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A QUENCHED CENTRAL LIMIT THEOREM FOR REVERSIBLE RANDOM WALKS IN A RANDOM ENVIRONMENT ON $\mathbb{Z}$

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Abstract

The main aim of this paper is to prove the quenched central limit theorem for reversible random walks in a stationary random environment on $\mathbb{Z}$ without having the integrability condition on the conductance and without using any martingale. The method shown here is particularly simple and was introduced by Depauw and Derrien [3]. More precisely, for a given realization ω of the environment, we consider the Poisson equation $(P_\omega - I)g = f$, and then use the pointwise ergodic theorem in [8] to treat the limit of solutions and then the central limit theorem will be established by the convergence of moments. In particular, there is an analogue to a Markov process with discrete space and the diffusion in a stationary random environment.

Keywords: Quenched central limit theorem; reversible random walk in random environment

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1. Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. The definition of a random walk in a random environment involves two ingredients:

1. The environment which is randomly chosen but remains fixed throughout the time evolution.
2. The random walk whose transition probability is determined by the environment.

The space Ω is interpreted as the space of environments. For each $\omega \in \Omega$, we define the random walk in the environment ω as the (time homogeneous) Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ with certain (random) transition probabilities

$$p(\omega, x, y) = \mathbb{P}_\omega\{X_1 = y / X_0 = x\}.$$

The probability measure $\mathbb{P}_\omega$ determines the distribution of the random walk in a given environment ω. In this paper we study only the random walk with the initial condition $X_0 = 0$,

$$\mathbb{P}^0_\omega\{X_0 = 0\} = 1.$$

The probability measure $\mathbb{P}^0_\omega$ indicates the distribution of the random walk in a given environment ω with the initial position of the walk referred to as *the Quenched law*. For more information about the random walk in a random environment and *the Annealed law* see [1], [6], and [9].

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Now we consider the following model for the random walk in a random environment. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and T be an invertible measure preserving transformation on Ω, which is ergodic. More precisely, T acts on Ω by

$$T : \Omega \times \mathbb{Z} \to \Omega \quad \text{and} \quad (\omega, k) \mapsto T^k\omega,$$

which is joint measurable and satisfies

1. For any $k, h \in \mathbb{Z}$: $T^{k+h} = T^k T^h$ and $T^0\omega = \omega$.
2. T preserves the measure μ: $\mu(T^k A) = \mu(A)$ for any $k \in \mathbb{Z}$.
3. T is ergodic: if $T^k A = A$ (up to null sets) for all $k \in \mathbb{Z}$ then $\mu(A)$ is equal to 0 or 1.

On the $\mathbb{Z}$ network, we assume that the conductivity of the edge between $\{k, k+1\}$ is equal to $c(T^k\omega)$, where c is a positive measurable function on Ω. Fix $\omega \in \Omega$ and consider a random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ where $X_0 = 0$ and the transition probability $p(\omega, k, h)$ is given by

$$p(\omega; k, k+1) = \mathbb{P}^0_\omega\{X_{n+1} = k+1 / X_n = k\} = c(T^k\omega)/\bar{c}(T^k\omega),$$

and

$$p(\omega; k, k-1) = \mathbb{P}^0_\omega\{X_{n+1} = k-1 / X_n = k\} = c(T^{k-1}\omega)/\bar{c}(T^k\omega),$$

where $\bar{c}(\omega) = c(\omega) + c(T^{-1}\omega)$. These random walks are reversible since for all adjacent vertices x, y in $\mathbb{Z}$, we have $\bar{c}(T^x\omega)p(\omega; x, y) = \bar{c}(T^y\omega)p(\omega; y, x)$. The Markov operator $f \mapsto P_\omega f$ is defined by

$$P_\omega f(k) = \frac{1}{\bar{c}(T^k\omega)}[c(T^{k-1}\omega) f(k-1) + c(T^k\omega) f(k+1)].$$

When c is integrable but c^{-1} not, Derriennic and Lin have proved, in an unpublished work, the annealed central limit theorem (CLT) with null variance: $\lim_{n\to\infty} n^{-1}\mathbb{E}_\omega(X_n^2) = 0$ in μ-measure, where $\mathbb{E}_\omega$ denotes the expectation relative to the randomness of the walk with the environment being fixed. For the quenched version Depauw and Derrien [3] considered a nonnegative solution f, defined on $\mathbb{Z}$, of the Poisson equation $(P_\omega - I)f = 1$ and that satisfies $f(0) = 0$ to obtain the limit of the variance of the reversible random walk $(X_n)_{n\geq 0}$ without using any martingale and without having any condition on function c except that $c > 0$.

Theorem 1. (Depauw and Derrien [3].) *For almost all environments ω,*

$$\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\frac{X_n^2}{n}\right\} = \left[\int \frac{1}{c}\,\mathrm{d}\mu \int c\,\mathrm{d}\mu\right]^{-1}.$$

This limit is null if at least one of the integrals is $+\infty$.

When both of c and c^{-1} are integrable, the quenched CLT, in the usual case, can be proved by the method of martingale, first introduced by Kozlov, [5]. Unfortunately, this method does not allow us to treat the case when c or c^{-1} are not integrable. The aim of this paper is to generalize Theorem 1 and to establish the quenched CLT without using any martingale and without having any condition on function c except that $c > 0$. In the case when at least one of c and c^{-1} is not integrable, $X_n/\sqrt{n}$ converges to the degenerate normal distribution. The second method is adapted from [3] and leads to the following theorem.

Theorem 2. *For almost all environments* ω,

$$\frac{X_n}{\sqrt{n}} \xrightarrow{\text{D}} \mathcal{N}\left(0, \left[\int \frac{1}{c}\,\mathrm{d}\mu \int c\,\mathrm{d}\mu\right]^{-1}\right) \quad \textit{as } n \to +\infty.$$

The limiting distribution being a degenerate normal distribution if at least one of the integrals is $+\infty$.

Throughout this paper, '$\xrightarrow{\text{D}}$' denotes the convergence in distribution and $\mathcal{N}(\lambda, \sigma^2)$ denotes the normal distribution with mean λ and variance σ^2.

This paper is organized as follows. In Section 2 we prove Theorem 2. In Section 3 there is an analogue to a Markov process with continuous time and discrete space. Finally, in Section 4 we consider the diffusion in a random environment. It is somewhat involved but we will show the proof of the CLT explicitly.

2. Proof of Theorem 2

Consider a normal distribution $Z \sim \mathcal{N}(0, \sigma^2)$, and for each $\ell = 1, 2, 3, \ldots$, we have

$$\mathbb{E}\{Z^\ell\} = 0 \quad \text{if } \ell = 2k-1 \quad \text{and} \quad \mathbb{E}\{Z^\ell\} = \frac{(2k)!}{k!\,2^k}\sigma^\ell \quad \text{if } \ell = 2k.$$

Using the method of moments introduced in [2, Theorem 30.2, page 390] to prove Theorem 2 it is sufficient to show that for almost all environments ω

$$\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_n}{\sqrt{n}}\right)^\ell\right\} = \mathbb{E}\{Z^\ell\} = \begin{cases} 0 & \text{if } \ell = 2k-1 \\ \dfrac{(2k)!}{k!\,2^k}\sigma^\ell & \text{if } \ell = 2k \end{cases}$$

for each $\ell = 1, 2, 3, \ldots$. In this case $\sigma^2 = [\int (1/c)\,\mathrm{d}\mu \int c\,\mathrm{d}\mu]^{-1}$.

We begin with the following elementary lemma.

Lemma 1. *Let* u_n, v_n *be two sequences of positive real numbers and a nonnegative integer* $\alpha \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty}(1/n)\sum_{\ell=1}^n u_\ell = u > 0$ *and* $\lim_{n\to\infty} v_n = v > 0$. *If both of* u *and* v *are finite*

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+1}} \sum_{\ell=1}^n \ell^\alpha u_\ell v_\ell = \frac{uv}{\alpha+1} \tag{1}$$

else if at least one of u *and* v *is infinite then the limit in* (1) *being* $+\infty$.

Proof. We will prove that both of u and v are finite, with the other cases left to the reader. For $\alpha = 0$, we will show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^n u_\ell v_\ell = uv. \tag{2}$$

We have

$$\begin{aligned} \left|\frac{1}{n}\sum_{\ell=1}^n u_\ell v_\ell - uv\right| &\le \left|\frac{1}{n}\sum_{\ell=1}^n u_\ell(v_\ell - v)\right| + \left|\frac{1}{n}\sum_{\ell=1}^n (u_\ell - u)v\right| \\ &\le \frac{1}{n}\sum_{\ell=1}^n u_\ell|v_\ell - v| + v\left|\frac{1}{n}\sum_{\ell=1}^n u_\ell - u\right| \\ &< \varepsilon \end{aligned}$$

for any $\varepsilon > 0$ when n large enough, completing the proof of (2).

Now assume that (1) is true for $\alpha \geq 0$. We claim that it also holds for $\alpha + 1$, that is

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n} \ell^{\alpha+1} u_\ell v_\ell = \frac{uv}{\alpha+2}. \tag{3}$$

Let $W_n = \sum_{\ell=1}^{n} \ell^{\alpha} u_\ell v_\ell$, using Abel's transformation

$$\frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n} \ell^{\alpha+1} u_\ell v_\ell = -\frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n-1} W_\ell + \frac{1}{n^{\alpha+1}} W_n = -I_1 + I_2.$$

By the assumption $\lim_{n\to\infty} I_2 = \lim_{n\to\infty}(1/n^{\alpha+1})W_n = uv/(\alpha+1)$, we have

$$\begin{aligned}\left| I_1 - \frac{uv}{(\alpha+1)(\alpha+2)} \right| &\leq \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n-1} \ell^{\alpha+1} \left| \frac{W_\ell}{\ell^{\alpha+1}} - \frac{uv}{\alpha+1} \right| \\ &\quad + \left| \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n-1} \ell^{\alpha+1} - \frac{1}{\alpha+2} \right| \frac{uv}{\alpha+1} \\ &< \varepsilon\end{aligned}$$

for any $\varepsilon > 0$ when n is large enough since $(1/n^{\alpha+2}) \sum_{\ell=1}^{n-1} \ell^{\alpha+1} = (1/n) \sum_{\ell=1}^{n-1} (\ell/n)^{\alpha+1}$ and as n goes to ∞ we will have the limit that is equal to $\int_0^1 x^{\alpha+1}\,\mathrm{d}x = 1/(\alpha+2)$. It follows that $\lim_{n\to\infty} I_1 = uv/((\alpha+1)(\alpha+2))$. Hence,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n} \ell^{\alpha+1} u_\ell v_\ell = -\frac{uv}{(\alpha+1)(\alpha+2)} + \frac{uv}{\alpha+1} = \frac{uv}{\alpha+2},$$

which completes the proof of (3).

Lemma 2. *Given a function* $\psi : \mathbb{Z} \to \mathbb{R}$ *there exists a unique function* $\phi : \mathbb{Z} \to \mathbb{R}$ *such that*

$$(P_\omega - I)\phi = \psi, \qquad \phi(1) = a, \qquad \phi(-1) = b. \tag{4}$$

Proof. Since

$$(P_\omega - I)\phi(0) = \psi(0)$$

we have

$$c(T^{-1}\omega)\phi(-1) + c(\omega)\phi(1) - \bar{c}(\omega)\phi(0) = \bar{c}(\omega)\psi(0). \tag{5}$$

This determines $\phi(0)$.

For $m \geq 2$, we consider

$$(P_\omega - I)\phi(m-1) = \psi(m-1).$$

This is equivalent to

$$c(T^{m-1}\omega)[\phi(m) - \phi(m-1)] = c(T^{m-2}\omega)[\phi(m-1) - \phi(m-2)] + \bar{c}(T^{m-1}\omega)\psi(m-1)$$

and then by induction on m

$$\phi(m) = \phi(1) + [\phi(1) - \phi(0)] \sum_{\ell=1}^{m-1} \frac{c(\omega)}{c(T^\ell\omega)} + \sum_{\ell=1}^{m-1} \frac{1}{c(T^\ell\omega)} \sum_{s=1}^{\ell} \bar{c}(T^s\omega)\psi(s).$$

Similarly for $m \leq -2$, we have

$$\phi(m) = \phi(-1) + [\phi(-1) - \phi(0)] \sum_{\ell=2}^{-m} \frac{c(T^{-1}\omega)}{c(T^{-\ell}\omega)} + \sum_{\ell=2}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum_{s=1}^{\ell-1} \bar{c}(T^{-s}\omega)\psi(-s).$$

We have thus proved that ϕ is a unique solution of (4). We also deduce that a particular solution ϕ of the Poisson equation $(P_\omega - I)\phi = \psi$ is characterized by the values $\phi(-1)$, $\phi(0)$ and $\phi(1)$ such that they satisfy (5).

Proposition 1. *For almost all environments ω, we have*

$$\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_n}{\sqrt{n}}\right)^{2k}\right\} = \frac{(2k)!}{2^k k!}\sigma^{2k}$$

for each $k \geq 1$. This limit is null if at least one of c and c^{-1} is not integrable.

This is the generalization of Theorem 1 (or [3, Theorem 0.1]).

Proof. Fixing $\omega \in \Omega$ we consider a sequence of functions $f_k \geq 0$, defined on $\mathbb{Z}$, such that

$$(P_\omega - I)f_k \equiv f_{k-1} \quad \text{for } k \geq 1, \qquad f_0 \equiv 1, \qquad f_k(0) = 0 \quad \text{for } k \geq 1.$$

By Lemma 2 we can determine the function f_1 which satisfies $f_1(-1) = \bar{c}(\omega)/c(T^{-1}\omega)$ and $f_1(1) = 0$

$$f_1(m) = \begin{cases} \sum_{\ell=1}^{m-1} \frac{1}{c(T^\ell\omega)} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) & \text{if } m \geq 2, \\ 0 & \text{if } m = 0, 1, \\ \sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum_{s=0}^{\ell-1} \bar{c}(T^{-s}\omega) & \text{if } m \leq -1 \end{cases}$$

and for $k \geq 2$ the function f_k which satisfies $f_k(1) = f_k(-1) = 0$

$$f_k(m) = \begin{cases} \sum_{\ell=1}^{m-1} \frac{1}{c(T^\ell\omega)} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) f_{k-1}(s) & \text{if } m \geq 2, \\ 0 & \text{if } m = -1, 0, 1, \\ \sum_{\ell=2}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum_{s=1}^{\ell-1} \bar{c}(T^{-s}\omega) f_{k-1}(-s) & \text{if } m \leq -2. \end{cases}$$

Then, for any integer m and for $k \geq 1$, we have

$$(P_\omega - I)f_k(m) = f_{k-1}(m).$$

Replace m by X_n and take the expectation to obtain

$$\mathbb{E}_\omega\{f_k(X_{n+1})\} = \mathbb{E}_\omega\{f_k(X_n)\} + \mathbb{E}_\omega\{f_{k-1}(X_n)\}$$

for any $n \geq 0$. It follows that for each $k \geq 1$

$$\mathbb{E}_\omega\{f_k(X_n)\} \sim \frac{n^k}{k!} \tag{6}$$

when n is large enough since $f_k(0) = 0$ by the definition of f_k and $X_0 = 0$ by the assumption of the random walk X_n. The proof of (6) is by induction on k.

Equation (6) can be rewritten as

$$\mathbb{E}_\omega\left\{\frac{f_k(X_n)}{X_n^{2k}}\frac{X_n^{2k}}{n^k}\right\} \sim \frac{1}{k!}.$$

We see that $\lim_{m\to\infty} f_k(m)/m^{2k}$ exists and so $\lim_{n\to+\infty}\mathbb{E}_\omega\{X_n^{2k}/n^k\}$ exists, completing the proof.

In the next step we will compute the limit of $f_k(m)/m^{2k}$ by using the pointwise ergodic theorem (p.e.t.) and Lemma 1.

Lemma 3. *For each $k \geq 1$, with f_k defined as above, we have*

$$\lim_{m\to\pm\infty}\frac{f_k(m)}{m^{2k}} = \frac{2^k}{(2k)!}\sigma^{-2k}. \tag{7}$$

This limit is $+\infty$ if at least one of c and c^{-1} is not integrable. It is denoted by c_k.

Proof. This limit is true for $k = 1$. Indeed, for the case $m > 0$, we have

$$\frac{f_1(m)}{m^2} = \frac{1}{m}\sum_{\ell=1}^{m-1}\left(\frac{\ell}{m}\right)\frac{1}{c(T^\ell\omega)}\frac{1}{\ell}\sum_{s=1}^{\ell}\bar{c}(T^s\omega).$$

Applying Lemma 1 and the p.e.t. for $u_\ell = 1/c(T^\ell\omega)$, $v_\ell = (1/\ell)\sum_{s=1}^{\ell}\bar{c}(T^s\omega)$ and $\alpha = 1$ this tends to $\sigma^{-2} = [\int(1/c)\,\mathrm{d}\mu\int c\,\mathrm{d}\mu]$. The point is that, since $c > 0$, the convergence is still satisfied if one of these integrals is $+\infty$ (see [3]).

Assume that (7) is also true for $k \geq 1$, we claim that it holds for $k + 1$, that is

$$\lim_{m\to\infty}\frac{f_{k+1}(m)}{m^{2(k+1)}} = \frac{2^{k+1}}{(2k+2)!}\sigma^{-2(k+1)}. \tag{8}$$

We have

$$\frac{1}{\ell^{2k+1}}\sum_{s=1}^{\ell}\bar{c}(T^s\omega)f_k(s) = \frac{1}{\ell}\sum_{s=1}^{\ell}\left(\frac{s}{\ell}\right)^{2k}\bar{c}(T^s\omega)\frac{1}{s^{2k}}f_k(s).$$

Again, applying Lemma 1 and the p.e.t. for $u_s = \bar{c}(T^s\omega)$, $v_s = (1/s^{2k})f_k(s)$ and $\alpha = 2k$, this tends to $\int_\Omega c\,\mathrm{d}\mu 2^{k+1}/(2k+1)!\,\sigma^{-2k}$. Moreover,

$$\frac{f_{k+1}(m)}{m^{2(k+1)}} = \frac{1}{m}\sum_{\ell=1}^{m-1}\left(\frac{\ell}{m}\right)^{2k+1}\frac{1}{c(T^\ell\omega)}\frac{1}{\ell^{2k+1}}\sum_{s=1}^{\ell}\bar{c}(T^s\omega)f_k(s).$$

Again, applying Lemma 1 and the p.e.t. for $u'_\ell = 1/c(T^\ell\omega)$, $v'_\ell = (1/\ell^{2k+1})\sum_{s=1}^{\ell}\bar{c}(T^s\omega)f_k(s)$ and $\alpha = 2k+1$, this tends to $2^{k+1}/(2(k+1))!\,\sigma^{-2(k+1)}$ which completes the proof of (8).

Similarly, we have the same result for the case $m < 0$.

From Lemma 3, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $m > M$

$$\left|\frac{m^{2k}}{f_k(m)} - \frac{1}{c_k}\right| < \varepsilon/2. \tag{9}$$

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Decomposing $\Omega = \{|X_n| \le M\} \cup \{|X_n| > M\}$ and combining (6) and (9) we obtain

$$\begin{aligned}\left|\mathbb{E}_\omega\left\{\left(\frac{X_n}{\sqrt{n}}\right)^{2k}\right\} - \frac{1}{k!\,c_k}\right| &\approx \left|\mathbb{E}_\omega\left\{\left(\frac{X_n^{2k}}{f_k(X_n)} - \frac{1}{c_k}\right)\frac{f_k(X_n)}{n^k}\mathbf{1}_{\{|X_n|>M\}}\right\}\right. \\ &\qquad \left. + \frac{1}{n^k}\mathbb{E}_\omega\left\{\left(X_n^{2k} - \frac{1}{c_k} f_k(X_n)\right)\mathbf{1}_{\{|X_n|\le M\}}\right\}\right| \\ &\le \mathbb{E}_\omega\left\{\left|\frac{X_n^{2k}}{f_k(X_n)} - \frac{1}{c_k}\right|\frac{f_k(X_n)}{n^k}\mathbf{1}_{\{|X_n|>M\}}\right\} \\ &\qquad + \frac{1}{n^k}\mathbb{E}_\omega\left\{\left|X_n^{2k} - \frac{1}{c_k} f_k(X_n)\right|\mathbf{1}_{\{|X_n|\le M\}}\right\} \\ &< \varepsilon\end{aligned}$$

for n large enough. Since ε is as small as we need, we obtain the desired result, which completes Proposition 1.

Proposition 2. *For almost all environments ω, it holds that*

$$\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_n}{\sqrt{n}}\right)^{2k-1}\right\} = 0 \quad \textit{for each } k \ge 1.$$

Proof. Fixing $\omega \in \Omega$ we consider a sequence of functions g_k, defined on $\mathbb{Z}$, such that

$$(P_\omega - I)g_k \equiv g_{k-1} \quad \text{for } k \ge 1, \qquad g_0 \equiv 0, \qquad g_k(0) = 0 \quad \text{for } k \ge 1.$$

Again, by Lemma 2 we can determine the function g_1 which satisfies $g_1(1) = 1/c(\omega)$ and $g_1(-1) = -1/c(T^{-1}\omega)$

$$g_1(m) = \begin{cases} \displaystyle\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} & \text{if } m \ge 1, \\ 0 & \text{if } m = 0, \\ \displaystyle -\sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} & \text{if } m \le -1 \end{cases}$$

and for $k \ge 2$ the function g_k satisfies $g_k(1) = g_k(-1) = 0$

$$g_k(m) = \begin{cases} \displaystyle\sum_{\ell=1}^{m-1} \frac{1}{c(T^\ell\omega)} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) g_{k-1}(s) & \text{if } m \ge 2, \\ 0 & \text{if } m = -1, 0, 1, \\ \displaystyle\sum_{\ell=2}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum_{s=1}^{\ell-1} \bar{c}(T^{-s}\omega) g_{k-1}(-s) & \text{if } m \le -2. \end{cases}$$

Then for any integer m and for $k \ge 1$

$$(P_\omega - I)g_k(m) = g_{k-1}(m).$$

Replace m by X_n and then take the expectation to obtain

$$\mathbb{E}_\omega\{g_k(X_{n+1})\} = \mathbb{E}_\omega\{g_k(X_n)\} + \mathbb{E}_\omega\{g_{k-1}(X_n)\} \quad \text{for any } n \ge 0.$$

It is straightforward to see that for each $k \geq 1$

$$\mathbb{E}_\omega\{g_k(X_n)\} = 0 \quad \text{for any } n \geq 0. \tag{10}$$

Equation (10) can be rewritten as

$$\mathbb{E}_\omega\left\{\frac{g_k(X_n)}{X_n^{2k-1}} \times \frac{X_n^{2k-1}}{(\sqrt{n})^{2k-1}}\right\} = 0.$$

We see that the limit of $g_k(m)/m^{2k-1}$ exists and so the limit of $\mathbb{E}_\omega\{(X_n/\sqrt{n})^{2k-1}\}$ equals 0.

In the next step we will compute the limit of $g_k(m)/m^{2k-1}$ by using the pointwise ergodic theorem and Lemma 1.

Lemma 4. *For each $k \geq 1$ and g_k defined as above, we have*

$$\lim_{m\to\pm\infty} \frac{g_k(m)}{m^{2k-1}} = \frac{2^{k-1}}{(2k-1)!}\sigma^{-2k+2}\int_\Omega \frac{1}{c}\,\mathrm{d}\mu.$$

This limit is $+\infty$ if at least one of c and c^{-1} is not integrable. It is denoted by d_k.

The proof of Lemma 4 is left to the reader. From Lemma 4, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $|m| > M$

$$\left|\frac{g_k(m)}{m^{2k-1}d_k} - 1\right| < \varepsilon. \tag{11}$$

Decomposing $\Omega = \{|X_n| \leq M\} \cup \{|X_n| > M\}$ and combining (10) and (11), we have

$$\begin{aligned}
\left|\mathbb{E}_\omega\left\{\left(\frac{X_n}{\sqrt{n}}\right)^{2k-1}\right\}\right| &= \left|\mathbb{E}_\omega\left\{\frac{1}{(\sqrt{n})^{2k-1}}\left(X_n^{2k-1} - \frac{g_k(X_n)}{d_k}\right)\mathbf{1}_{\{|X_n|\leq M\}}\right\}\right.\\
&\quad \left.+ \mathbb{E}_\omega\left\{\frac{1}{(\sqrt{n})^{2k-1}}\left(X_n^{2k-1} - \frac{g_k(X_n)}{d_k}\right)\mathbf{1}_{\{|X_n|> M\}}\right\}\right|\\
&\leq \mathbb{E}_\omega\left\{\frac{1}{(\sqrt{n})^{2k-1}}\left|X_n^{2k-1} - \frac{g_k(X_n)}{d_k}\right|\mathbf{1}_{\{|X_n|\leq M\}}\right\}\\
&\quad + \mathbb{E}_\omega\left\{\left(\frac{|X_n|}{\sqrt{n}}\right)^{2k-1}\left|1 - \frac{g_k(X_n)}{X_n^{2k-1}d_k}\right|\mathbf{1}_{\{|X_n|> M\}}\right\}\\
&\leq \varepsilon + \varepsilon\sqrt{\mathbb{E}_\omega\left\{\left(\frac{X_n}{\sqrt{n}}\right)^{2(2k-1)}\right\}}
\end{aligned}$$

for n large enough. By Proposition 1 the limit $\lim_{n\to\infty}\mathbb{E}_\omega\{(X_n/\sqrt{n})^{2(2k-1)}\}$ exists, and since ε is as small as we need, we then obtain the desired result which completes Proposition 2.

3. An analogue to a Markov process with discrete space

We consider a Markov process $(X_t)_{t\in[0,+\infty)}$ on $\mathbb{Z}$ with $X_0 = 0$. The infinitesimal generator is defined by

$$L_\omega f(k) = c(T^{k-1}\omega)f(k-1) + c(T^k\omega)f(k+1) - \bar{c}(T^k\omega)f(k).$$

We will now establish a CLT for a Markov process $(X_t)_{t\in[0,+\infty)}$ without the use of a martingale.

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Theorem 3. *For almost all environments ω,*

$$\frac{X_t}{\sqrt{t}} \xrightarrow{\mathrm{D}} \mathcal{N}\left(0, 2\left[\int \frac{1}{c}\,\mathrm{d}\mu\right]^{-1}\right) \quad \text{as } t \to +\infty.$$

The limiting distribution being a degenerate normal distribution if the integral is $+\infty$.

This problem was also considered by Kawazu and Kesten, [4]. They did not use the method of martingale but instead used a time change of a Brownian motion.

To prove Theorem 3 it is sufficient to show that for almost all environments ω

$$\lim_{t\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_t}{\sqrt{t}}\right)^\ell\right\} = 0 \quad \text{if } \ell = 2k-1, \qquad \lim_{t\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_t}{\sqrt{t}}\right)^\ell\right\} = \frac{(2k)!}{k!\,2^k}\sigma^\ell \quad \text{if } \ell = 2k$$

for each $\ell = 1, 2, 3, \ldots$. In this case $\sigma^2 = 2[\int (1/c)\,\mathrm{d}\mu]^{-1}$.

Lemma 5. *Given a function $\psi\colon \mathbb{Z} \to \mathbb{R}$ there exists an unique function $\phi\colon \mathbb{Z} \to \mathbb{R}$ such that*

$$L_\omega \phi = \psi, \qquad \phi(1) = a, \qquad \phi(-1) = b. \tag{12}$$

Proof. By a similar argument as in Lemma 2 we can show that (12) has a unique solution ϕ such that

- $\phi(1) = a$ and $\phi(-1) = b$;
- $\phi(0)$ satisfies $c(T^{-1}\omega)\phi(-1) + c(\omega)\phi(1) - \bar{c}(\omega)\phi(0) = \psi(0)$;
- $\phi(m) = \phi(1) + [\phi(1) - \phi(0)] \sum_{\ell=1}^{m-1} \frac{c(\omega)}{c(T^\ell\omega)} + \sum_{\ell=1}^{m-1} \frac{1}{c(T^\ell\omega)} \sum_{s=1}^{\ell} \psi(s)$ if $m \ge 2$;
- $\phi(m) = \phi(-1) + [\phi(-1) - \phi(0)] \sum_{\ell=2}^{-m} \frac{c(T^{-1}\omega)}{c(T^{-\ell}\omega)} + \sum_{\ell=2}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum_{s=1}^{\ell-1} \psi(-s)$ if $m \le -2$.

Proposition 3. *For almost all environments ω, we have*

$$\lim_{t\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_t}{\sqrt{t}}\right)^{2k}\right\} = \frac{(2k)!}{k!\,2^k}\sigma^{2k} \quad \text{for each } k \ge 1.$$

This limit is null if c^{-1} is not integrable.

Proof. Fixing $\omega \in \Omega$, we consider a sequence of functions $f_k \ge 0$, defined on $\mathbb{Z}$, such that

$$L_\omega f_k \equiv f_{k-1} \quad \text{for } k \ge 1, \qquad f_0 \equiv 1, \qquad f_k(0) = 0 \quad \text{for } k \ge 1.$$

By Lemma 5 we determine a function f_1 which satisfies $f_1(-1) = 1/c(T^{-1}\omega)$ and $f_1(1) = 0$

$$f_1(m) = \begin{cases} \sum_{\ell=1}^{m-1} \frac{\ell}{c(T^\ell\omega)} & \text{if } m \ge 2, \\ 0 & \text{if } m = 0, 1 \\ \sum_{\ell=1}^{-m} \frac{\ell}{c(T^{-\ell}\omega)} & \text{if } m \le -1 \end{cases}$$

and for $k \geq 2$ a function f_k which satisfies $f_k(1) = f_k(-1) = 0$

$$f_k(m) = \begin{cases} \sum_{\ell=1}^{m-1} \frac{1}{c(T^\ell \omega)} \sum_{s=1}^{\ell} f_{k-1}(s) & \text{if } m \geq 2, \\ 0 & \text{if } m = -1, 0, 1, \\ \sum_{\ell=2}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum_{s=1}^{\ell-1} f_{k-1}(-s) & \text{if } m \leq -2. \end{cases}$$

Then $L_\omega f_k(m) = f_{k-1}(m)$ for any integer m and for $k \geq 1$. Replace m by X_t to obtain

$$L_\omega f_k(X_t) = f_{k-1}(X_t) \quad \text{for any } t \geq 0.$$

Hence we can show that for each $k \geq 1$

$$\mathbb{E}_\omega\{f_k(X_t)\} = \frac{t^k}{k!} \quad \text{for any } t \geq 0. \tag{13}$$

Indeed, if $h_1(t) = \mathbb{E}_\omega\{f_1(X_t)\}$ then

$$\begin{aligned} \mathbb{E}_\omega\{L_\omega f_1(X_t)\} &= \lim_{s\to 0} \mathbb{E}_\omega\left\{\frac{\mathbb{E}_\omega\{f_1(X_{t+s})/X_t\} - f_1(X_t)}{s}\right\} \\ &= \lim_{s\to 0} \mathbb{E}_\omega\left\{\frac{f_1(X_{t+s}) - f_1(X_t)}{s}\right\} \\ &= \lim_{s\to 0} \frac{h_1(t+s) - h_1(t)}{s} = h_1'(t). \end{aligned}$$

Since $\mathbb{E}_\omega\{L_\omega f_1(X_t)\} = 1$ implies $h_1(t) = t + \beta$ for $t \geq 0$. And $h_1(0) = \mathbb{E}_\omega\{f_1(X_0)\} = 0$ implies $\beta = 0$, hence $h_1(t) = \mathbb{E}_\omega\{f_1(X_t)\} = t$. By induction on k we obtain the proof of (13).

The expansion is similar to Proposition 1 in Section 2. The proof of Proposition 3 is complete.

Proposition 4. *For almost all environments ω, we have*

$$\lim_{t\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_t}{\sqrt{t}}\right)^{(2k-1)}\right\} = 0 \quad \textit{for each } k \geq 1.$$

Proof. Fixing $\omega \in \Omega$ we consider a sequence of functions g_k, defined on $\mathbb{Z}$, such that

$$L_\omega g_k \equiv g_{k-1} \quad \text{for } k \geq 1, \qquad g_0 \equiv 0, \qquad g_k(0) = 0 \quad \text{for } k \geq 1.$$

By Lemma 5 we can determine a function g_1 which satisfies $g_1(-1) = -1/c(T^{-1}\omega)$ and $g_1(1) = 1/c(\omega)$

$$g_1(m) = \begin{cases} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} & \text{if } m \geq 1, \\ 0 & \text{if } m = 0, \\ -\sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} & \text{if } m \leq -1 \end{cases}$$

and for $k \geq 2$ a function g_k which satisfies $g_k(1) = g_k(-1) = 0$

$$g_k(m) = \begin{cases} \sum_{\ell=1}^{m-1} \frac{1}{c(T^\ell \omega)} \sum_{s=1}^{\ell} g_{k-1}(s) & \text{if } m \geq 2, \\ 0 & \text{if } m = -1, 0, 1, \\ \sum_{\ell=2}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum_{s=1}^{\ell-1} g_{k-1}(-s) & \text{if } m \leq -2. \end{cases}$$

The expansion is similar to that in Proposition 2 in Section 2. The proof of Proposition 4 is thus complete.

Remark 1. We introduce the sequence of transition times, $(\tau_n)_{n\in\mathbb{N}}$, that are the times when X_t jumps i.e. $\tau_0 = 0$ and

$$\tau_n = \inf\{t \geq \tau_{n-1} \colon X_t \neq X_{\tau_{n-1}}\}.$$

The times between transition times $\tau_{n+1} - \tau_n$ are called waiting times. It is well known that the waiting times have an exponential distribution with the parameter $\bar{c}(T^{X_{\tau_n}}\omega)$ that depends only on the position of X_t at time τ_n. In this case $\tau_n \to \infty$ as $n \to \infty$.

The embedded process is given by $Y_0 = X_0 = 0$ and

$$Y_n = X_{\tau_n}.$$

Then $(Y_n)_{n\geq 0}$ is also the random walk with transition probabilities as in the previous section. Hence we can deduce that $(X_{\tau_n})_{n\geq 0}$ satisfies Theorem 2.

4. Diffusion in a stationary random environment

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space equipped with a flow $(T_x)_{x\in\mathbb{R}}$ that is ergodic and preserves the measure μ. We consider two random variables $a, b > 0$ such that the functions $x \mapsto a(T_x\omega)$ and $x \mapsto b(T_x\omega)$ are continuous. Now we look at where ω is fixed. The process of the infinitesimal generator is defined by

$$L_\omega f(x) = \frac{1}{2a(T_x\omega)} \frac{\mathrm{d}}{\mathrm{d}x}\left(b(T_x\omega)\frac{\mathrm{d}f}{\mathrm{d}x}\right)$$

with the initial condition $X_0 = 0$.

This describes the problem associated to the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \sigma_\omega(X_t)\,\mathrm{d}B_t + \mu_\omega(X_t)\,\mathrm{d}t, \tag{14}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion, the coefficient of diffusion $\sigma_\omega^2(x) = b(T_x\omega)/a(T_x\omega)$ and the drift $\mu_\omega(x) = (2a(T_x\omega))^{-1}(\mathrm{d}/\mathrm{d}x)(b(T_x\omega))$.

Theorem 4. *Suppose that, almost anywhere $\omega \in \Omega$, the functions $\sigma_\omega^2(x)$ and $\mu_\omega(x)$ are locally Lipschitz. Then, for almost all $\omega \in \Omega$, the solution $(X_t)_{t\geq 0}$ of SDE (14) satisfies*

$$\frac{X_t}{\sqrt{t}} \xrightarrow{\mathrm{D}} \mathcal{N}\left(0, \left[\int_\Omega a\,\mathrm{d}\mu \int_\Omega \frac{1}{b}\,\mathrm{d}\mu\right]^{-1}\right) \quad \text{as } t \to +\infty.$$

The limiting distribution is a degenerate normal distribution if at least one of the integrals is $+\infty$.

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Papanicolaou and Varadhan [7] established the CLT for the elliptic case in dimension $d \geq 1$.

As in Theorem 2 we do not use any martingale to prove Theorem 4. We begin with a continuous version of Lemma 1.

Lemma 6. *Let $u(x)$ and $v(x)$ be two positive continuous functions and α a nonnegative integer. Assume that $\lim_{y\to+\infty}(1/y)\int_0^y u(x)\,\mathrm{d}x = \bar{u} > 0$ and $\lim_{x\to+\infty} v(x) = \bar{v} > 0$. If both of $\bar{u}$ and $\bar{v}$ are finite then*

$$\lim_{y\to+\infty} \frac{1}{y^{\alpha+1}} \int_0^y x^\alpha u(x) v(x)\,\mathrm{d}x = \frac{\bar{u}\bar{v}}{\alpha+1} \tag{15}$$

else if at least one of $\bar{u}$ and $\bar{v}$ is infinite then the limit in (15) is $+\infty$.

The proof of Lemma 6 is left to the reader.

Now we are going to show that for almost all environments ω

$$\lim_{t\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_t}{\sqrt{t}}\right)^\ell\right\} = \begin{cases} 0 & \text{if } \ell = 2k-1, \\ \dfrac{(2k)!}{k!\,2^k}\sigma^\ell & \text{if } \ell = 2k \end{cases}$$

for each $\ell = 1, 2, 3, \dots$. In this case $\sigma^2 = [\int_\Omega a\,\mathrm{d}\mu \int_\Omega (1/b)\,\mathrm{d}\mu]^{-1}$.

Proposition 5. *For almost all environments ω, we have*

$$\lim_{t\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_t}{\sqrt{t}}\right)^{2k}\right\} = \frac{(2k)!}{k!\,2^k}\sigma^{2k}$$

for each $k \geq 1$. This limit is null if at least one of a and $1/b$ is not integrable.

This is the generalization of Theorem 4.1 in [3].

Proof. Fixing $\omega \in \Omega$ we consider a sequence of functions f_k, defined on $\mathbb{R}$, such that

$$L_\omega f_k \equiv f_{k-1} \quad \text{for } k \geq 1, \qquad f_0 \equiv 1, \qquad f_k(0) = 0 \quad \text{for } k \geq 1.$$

For example, we can take

$$f_1(x) = \begin{cases} \displaystyle\int_{v=0}^{x} \frac{1}{b(T_v\omega)} \int_{u=0}^{v} 2a(T_u\omega)\,\mathrm{d}u\,\mathrm{d}v & \text{if } x \geq 0, \\ \displaystyle\int_{v=x}^{0} \frac{1}{b(T_v\omega)} \int_{u=v}^{0} 2a(T_u\omega)\,\mathrm{d}u\,\mathrm{d}v & \text{if } x < 0 \end{cases}$$

and for $k \geq 2$

$$f_k(x) = \begin{cases} \displaystyle\int_{v=0}^{x} \frac{1}{b(T_v\omega)} \int_{u=0}^{v} 2a(T_u\omega) f_{k-1}(u)\,\mathrm{d}u\,\mathrm{d}v & \text{if } x \geq 0, \\ \displaystyle\int_{v=x}^{0} \frac{1}{b(T_v\omega)} \int_{u=v}^{0} 2a(T_u\omega) f_{k-1}(u)\,\mathrm{d}u\,\mathrm{d}v & \text{if } x < 0. \end{cases}$$

Lemma 7. *For each $k \geq 1$, we have*

$$\lim_{x\to\pm\infty} \frac{f_k(x)}{x^{2k}} = \frac{2^k}{(2k)!}\sigma^{-2k}.$$

This limit is $+\infty$ if at least one of a and $1/b$ is not integrable. It is denoted by ℓ_k.

The proof of Lemma 7 is left to the reader as a continuous version of Lemma 3.

Moreover, by the hypothesis of the theorem, function $f_k \in \mathcal{C}^2$ for each $k \geq 1$, the process $Y_t^{(k)}$, defined by $Y_t^{(k)} = f_k(X_t)$, satisfies SDE (14) by using Itô's lemma. We recall that

$$\mathrm{d} f_k(X_t) = f_k'(X_t)\sigma_\omega(X_t)\,\mathrm{d}B_t + [f_k'(X_t)\mu_\omega(X_t) + \tfrac{1}{2} f_k''(X_t)\sigma_\omega^2(X_t)]\,\mathrm{d}t.$$

By calculating, $\mathrm{d}Y_t^{(k)} = c_\omega(X_t)\,\mathrm{d}B_t + f_{k-1}(X_t)\,\mathrm{d}t$ for each $k \geq 1$ where $f_0 \equiv 1$. Hence, it follows that

$$\mathbb{E}_\omega\{f_k(X_t)\} = \frac{t^k}{k!} \quad \text{for each } k \geq 1. \tag{16}$$

Combining Lemma 7 and (16) we obtain

$$\left|\mathbb{E}_{\lambda,\omega}\left\{\left(\frac{X_t^2}{t}\right)^k\right\} - \frac{1}{k!\,\ell_k}\right| < \varepsilon$$

for any $\varepsilon > 0$ and for t large enough. And hence the proof of Proposition 5 is complete.

Proposition 6. *For almost all environments ω, we have*

$$\lim_{t\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_t}{\sqrt{t}}\right)^{(2k-1)}\right\} = 0 \quad \textit{for each } k \geq 1.$$

Proof. Fixing $\omega \in \Omega$ we consider a sequence of functions g_k, defined on $\mathbb{R}$, such that

$$L_\omega g_k \equiv g_{k-1} \quad \text{for } k \geq 1, \qquad g_0 \equiv 0, \qquad g_k(0) = 0, \quad \text{for } k \geq 1.$$

For instance, we can take

$$g_1(x) = \int_{v=0}^{x} \frac{1}{b(T_v\omega)}\,\mathrm{d}v \quad \text{if } x \geq 0, \quad \text{and} \quad g_1(x) = -\int_{v=x}^{0} \frac{1}{b(T_v\omega)}\,\mathrm{d}v \quad \text{if } x < 0$$

and for $k \geq 2$

$$g_k(x) = \begin{cases} \displaystyle\int_{v=0}^{x} \frac{1}{b(T_v\omega)} \int_{u=0}^{v} 2a(T_u\omega) g_{k-1}(u)\,\mathrm{d}u\,\mathrm{d}v & \text{if } x \geq 0, \\ \displaystyle\int_{v=x}^{0} \frac{1}{b(T_v\omega)} \int_{u=v}^{0} 2a(T_u\omega) g_{k-1}(u)\,\mathrm{d}u\,\mathrm{d}v & \text{if } x < 0. \end{cases}$$

Lemma 8. *For each $k \geq 1$, we have*

$$\lim_{x\to\pm\infty} \frac{g_k(x)}{x^{2k-1}} = \frac{2^{k-1}}{(2k-1)!}\sigma^{-2k+2} \int \frac{1}{b}\,\mathrm{d}\mu.$$

This limit is $+\infty$ if at least one of a and $1/b$ is not integrable.

The proof of Lemma 8 is left to the reader as a continuous version of Lemma 4.

Moreover, by the hypothesis of the theorem, function $g_k \in \mathcal{C}^2$ for each $k \geq 1$, the process $Z_t^{(k)}$, defined by $Z_t^{(k)} = g_k(X_t)$, satisfies SDE (14) by using Itô's lemma. We recall that

$$\mathrm{d}g_k(X_t) = g_k'(X_t)\sigma_\omega(X_t)\,\mathrm{d}B_t + [g_k'(X_t)\mu_\omega(X_t) + \tfrac{1}{2} g_k''(X_t)\sigma_\omega^2(X_t)]\,\mathrm{d}t.$$

By calculating, $\mathrm{d}Z_t^{(k)} = d_\omega(X_t)\,\mathrm{d}B_t + g_{k-1}(X_t)\,\mathrm{d}t$ for each $k \geq 1$ and where $g_0 \equiv 0$. Hence, it follows that

$$\mathbb{E}_\omega\{g_k(X_t)\} = 0 \quad \text{for each } k \geq 1. \tag{17}$$

Combining Lemma 8 and (17) we obtain

$$\left|\mathbb{E}_{\lambda,\omega}\left\{\left(\frac{X_t}{\sqrt{t}}\right)^{2k-1}\right\}\right| < \varepsilon$$

for any $\varepsilon > 0$ and for t large enough. And hence the proof of Proposition 6 is complete.

Remark 2. In our model, there is no explosion in finite time under the assumptions concerning $\sigma_\omega^2(x)$ and $\mu_\omega(x)$ for almost all $\omega \in \Omega$. Indeed, we assume that there exists $\Gamma > 0$ such that $\lim_{t\nearrow\Gamma} X_t = \infty$, where X_t is a solution of SDE (14). Lemma 7 ensures that $f_k(X_\gamma) > \Gamma^k/k!$ for some $\gamma \le \Gamma$ and $k \ge 1$. This is a contradiction since we know from (16) that

$$\mathbb{E}_\omega\{f_k(X_\gamma)\} = \frac{\gamma^k}{k!} \le \frac{\Gamma^k}{k!} \quad \text{for each } k \ge 1.$$

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