A REMARK ON COMPLETELY MONOTONIC SEQUENCES, WITH AN APPLICATION TO SUMMABILITY

Lee Lorch and Leo Moser

(received June 1, 1962)

A sequence of non-negative numbers, $\mu_0, \mu_1, \ldots, \mu_n, \ldots$, is called completely monotonic [5, p. 108] if $(-1)^n \Delta^n \mu_k \geqq 0$ for $n, k = 0, 1, 2, \ldots$. Such sequences occur in many connexions, such as the Hausdorff moment problem and Hausdorff summability [1, Chapter XI, 5, Chapters III and IV].

It is natural to inquire as to the circumstances under which the inequality "$\geqq$" above can be strengthened to "$>$". As it happens, this can be done always, except for sequences all of whose terms past the first are identical. The formal statement follows. (In it, as above, $\Delta^n \mu_k$ is the n-th forward difference, i.e., $\Delta^0 \mu_k = \mu_k$; $\Delta^n \mu_k = \Delta^{n-1} \mu_{k+1} - \Delta^{n-1} \mu_k$.)

If $\{\mu_k\}_0^\infty$ is a completely monotonic sequence, then $(-1)^n \Delta^n \mu_k > 0$, for $n, k = 0, 1, 2, \ldots$, unless $\mu_1 = \mu_2 = \ldots = \mu_n = \ldots$.

Proof. Suppose that there is a pair of integers n and k such that $\Delta^n \mu_k = 0$ and suppose that the zero elements of lowest order in $\{\Delta^n \mu_k\}_{k, n}$ have order N and that the first

Canad. Math. Bull. vol. 6, no. 2, May 1963.

https://doi.org/10.4153/CMB-1963-016-3 Published online by Cambridge University Press

such zero element in the sequence $\{\Delta^N \mu_k\}_k$ occurs for $k = K$. Consider first the case N even. The sequence $\{\Delta^N \mu_k\}$, $k = 0, 1, \ldots,$ is non-increasing and non-negative, so that $\Delta^N \mu_{K+j} = 0$, $j = 0, 1, 2, \ldots$. Then for $K > 0$, $\Delta^N \mu_{K-1} > 0$. (The case $K = 0$ leads to a polynomial sequence, which, being completely monotonic, must be bounded and hence constant.) Let the positive element $\Delta^N \mu_{K-1}$ be denoted by b. If $K > 1$, i.e., if there be a previous element, say $a = \Delta^N \mu_{K-2}$, it must be positive. Then $\Delta^{N+2m} \mu_{K-2} = \Delta^{2m} a = a - 2mb \geq 0$; $m = 1, 2, \ldots$. But this is clearly impossible, since a and b are fixed positive constants. Thus, there can be at most one element preceding the first zero in the sequence $\{\Delta^N \mu_{K+j}\}$, $j = 0, 1, \ldots,$ and the result is immediate.

The same argument holds when N is odd, so that the result is established.

This result yields an alternative proof of W. W. Rogosinski's remark [4, p. 170] that the moment sequence generating a totally regular[1] Hausdorff summation method contains no zeros.

To see this, we note that results of W. A. Hurwitz [2, esp. p. 243] show that for such a method the moment sequence is completely monotonic. If any of the moments were zero, then, by the result above, all but the first would also have to be zero, so that the method could not be regular [1, Theorem 202, p. 256].

Rogosinski's remark can be rephrased (in view of [1, Theorem 199, p. 250]) to state that the diagonal elements in the matrix of a totally regular Hausdorff summation method are all strictly positive. Thus, such methods are "normal".

---

1. A sequence-to-sequence summation method is "totally regular" if the transformed sequence of $\{s_n\}$ approaches s whenever $\{s_n\}$ does both for s finite and s infinite.

https://doi.org/10.4153/CMB-1963-016-3 Published online by Cambridge University Press

Our result on completely monotonic sequences is equivalent to one on completely monotonic functions on $(0, \infty)$ (i. e., functions $p(x)$ such that $(-1)^n p^{(n)}(x) \geqq 0$, $n = 0, 1, \ldots,$ $0 < x < \infty$), to which the transition can be made in a familiar way, using [5, Chapter IV, § 14].

The statement in question reads as follows: If $p(x)$ is completely monotonic, $0 < x < \infty$, then $(-1)^n p^{(n)}(x) > 0$, $0 < x < \infty$, unless $p(x)$ is identically constant. A direct proof of this last assertion is given in [3 § 9] where it is used as a lemma.

REFERENCES

1. G. H. Hardy, Divergent Series, Oxford, 1949.

2. W. A. Hurwitz, Some properties of methods of evaluation of divergent sequences, Proc. London Math. Soc. (2), vol. 26 (1927), pp. 231-248.

3. Lee Lorch and Peter Szego, Higher monotonicity properties of certain Sturm-Liouville functions, Acta Math., (to appear).

4. W. W. Rogosinski, On Hausdorff's methods of summability, Proc. Cambridge Philos. Soc., vol. 38 (1942), pp. 166-192.

5. D. V. Widder, The Laplace Transform, Princeton, 1941.

University of Alberta

https://doi.org/10.4153/CMB-1963-016-3 Published online by Cambridge University Press