [15] of Iséki and Tanaka give excellent introductions to their ideal theory and first-order theory, respectively, while Iséki's survey [12] contains many references. Recently, Traczyk [26] and

Received 19 June 1980.

https://doi.org/10.1017/S0004972700006730 Published online by Cambridge University Press

Romanowska and Traczyk [23] have done much to elucidate the nature of the so-called commutative BCK-algebras. Independent of these developments, Komori [18], [19], at Shizuoka, has considered the subdirectly irreducible algebras in a variety whose members are the groupoid-opposites and order-duals of the algebras in an important subvariety of commutative BCK-algebras. This work was done in connection with his investigations of the Lukasiewicz many-valued logics. In [3] and [4], the present author considered an interaction between BCK-algebras, Universal Algebra and Lattice Theory.

Section 1 is devoted to showing that the BCK-algebras which satisfy the identity (E_n) form a variety. Section 2 uses Malcev conditions to determine congruence-phenomena in this variety. Section 3 contains examples and is concerned with commutative BCK-algebras. We exploit the connection with Komori's work, and use the information on congruences to give an alternative proof to the main result of Romanowska and Traczyk [23], which determines the nature of finite bounded commutative BCK-algebras.

1. The variety $\underline{\underline{E}}_n$

Let $(A; 0)$ be a groupoid with a distinguished element 0 ; the multiplication of the groupoid is denoted by juxtaposition. On the underlying set, a derived binary relation is defined by

(1.1) $x \le y$ if and only if $xy = 0$.

Then, $(A; 0)$ is a *BCK-algebra* if it satisfies the following universally quantified sentences:

(1.2) $(xy)(xz) \le zy$;

(1.3) $x(xy) \le y$;

(1.4) $x \le x$;

(1.5) $0 \le x$;

(1.6) if $x \le y$ and $y \le x$, then $x = y$.

Thus, a BCK-algebra is an algebra of type $(2, 0)$ which satisfies the identities (1.2)-(1.5) and the quasi-identity (1.6). It is customary to regard the nullary operation as a fundamental operation even though it is

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given equationally by (1.4), that is, $xx = 0$.

From (1.2) and (1.5), it follows that

(1.7) $y \le z$ implies $xz \le xy$.

Using this and (1.2), we obtain

(1.8) $x \le y$ and $y \le z$ imply $x \le z$.

Thus, (1.4)-(1.6) and (1.8) say that $(A; \le, 0)$ is a partially ordered set with 0 as the smallest element. It is possible to interpret the above information in terms of Galois connections; Shmuely [25] is a good up-to-date reference. Recall a Galois connection between two partially ordered sets P and Q is a pair (t, g) of mappings $t : P \to Q$, $g : Q \to P$ such that

(i) t and g are antitone, and

(ii) for each $p \in P$ and $q \in Q$, $gt(p) \ge p$ and $tg(q) \ge q$.

Thus, let x be an arbitrary element in a BCK-algebra A , $t_x : A \to A$ be given by $t_x(y) = xy$ for each $y \in A$, and $\overline{A}$ denote the order-dual of the partially ordered set $(A; \le)$. Then, because of (1.3), the pair (t_x, t_x) is a Galois connection between $\overline{A}$ and itself. Thus we must have $t_x^3 = t_x$, that is, $x(x(xy)) = xy$ for all x and y in the BCK-algebra.

Other important consequences of the axioms are:

(1.9) $x0 = x$;

(1.10) $y \le z$ implies $yx \le zx$;

and the crucial identity

(1.11) $(xy)z = (xz)y$.

We also have

(1.12) $xy \le x$.

The details can be found in Iséki and Tanaka [15]. It is the anti-symmetry property of (1.6) which forces us to say that the class of BCK-algebras is merely a quasivariety, although it is unknown whether this class is equationally definable.

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For any integer $n \geq 1$, we define the polynomials xy^n inductively by: $xy^1 = xy$, $xy^{k+1} = (xy^k)y$ for $k \geq 1$. Their behaviour is summarized below.

LEMMA 1.1. *For any integers $m, n \geq 1$, the following are BCK-identities:*

(i) $0x^n = 0$;

(ii) $x0^n = x$;

(iii) $xx^n = 0$;

(iv) $(xy^n)y^m = xy^{n+m} = (xy^m)y^n$;

(v) $(xy^n)z^m = (xz^m)y^n$;

(vi) $(xy^n)(xz) \leq zy^n$;

(vii) $(xz^n)(yz^n) \leq xy$;

(viii) $xy^m \leq xy^n$, *when* $m \geq n$.

Proof. *(i)* follows from (1.5); *(ii)* follows from (1.9) and induction; *(iii)* is a consequence of (1.4), (1.5) and induction; both *(iv)* and *(v)* follow from (1.11).

(vi) When $n = 1$, *(vi)* is (1.2). Suppose *(vi)* holds for $n = k$. Then

$$(xy^{k+1})(xz) = ((xy^k)y)(xz) = ((xy^k)(xz))y \leq (zy^k)y = zy^{k+1} ,$$

by (1.11) and (1.2).

(vii) Because of (1.2) and (1.11), $(xz)(yz) \leq xy$, that is *(vii)* holds when $n = 1$. Suppose *(vii)* is an identity when $n = k$. Then, by *(iv)* above, we obtain

$$(xz^{k+1})(yz^{k+1}) = ((xz)z^k)((yz)z^k) \leq (xz)(yz) \leq xy .$$

(viii) Suppose $m > n$ and so $m = n + k$ for a suitable $k \geq 1$. Then

$$(xy^m)(xy^n) = (xy^{n+k})(xy^n) = ((xy^n)y^k)(xy^n) = ((xy^n)(xy^n))y^k = 0y^k = 0 ,$$

by *(iii)*, (1.11), (1.4) and *(i)*. Due to (1.1), $xy^m \le xy^n$.

For any integer $n \ge 1$, we introduce the identity

$$(\mathrm{E}_n) \qquad xy^n = xy^{n+1} .$$

We now give some identities which are equivalent to (E_n) .

PROPOSITION 1.2. *A BCK-algebra satisfies the identity* (E_n) *if and only if it satisfies any one of the following identities:*

(i) $(xy^n)y^n = xy^n$;

(ii) $(xy^n)y^m = xy^n$, *for any fixed* $m \ge 1$;

(iii) $x\big((xy^n)y^n\big) = x(xy^n)$;

(iv) $(xy)z^n = (xz^n)(yz^n)$.

Proof. *(ii)* is an immediate consequence of (E_n) and *(i)* is an instance of *(ii)*. Due to the *(viii)* of Lemma 1.1, $(xy^n)y^m \le xy^{n+1}$ and $xy^{n+1} \le xy^n$. Hence, *(ii)* implies (E_n) .

Of course, *(i)* implies *(iii)*. Conversely, assume that *(iii)* holds. Then *(iii)* yields $\big(x\big((xy^n)y^n\big)\big)y^n = \big(x(xy^n)\big)y^n$, and due to *(v)* of Lemma 1.1, we obtain $(xy^n)\big((xy^n)y^n\big) = (xy^n)(xy^n) = 0$. Due to (1.1), $xy^n \le (xy^n)y^n$. By Lemma 1.1 *(viii)*, the reverse inequality always holds. Hence we obtain *(i)*.

Of course, *(iv)* yields an instance of *(ii)*. The proof that *(iv)* follows from (E_n) is along the lines of the proof of Theorem 8 in [15]. We will include the details. Firstly, the inequality $(xy)z^n \le (xz^n)(yz^n)$ always holds. Indeed, due to *(v)* of Lemma 1.1, (1.2) and (1.12),

$$\big((xy)z^n\big)\big((xz^n)(yz^n)\big) = \big((xz^n)y\big)\big((xz^n)(yz^n)\big) \le (yz^n)y = 0 .$$

Secondly, using (1.2) and (1.11), we get identity (31) of [15], namely $\big((xy)u\big)(xz) \le (zy)u$. Now replace the role of x by xz^n , y by yz^n , z by $(xz^n)z^n$, and u by $(xy)z^n$ to obtain:

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$$\begin{aligned}[((xz^n)(yz^n))((xy)z^n)][(xz^n)((xz^n)z^n)] &\le [((xz^n)z^n)(yz^n)][(xy)z^n] \\ &\le [(xz^n)y][(xy)z^n] \quad \text{by (vii) of Lemma 1.1,} \\ &= [(xy)z^n][(xy)z^n] = 0 .\end{aligned}$$

Due to (E_n), or rather *(i)*, $xz^n = (xz^n)z^n$. Hence (1.4) and the above inequality gives

$$[((xz^n)(yz^n))((xy)z^n)]0 = 0 .$$

Due to (1.9), we have $((xz^n)(yz^n))((xy)z^n) = 0$, which is equivalent to the desired reverse inequality.

The next lemma can be regarded as a generalization of Proposition 5 in Iséki and Tanaka [15]; it is vital to both this section and the next.

LEMMA 1.3. *If a BCK-algebra satisfies the identity* (E_n), *then it also satisfies*

(C_n) $$(x(xy)^n)(yx)^n = (y(yx)^n)(xy)^n .$$

Proof. Due to (E_n), *(v)* and *(vi)* of Lemma 1.1, and (1.10),

$$\begin{aligned}(x(xy)^n)(yx)^n = (x(xy)^{n+1})(yx)^n &= ((x(xy)^n)(xy))(yx)^n \\ &\le (y(xy)^n)(yx)^n = (y(yx)^n)(xy)^n.\end{aligned}$$

By symmetry, we get the reverse inequality and so (1.6) ensures that (C_n) holds.

Let $\underline{\underline{E}}_n$ and $\underline{\underline{C}}_n$ denote the classes of all BCK-algebras which satisfy (E_n) and (C_n), respectively.

THEOREM 1.4. *The classes* $\underline{\underline{C}}_n$ *and* $\underline{\underline{E}}_n$ *are varieties. The following identities form a base for the variety* $\underline{\underline{C}}_n$:

(i) $((xy)(xz))(zy) = 0$,

(ii) $0x = 0$,

(iii) $x0 = x$,

(iv) (C_n) .

These identities together with (E_n) *form a base for* $\underline{\underline{E}}_n$.

Proof. We must show that (1.3), (1.4) and (1.6) follow from *(i)-(iv)*, above. Putting $y = z = 0$ in *(i)*, yields (1.4) *via (ii)* and *(iii)*. Replacing y by 0 in *(i)*, yields (1.3). Finally, suppose $x \leq y$ and $y \leq x$, that is, $xy = 0 = yx$. Substituting in (C_n) , we obtain $(x0^n)0^n = (y0^n)0^n$. Induction and *(iii)* enables us to deduce that $x = y$.

The technique of the above proof is related to that of Yutani [27] in the proof of his Theorem 1.

We will defer giving examples until Section 3. The next section is devoted to congruence-properties of the varieties $\underline{\underline{E}}_n$ and $\underline{\underline{C}}_n$.

2. Congruences

Let A be a finitary algebra, $\mathrm{Con}(A)$ its lattice of congruences and $n \geq 2$ be an integer. Then A is *n-permutable* if for any $\Theta, \Phi \in \mathrm{Con}(A)$, the n-fold alternating relational products $\Theta\Phi \ldots$ and $\Phi\Theta \ldots$ are equal. This concept is a generalization of permutability (equals 2-permutability). A variety is called n-permutable if each of its members is n-permutable. In [9], Hagemann and Mitschke characterized n-permutable varieties in terms of the existence of $n - 1$ ternary polynomials satisfying certain identities. In particular, a variety is 3-permutable if and only if there are two ternary polynomials $r(x, y, z)$ and $s(x, y, z)$ such that each algebra in the variety satisfies the identities $r(x, z, z) = x$, $s(x, x, z) = z$ and $r(x, x, z) = s(x, z, z)$. While weaker than permutability, 3-permutability still implies modularity of the congruence lattice and a number of other properties. The author has already considered 3-permutability in relation to BCK-algebras and universal algebras in [4] and we refer to that paper for details and additional references.

A variety is *congruence-distributive* if the lattice of congruences of each of its algebras is distributive. In [16, Theorem 2.1], Jónsson showed that a variety is congruence-distributive if and only if it is *congruence-n-distributive*, or more briefly *n-distributive*, in the sense that there exists an integer $n \geq 2$ and $n - 1$ ternary polynomials satisfying certain identities. For example, a variety is 2-distributive if and only if there is a polynomial $m(x, y, z)$ such that

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x$$

on each member of the variety. More importantly for us, a variety is 3-distributive if and only if there exist polynomials $t_1(x, y, z)$, $t_2(x, y, z)$ such that each algebra in the variety satisfies the identities:

$$t_1(x, y, x) = x = t_2(x, y, x) \ ; \quad t_1(x, x, z) = x \ ; \quad t_2(x, x, z) = z \ ;$$
$$t_1(x, z, z) = t_2(x, z, z) \ .$$

In [4, Theorem 2.6], the author showed that an n-permutable variety is congruence-distributive if and only if it is n-distributive. Important for our aim is Theorem 1 of Padmanabhan and Quackenbush [21], which states that a finitely based n-distributive variety is n-based. Combining the above results and notation with Theorem 1.4, we can now give the following result whose proof amounts to checking that the given polynomials satisfy the identities that ensure 3-permutability and 3-distributivity. It should be noted that it is the identity (C_n) which ensures the non-trivial identities $r(x, x, z) = s(x, z, z)$ and $t_1(x, z, z) = t_2(x, z, z)$.

THEOREM 2.1. *The variety* $\underline{\underline{C}}_n$, *and so each of its subvarieties and, in particular, the variety* $\underline{\underline{E}}_n$, *is* 3-*permutable,* 3-*distributive and* 3-*based.*

The polynomials which ensure 3-*permutability are*

$$r(x, y, z) = \big(x(yz)^n\big)(xy)^n \quad \textit{and} \quad s(x, y, z) = r(z, y, x) = \big(z(yx)^n\big)(xy)^n \ .$$

The polynomials which ensure 3-*distributivity are*

$$t_1(x, y, z) = \big(x\big((xy)(zy)\big)^n\big)\big((yx)(yz)\big)^n$$

and

$$t_2(x, y, z) = \big(z\big((yx)(yz)\big)^n\big)\big((xy)(zy)\big)^n \ .$$

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We now turn to finite BCK-algebras. Because of *(viii)* of Lemma 1.1, we must have, for any ordered pair (a, b) of elements in a finite BCK-algebra A, an integer $n(a, b) \geq 1$ such that $ab^{n(a,b)} = ab^{n(a,b)+1}$. Put $n = \max\{n(a, b) : (a, b) \in A \times A\}$. Then A satisfies the identity (E_n). This has also been observed by Iséki [*13*]. It follows that a finite BCK-algebra generates a variety (and not just a quasivariety) of BCK-algebras, which is a subvariety of a suitable variety $\underline{\underline{C}}_n$, or $\underline{\underline{E}}_n$. Due to Theorem 2.1, this variety is congruence-distributive and even n-distributive. Then Baker's Theorem ensures that the variety is finitely based; for a proof of Baker's Theorem, and references to other proofs, we refer to Burris [*1*]. We thus arrive at

THEOREM 2.2. *Any finite BCK-algebra generates a variety of BCK-algebras, which is* 3-*permutable,* 3-*distributive and* 3-*based.*

In connection with Theorems 2.1 and 2.2, we should mention that *no non-trivial variety of BCK-algebras is either permutable or* 2-*distributive*. The reason for this is as follows. Firstly, any non-trivial BCK-algebra must contain the 2-element BCK-algebra $\{0, a : 0a = aa = 00 = 0, a0 = a\}$. The variety generated by this 2-element algebra is the variety of so-called *implicative* BCK-algebras; it can be regarded as the subvariety of $\underline{\underline{C}}_n$ (or $\underline{\underline{E}}_n$) of all algebras which satisfy the additional identity $x(yx) = x$. Its members are simply subalgebras of Boolean algebras $(B; \wedge, \vee, ', 0, 1)$ with respect to the derived operation $ab = a \wedge b'$; for a proof and a history see [*2*]. And, in effect, Mitschke [*20*] showed that this variety is neither permutable nor 2-distributive; see also [*8*, Theorems 3.14, 3.15].

We now turn to another congruence-property of the variety $\underline{\underline{E}}_n$. The following results generalize some of those in [*4*]; their importance rests in their wide range of applicability.

An ideal of a BCK-algebra A is a subset K of A such that

(i) $0 \in K$ and

(ii) $a \in K$ whenever $ab, b \in K$.

The ideals of A form a complete lattice $J(A)$. Because of Iséki and

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Tanaka [14, Theorem 2] the ideal $\langle a_1, \ldots, a_t \rangle$ of A generated by $a_1, \ldots, a_t$ is the set of all $d \in A$ such that

$$\left(\left(\ldots\left(\left(db_1\right)b_2\right)\ldots\right)b_{k-1}\right)b_k = 0$$

for suitable $b_1, b_2, \ldots, b_k \in \{a_1, \ldots, a_t\}$. When A is within $\underline{\underline{E}}_n$, we can give a much better description of this ideal.

LEMMA 2.3. *Let* $A \in \underline{\underline{E}}_n$, $K \in J(A)$ *and* $a, a_1, \ldots, a_t \in A$. *Then the supremum* $K \vee \langle a \rangle$ *in* $J(A)$ *is* $\{b \in A : ba^n \in K\}$. *Consequently*

$$\langle a_1, \ldots, a_t \rangle = \left\{b \in A : \left(\ldots\left(\left(ba_1^n\right)a_2^n\right)\ldots\right)a_t^n = 0\right\}.$$

Proof. Because of *(iv)* in Proposition 1.2, it is easy to check that $\{b \in A : ba^n \in K\}$ is an ideal. Of course, this ideal is within any ideal which contains both a and K, and so it is the supremum in the ideal-lattice. The second assertion follows from the first *via* induction.

Any ideal $K \in J(A)$ gives rise to a congruence $\Theta(K)$ on A, defined by $a \equiv b(\Theta(K))$ if and only if $ab, ba \in K$. Moreover, the quotient algebra is a BCK-algebra; see [14, Theorem 2]. On the other hand, when $\Phi \in \mathrm{Con}(A)$, $\ker(\Phi) = \{a \in A : a \equiv 0(\Phi)\}$ is an ideal, but the quotient algebra may not be a BCK-algebra. When the quotient algebra is a BCK-algebra, the validity of (1.6) in the quotient ensures that $a \equiv b(\Phi)$ $(a, b \in A)$ *if* and only if $ab, ba \in \ker(\Phi)$. Of course, this hypothesis is ensured when A is within a variety of BCK-algebras. Hence, if a BCK-algebra A is within a variety of BCK-algebras, the maps $K \to \Theta(K)$ and $\Phi \to \ker(\Phi)$ are mutually inverse lattice-isomorphisms between the ideal-lattice $J(A)$ and the congruence-lattice $\mathrm{Con}(A)$. It is Theorem 1.4 which makes this applicable to algebras satisfying (E_n).

THEOREM 2.4. *Let* $A \in \underline{\underline{E}}_n$, $a, b, c, d \in A$, *and* $\Theta(a, b)$ *denote the smallest congruence identifying* a *and* b. *Then* $c \equiv d(\Theta(a, b))$ *if and only if*

$$\left((cd)(ab)^n\right)(ba)^n = 0 = \left((dc)(ab)^n\right)(ba)^n .$$

Proof. Because of our preceding remarks, $c \equiv d(\Theta(a, b))$ if and only if $cd, dc \in \langle ab, ba \rangle$. Hence Lemma 2.3 yields the result.

Following Köhler and Pigozzi [17], a variety $\underline{\underline{V}}$ has *strongly equationally definable principal congruences* if there exists a set $\{(p_i, q_i) : i \in I\}$ of pairs of quaternary polynomials such that, for all $A \in \underline{\underline{V}}$ and all $a, b, c, d \in A$, $c \equiv d(\Theta(a, b))$ if and only if $p_i(a, b, c, d) = q_i(a, b, c, d)$ for each $i \in I$. Thus Theorem 2.4 says that the variety $\underline{\underline{E}}_n$ has strongly equalionally definable principal congruences. As these authors mention, strongly equaltionally definable principal congruences implies the *congruence extension property* due to a well known result of Day [5]. A class $\underline{\underline{H}}$ of algebras has congruence extension property if each congruence on a subalgebra of an algebra $A \in \underline{\underline{H}}$ is the restriction of a congruence on A; see Fried [8] for some recent results on congruence extension properties.

The main result of Köhler and Pigozzi [17] states that a variety has strongly equaltionally definable principal congruences if and only if the compact congruences on each algebra in the variety form a (dual) relatively pseudocomplemented upper semilattice, and from this the congruence-distributivity of the variety can be inferred. In connection with this, recall that an upper semilattice $(S; \vee)$ is (dual) relatively pseudo-complemented if, for each $a, b \in S$, the subset $\{c \in S : a \leq b \vee c\}$ has a (necessarily unique) smallest element, which is denoted by ab. Here there is an important link with BCK-algebras. For if $(S; \vee)$ is such a semilattice and $0 = aa$ for any $a \in S$, then $(S; 0)$, with respect to the above product ab, is an E_1-BCK-algebra - a detailed analysis can be found in the author's paper [4].

Thus, there are entirely different reasons for the congruence-distributivity of $\underline{\underline{E}}_n$. We will not state the obvious consequence for $\underline{\underline{E}}_n$ of Köhler and Pigozzi's Theorem. Instead, we give a related ideal-theoretic result which extends part of Theorem 1.3 in [4]; it is, in fact, a direct consequence of Lemma 2.3, above.

THEOREM 2.5. *Let* $A \in \underline{\underline{E}}_n$ *and* $H = \langle a_1, \ldots, a_t \rangle$, $K = \langle b_1, \ldots, b_r \rangle$ *be two finitely generated ideals of* A. *For* $i = 1, \ldots, t$, *let* $d_i = \left(\ldots\left(a_i b_1^n\right)\ldots\right) b_r^n$. *Then the (dual) relative pseudocomplement,* HK *of* H *and* K *in the upper semilattice of finitely*

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generated ideals is the ideal $\langle d_1, \ldots, d_t \rangle$.

3. Commutative BCK-algebras

A BCK-algebra $(A; 0)$ is called *bounded* if the underlying partially ordered set $(A; \leq)$ has a largest element, which is denoted by 1 . In other words, there is an element $1 \in A$ such that

(B) $$x1 = 0 ,$$

for all $x \in A$. When dealing with bounded BCK-algebras, we shall consider then as algebras $(A; 0, 1)$ of type $(2, 0, 0)$; that is, 1 becomes a nullary operation and (B) becomes an identity satisfied by the bounded algebra.

A *commutative* BCK-algebra, or *Tanaka algebra*, is a BCK-algebra which satisfies the identity

(T) $$x(xy) = y(yx) .$$

When the derived operation $x \wedge y = x(xy)$ is introduced, a commutative BCK-algebra $(A; 0)$ has, as a reduct, the lower semilattice $(A; \wedge)$ and the partial order of (1.1) is consistent with the semilattice-order; that is, for any $a, b \in A$, $a \leq b$ when and only when $a = a \wedge b$. When $(A; 0, 1)$ is a bounded commutative BCK-algebra, the algebra $(A; \wedge, \vee, \sim, 0, 1)$ is a bounded lattice with an involution, wherein the supremum is $x \vee y = \sim(\sim x \wedge \sim y)$ and the involution is $\sim x = 1x$; this is a fundamental result of Iséki and Tanaka [*15*, Theorem 6]. Actually this lattice is distributive and $x \wedge \sim x \leq y \vee \sim y$ is an identity; see Traczyk [*26*] and [*3*, Theorems 3.9, 3.11].

The class $\underline{\underline{T}}$ of all commutative BCK-algebras is a variety; identities *(i)*, *(ii)* and *(iii)* of Theorem 1.4, together with (T), provide an equational base. In [*3*] it was shown that this variety is 3-permutable and 3-distributive. On the other hand, the variety $\underline{\underline{T}}^1$ of bounded commutative BCK-algebras is permutable; $p(x, y, z) = x(yz) \vee z(yx)$ is a suitable (2/3)-minority polynomial; *cf.* [*3*, Lemma 1.6].

In the presence of commutativity, we can add to Proposition 1.2.

PROPOSITION 3.1. *A commutative BCK-algebra satisfies* (E_n) *if and only if it satisfies* (*i*) $x \wedge (yx^n) = 0$.

A bounded commutative BCK-algebra satisfies (E_n) *if and only if it satisfies any one of the following identities:*

(*ii*) $1x^n = 1x^{n+1}$,

(*iii*) $x \wedge (1x^n) = 0$,

(*iv*) $x(1x^n) = x$.

Proof. It is easy to see that in any commutative BCK-algebra, $x \wedge y = 0$ if and only if $x = xy$, or alternatively $y = yx$. Hence, (*i*) is equivalent to $yx^n = (yx^n)x$; that is, (E_n) . For the same reason, (*ii*), (*iii*), and (*iv*) are equivalent.

Of course, (*ii*) is a specialization of (E_n) , and so it remains to prove that (*ii*) implies (E_n) .

Because of (B) and (T), $x = 1(1x)$. Hence, (*ii*) and Lemma 1.1 (*v*) imply

$$xy^n = (1(1x))y^n = (1y^n)(1x) = (1y^{n+1})(1x) = (1(1x))y^{n+1} = xy^{n+1} .$$

COROLLARY 3.2. *A subdirectly commutative algebra in* $\underline{\underline{E}}_n$ *is simple.*

Proof. Suppose B is such an algebra and a is a non-zero element of B . Let b be any element of B . Then $a \wedge (ba^n) = 0$. As ideals are herditary, $\{0\} = \langle a \rangle \cap \langle ba^n \rangle$. Due to the correspondence between ideal and congruences and the fact that B is subdirectly irreducible, $\langle ba^n \rangle = \{0\}$. Hence $b \in \langle a \rangle$. Thus B has only two ideals and is, thus, simple.

All of the hypotheses of the above corollary are necessary. Indeed, let us firstly consider the variety $\underline{\underline{E}}_1$ of so-called *positive implicative* BCK-algebra; it is the class of implicational models of Henkin [*10*]. As Iséki and Tanaka observed in [*14*, Example 7, p. 356], any partially ordered set $(A; \leq, 0)$ with a smallest element 0 can be converted into a BCK-algebra by defining $ab = 0$ when $a \leq b$ and $ab = a$ when $a \nleq b$. The

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resulting algebra is positive implicative. Moreover, it is easy to see that the ideals of this algebra are precisely hereditary subsets of the original poset. Hence we get a subdirectly irreducible algebra which is not simple when the poset has at least three elements and a unique atom. On the other hand, subdirectly irreducible commutative BCK-algebras, which are not simple, are hard to come by. We now describe an example.

Let A be a chain $a_0 < a_1 < \ldots < a_n < \ldots$ of order type ω, $\bar{A}$ be its dual $\ldots \bar{a}_n < \ldots < \bar{a}_1 < \bar{a}_0$, and A_ω be the ordinal sum $A \oplus \bar{A}$. The BCK-multiplication is defined on A_ω by:

$$a_n a_m = a_{\max(n-m,0)} ,$$

$$a_n \bar{a}_m = 0 = a_0 ,$$

$$\bar{a}_n \bar{a}_m = a_{\max(m-n,0)} ,$$

$$\bar{a}_n a_m = \bar{a}_{n+m} .$$

The resulting algebra turns out to be in $\underline{\underline{T}}^1$ and as a $\underline{\underline{T}}^1$-algebra it is generated by a_1; $a_n = 1\left(1a_1^n\right)$, $\bar{a}_n = 1a_n$, where $1 = \bar{a}_0$. The algebra is subdirectly irreducible and not simple; its non-trivial smallest ideal is $A = \langle a_1 \rangle = \{a_n : n \in \omega\}$.

In this connection, let A_n be the $\underline{\underline{T}}$-subalgebra whose underlying poset is the chain $a_0 < \ldots < a_n$ of length $n \geq 1$. We also let A_n^1 denote the associated $\underline{\underline{T}}^1$-algebra. These algebras are important in the study of the varieties $\underline{\underline{E}}_n$. Indeed, using part *(ii)* of Proposition 3.1, it is easy to see that $A_m \in \underline{\underline{E}}_n$ if and only if $m \leq n$, for any $m, n \geq 1$. As $\underline{\underline{E}}_m \subseteq \underline{\underline{E}}_n$ whenever $m \leq n$, the varieties $\underline{\underline{E}}_n$, $\underline{\underline{E}}_n \cap \underline{\underline{T}}$, and $\underline{\underline{E}}_n \cap \underline{\underline{T}}^1$ each form an increasing infinite chain.

Before continuing, we will tidy up a connection between chains and subdirectly irreducible $\underline{\underline{T}}$-algebras. At the end of the paper [3], we showed a theory of prime ideals could be developed for commutative

BCK-algebras. The relevant part for us here is as follows:

An ideal P of a commutative BCK-algebra A is called *prime* if $P \neq A$ and either $a \in P$ or $b \in P$, whenever $a \wedge b \in P$. Then, when A is not trivial, $\bigcap\{P : P \text{ is a prime ideal}\} = \{0\}$. Hence, with the notation of Section 2, A becomes a subdirect product of the quotient algebras $A/\Theta(P)$. We now easily obtain:

THEOREM 3.3. *Let A be a commutative BCK-algebra which satisfies the identity*

$$(L) \qquad (xy) \wedge (yx) = 0 .$$

Then an ideal $P \neq A$ is prime if and only if its associated quotient is a chain.

Hence, a commutative BCK-algebra satisfies (L) *if and only if it is isomorphic to a subdirect product of totally ordered algebras.*

As a matter of fact there are simple $\underline{\underline{T}}$-algebras which are not chains. Let I be an index set with at least two elements and A be the tree $\{0, a, a_i : 0 < a < a_i, a_i \| a_j \text{ for any } i \neq j, i, j \in I\}$. Then Setō [24] showed that A can be converted into a $\underline{\underline{T}}$-algebra by defining the products $a_i a_j = a$ when $i \neq j$ and the others in the obvious manner. The resulting algebra is simple and in $\underline{\underline{E}}_2$. Consequently, the variety $\underline{\underline{E}}_2 \cap \underline{\underline{T}}$ is *not residually small*; that is, it does not possess a set of subdirectly irreducible algebras. For any $n \geq 2$, the algebras of Example 5 in Iséki and Tanaka [14] provide another class, as opposed to set, of simple algebras, which are trees but not chains, in the variety $\underline{\underline{E}}_{n+1} \cap \underline{\underline{T}}$.

In [18] and [19], Komori considered a variety of groupoids which turn out to be the groupoid-duals (opposites) and order-duals of commutative BCK-algebras satisfying (L). His Theorem 2.10 in [18] thus states that the subdirectly irreducible BCK-algebras satisfying (T) and (L) are chains; the method in our Theorem 3.3 is quite different. The effect of the dual of equation *(i)* in Proposition 3.1 is considered in [19]. In fact, Theorem 3.13 of [19] can be interpreted as the following non-trivial important result.

LEMMA 3.4 (Komori [19]). *A commutative totally ordered $\underline{\underline{E}}_n$-algebra* *is isomorphic to the algebra* A_m *for some* $m \leq n$.

Combining the results of our results, we obtain

THEOREM 3.5. *The subvariety of* $\underline{\underline{E}}_n$ *determined by the identities* (T) *and* (L) *is the variety of BCK-algebras generated by* A_n .

We now turn to bounded algebras. Traczyk [*26*] has already proved that the subdirectly algebras in $\underline{\underline{T}}^1$ are totally ordered. In fact in the proof of his Theorem 3.3, he shows that a $\underline{\underline{T}}^1$-algebra satisfies (L). The demonstration of this identity is by no means trivial; it is intimately related with his method of establishing the distributivity of the underlying lattice of a $\underline{\underline{T}}^1$-algebra. For the purposes of emphasis, we state the result formally as

LEMMA 3.6 (Traczyk [*26*]). *A bounded commutative BCK-algebra satisfies the identity* (L).

We are now in a position to give an alternative proof of the central result of Romanowska and Traczyk [*23*]. Their proof is quite computational. Our proof is more in line with Universal Algebra.

THEOREM 3.7 (Romanowska and Traczyk [*23*]). *A finite bounded commutative BCK-algebra is isomorphic to the direct product of simple totally ordered BCK-algebras. Consequently, its congruence-lattice is a Boolean lattice.*

Proof. Because of the reasoning which preceded Theorem 2.2, we can consider the finite algebra to be in the variety $\underline{\underline{E}}_n \cap \underline{\underline{T}}^1$ for some suitable $n \geq 1$. Due to Corollary 3.2, Theorem 3.3 and Lemma 3.6, the algebra is isomorphic to a subdirect product of finitely many simple chains. But as we remarked prior to Proposition 3.1, the variety $\underline{\underline{T}}^1$ is permutable. Hence, the algebra becomes isomorphic to the direct product of some of these simple algebras; this is a well known result of Universal Algebra; see for example Foster and Pixley [*6*, Theorem 2.4]. Finally, either of the varieties $\underline{\underline{E}}_n$ and $\underline{\underline{T}}$ is congruence-distributive and so the congruence-lattice of a direct product of finitely many algebras in $\underline{\underline{E}}_n \cap \underline{\underline{T}}^1$ is naturally isomorphic to the direct product of the congruence-

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distributive and so the congruence-lattice of a direct product of finitely many algebras in $\underline{\underline{E}}_n \cap \underline{\underline{T}}^1$ is naturally isomorphic to the direct product of the congruence-lattices of the factors; *cf.* Fraser and Horn [7]. We now have the second assertion of the theorem.

In close relation to Komori's Lemma 3.4, above, Traczyk [26] showed that the algebras A_n^1 are the only finite subdirectly irreducibles in $\underline{\underline{T}}^1$. We are going to conclude this paper with a closer look at these algebras. We assume that the reader is familiar with Primal Algebra Theory, in particular with the notions of quasiprimal and semiprimal algebras. A perspective can be obtained from Quackenbush's survey [22]. Let us recall that the *ternary discriminator* on a set A is a function $t : A^3 \to A$ such that $t(a, b, c) = a$ if $a \neq b$ and $t(a, b, c) = c$ if $a = b$.

THEOREM 3.8. *For each divisor r of n, A_n^1 possesses a unique $\underline{\underline{T}}^1$-subalgebra and this is isomorphic to $A_{n/r}^1$; these are the only $\underline{\underline{T}}^1$-subalgebras of A_n^1. Consequently, the variety $\underline{\underline{E}}_n \cap \underline{\underline{T}}^1$ is generated by A_n^1 and the algebras A_s^1, where $1 < s < n$ and s is a non-divisor of n.*

A_n^1 is a semiprimal algebra. Consider the following $\underline{\underline{T}}^1$-polynomials:

$$e_n(x) = x(\sim x)^{n-1} = x(1x)^{n-1}, \quad d(x, y) = (xy) \vee (yx),$$

and

$$t_n(x, y, z) = \left(x \wedge \sim e_n\left(\sim d(x, y)\right)\right) \vee \left(z \wedge e_n\left(\sim d(x, y)\right)\right).$$

On A_n^1, $e_n\left(a_i\right) = 0$ if $i < n$ and $e_n\left(a_i\right) = 1$ if $i = n$. Hence $t_n(x, y, z)$ represents the ternary discriminator on A_n^1.

Proof. When r divides n, $A_{n/r}^1$ is isomorphic to the subalgebra

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$\{a_{kr} : 0 \le k \le n/r\}$. On the other hand, let B be any subalgebra and a_r be its atom. Suppose a_s is another non-zero element of B . Then r must divide s . Otherwise, $s = qr + t$ for some $0 < t < r$, and so $0 < a_t < a_r$ and $a_t \in B$, as $a_t = a_s a_r^q$. The nature of the variety $\underline{\underline{E}}_n \cap \underline{\underline{T}}^1$ then follows from Theorem 3.3, Lemma 3.4 and Lemma 3.6.

For any $i = 0, \ldots, n$,

$$\sim a_i = a_{n-i} , \quad a_i(\sim a_i) = a_{\max(2i-n,0)} , \quad a_i(\sim a_i)^2 = a_{\max(3i-2n,0)} ,$$

and, by induction, it follows that $e_n(a_i) = a_{\max(ni-(n-1)n,0)}$. But $ni + n - n^2 \le 0$ if and only if $n(i+1) \le n^2$. Hence e_n behaves as stated on A_n^1 . On any bounded commutative BCK-algebra, $d(x, y) = 0$ if and only if $x = y$. It now follows that t_n acts as the ternary discriminator. Hence A_n^1 is quasiprimal and even semiprimal because the only automorphisms between its subalgebras are identity-maps. Of course, we could have deduced the quasiprimality of A_n^1 from the simplicity of its subalgebras and the congruence-distributivity and permutability of $\underline{\underline{T}}^1$.

References

[1] Stanley Burris, "On Baker's finite basis theorem for congruence distributive varieties", *Proc. Amer. Math. Soc.* 73 (1979), 141-148.

[2] William H. Cornish, "A multiplier approach to implicative BCK-algebras", *Math. Sem. Notes Kobe Univ.* 8 (1980), 157-169.

[3] William H. Cornish, "3-permutability and quasicommutative BCK-algebras", *Math. Japon.* (to appear).

[4] William H. Cornish, "On positive implicative BCK-algebras", *Math. Sem. Notes Kobe Univ.* (to appear).

[5] Alan Day, "A note on the congruence extension property", *Algebra Universalis* 1 (1971), 234-235.

[6] Alfred L. Foster and Alden F. Pixley, "Semi-categorical algebras. II", *Math. Z.* 85 (1964), 169-184.

[7] Grant A. Fraser and Alfred Horn, "Congruence relations in direct products", *Proc. Amer. Math. Soc.* 26 (1970), 390-394.

[8] E. Fried, "A note on congruence extension property", *Acta Sci. Math.* 40 (1978), 261-263.

[9] J. Hagemann and A. Mitschke, "On n-permutable congruences", *Algebra Universalis* 3 (1973), 8-12.

[10] Leon Henkin, "An algebraic characterization of quantifiers", *Fund. Math.* 37 (1950), 63-74.

[11] Kiyoshi Iséki, "An algebra related with a propositional calculus", *Proc. Japan Acad.* 42 (1966), 26-29.

[12] Kiyoshi Iséki, "Topics of BCK-algebras", *Proceedings of the* 1st *Symposium on Semigroups*, 44-56 (Shimane University, Shimane-Ken, Japan, 1977).

[13] Kiyoshi Iséki, "On finite BCK-algebras", *Proceedings of the* 3rd *Symposium on Semigroups*, 11-12 (Inter-University Seminar House, Kansai, 1979).

[14] Kiyoshi Iséki and Shotaro Tanaka, "Ideal theory of BCK-algebras", *Math. Japon.* 21 (1976), 351-366.

[15] Kiyoshi Iséki and Shotaro Tanaka, "An introduction to the theory of BCK-algebras", *Math. Japon.* 23 (1978), 1-26.

[16] Bjarni Jónnson, "Algebras whose congruence lattices are distributive", *Math. Scand.* 21 (1967), 110-121.

[17] Peter Köhler and Don Pigozzi , "Varieties with equationally definable principal congruences", preprint.

[18] Yuichi Komori, "The separation theorem of the $\aleph_0$-valued Lukasiewicz propositional logic", *Rep. Fac. Sci. Shizuoka Univ.* 12 (1978), 1-5.

[19] Yuichi Komori, "Super-Łukasiewicz implicational logics", *Nagoya Math. J.* 72 (1978), 127-133.

[20] Aleit Mitschke, "Implication algebras are 3-permutable and 3-distributive", *Algebra Universalis* 1 (1971), 182-186.

[21] R. Padmanabhan and R.W. Quackenbush, "Equational theories of algebras with distributive congruences", *Proc. Amer. Math. Soc.* 41 (1973), 373-377.

[22] Robert W. Quackenbush, "Primality: the influence of Boolean algebras in Universal Algebra", *Universal algebra*, 2nd edition, Appendix 5, 401-416 (Springer-Verlag, New York, Heidelberg, Berlin, 1979).

[23] Anna Romanowska and Tadeusz Traczyk, "On commutative BCK-algebras", preprint.

[24] Yasuo Setō, "Some examples of BCK-algebras", *Math. Sem. Notes Kobe Univ.* 5 (1977), 397-400.

[25] Zahava Shmuely, "The structure of Galois connections", *Pacific J. Math.* 54 (1974), no. 2, 209-225.

[26] Tadeusz Traczyk, "On the variety of bounded commutative BCK-algebras", *Math. Japon.* 24 (1979), 283-292.

[27] Hiroshi Yutani, "Quasi-commutative BCK-algebras and congruence lattices", *Math. Sem. Notes Kobe Univ.* 5 (1977), 469-480.

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