Canad. Bull. Math. Vol. **37** (4), 1994 pp. 552–555

A TOPOLOGICAL BANACH FIXED POINT THEOREM FOR COMPACT HAUSDORFF SPACES

JURIS STEPRĀNS, STEPHEN WATSON AND WINFRIED JUST

ABSTRACT. We propose an analogue of the Banach contraction principle for connected compact Hausdorff spaces. We define a J-contraction of a connected compact Hausdorff space. We show that every contraction of a compact metric space is a J-contraction and that any J-contraction of a compact metrizable space is a contraction for some admissible metric. We show that every J-contraction has a unique fixed point and that the orbit of each point converges to this fixed point.

1. **Introduction.**

DEFINITION 1. If M is a metric space, then a contraction is a continuous mapping $T: M \longrightarrow M$ such that there is $k < 1$ such that $(\forall x, y \in M)\ \rho(Tx, Ty) \leq k\rho(x, y)$.

THEOREM 1 (BANACH). *Any contraction of a complete metric space has a unique fixed point.*

The natural generalizations of this principle lead into the theory of uniform spaces. We would like to suggest a topological approach for connected compact Hausdorff spaces.

DEFINITION 2. If $\mathcal{U}$ is an open cover of a topological space X and $T: X \longrightarrow X$ is a continuous mapping, then we say that $\mathcal{U}$ is J-contractive for T if $(\forall U \in \mathcal{U})\ (\exists U' \in \mathcal{U})\ T(\bar{U}) \subset U'$.

DEFINITION 3. If X is a compact Hausdorff space and $T: X \longrightarrow X$, then we say that T is a *J-contraction* if any open cover $\mathcal{U}$ has a finite J-contractive open refinement $\mathcal{V}$ for T.

We begin by proving some elementary facts about J-contractions.

PROPOSITION 1. *If $T: X \longrightarrow X$ is a J-contraction of a compact Hausdorff space and A is a closed subspace of T such that $T(A) \subset A$, then $T \restriction A$ is also a J-contraction.*

PROOF. Suppose that $\mathcal{U}$ is an open cover of A. Let $\mathcal{W}$ be an open cover of X whose restriction to A is $\mathcal{U}$. Suppose $\mathcal{Y}$ is a J-contractive open refinement of $\mathcal{W}$ for T. Let $\mathcal{V}$ be the restriction of $\mathcal{Y}$ to A. We claim that $\mathcal{V}$ is a J-contractive open refinement of $\mathcal{U}$ for $T \restriction A$. Suppose $V \in \mathcal{V}$. We must find $V' \in \mathcal{V}$ such that $T(\bar{V}) \subset V'$. Suppose $V = Y \cap A$, where $Y \in \mathcal{Y}$. We can find $Y' \in \mathcal{Y}$ such that $T(\bar{Y}) \subset Y'$. Now $T(\bar{V}) = T(\overline{Y \cap A}) \subset T(\bar{Y} \cap A) \subset T(\bar{Y}) \cap T(A) \subset Y' \cap A \in \mathcal{V}$. So let $V' = Y' \cap A$.

The second author has been supported by the Natural Sciences and Engineering Research Council of Canada.

The third author was partially supported by grant DMS-9016021 from the National Science Foundation.

Received by the editors April 15, 1993; revised April 5, 1994.

AMS subject classification: Primary: 54H25, 54E40; secondary: 54E45, 54D05, 54D30.

https://doi.org/10.4153/CMB-1994-081-0 Published online by Cambridge University Press

PROPOSITION 2. *If $T: X \longrightarrow X$ is a J-contraction of a compact Hausdorff space, then, for any $n > 0$, $T^n: X \longrightarrow X$ is also a J-contraction.*

PROPOSITION 3. *There is a J-contraction on any non-connected compact Hausdorff space X which has no fixed points.*

PROOF. Suppose K is clopen and we can choose $k \in K$ and $k' \in X - K$. Map K to k' and $X - K$ to k. This is a continuous mapping T with no fixed points. We shall show that T is a J-contraction. Any open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ all of whose elements are either contained in K or disjoint from K. If U is some element of $\mathcal{V}$, then $T(\bar{U})$ either equals $\{k\}$ or $\{k'\}$. Each of these is contained in some element of $\mathcal{V}$ and so we are done.

Thus to understand spaces in which J-contractions must have fixed points, we need only study connected compact Hausdorff space (*i.e.* Hausdorff continua).

Next we justify the terminology "J-contraction".

THEOREM 2. *Any contraction on a compact metric space is a J-contraction.*

PROOF. Let $T: M \longrightarrow M$ be a contraction of a compact metric space with metric μ with constant $k < 1$. Suppose $\mathcal{U}$ is an open cover of M. Suppose $\epsilon > 0$ is the Lebesgue number of $\mathcal{U}$. This means that any subset of M of diameter at most ϵ is a subset of some element of $\mathcal{U}$. Choose $F \in [M]^{<\omega}$ such that $(\forall m \in M)\,(\exists f \in F)\,\mu(m,f) < \frac{(1-k)\epsilon}{2}$. Let $\mathcal{V} = \{B_{\epsilon/2}(f) : f \in F\}$. The choice of ϵ implies that $\mathcal{V}$ is an open refinement of $\mathcal{U}$.

We shall show that for each $V \in \mathcal{V}$, $T(\bar{V})$ is a subset of some element of $\mathcal{V}$. Suppose $V = B_{\epsilon/2}(f)$. Choose $f' \in F$ such that $\mu\big(T(f), f'\big) < \frac{(1-k)\epsilon}{2}$. Suppose $x \in T(\bar{V})$. This means that $x = T(y)$ where $\mu(y,f) \leq \epsilon/2$. The definition of k says that $\mu\big(x, T(f)\big) = \mu\big(T(y), T(f)\big) \leq k \cdot \epsilon/2$. The triangle inequality implies that $\mu(x,f') < \epsilon/2$. Thus $x \in B_{\epsilon/2}(f')$ as required.

A converse to Theorem 2 also holds.

THEOREM 3. *If T is a J-contraction of a connected compact metrizable space X, then X admits a metric under which T is a contraction.*

PROOF. In 1967, Janos [2] (see also [1]) showed that if T is any continuous self-map on a compact metrizable space such that $\left|\bigcap\{T^n(X) : n \in \omega\}\right| = 1$, then X admits a metric under which T is a contraction. Theorem 5 (which does not rely on this theorem) says precisely that $\left|\bigcap\{T^n(X) : n \in \omega\}\right| = 1$, and so the proof is complete.

LEMMA 1. *If $f: X \longrightarrow X$ is an onto J-contraction and $\mathcal{U}$ is a finite J-contractive open cover of X, then there is a subcover $\mathcal{U}'$ of $\mathcal{U}$ such that* $(\exists n > 0)\,(\forall U \in \mathcal{U}')\,f^n(\bar{U}) \subset U$.

PROOF. Construct a function $\rho: \mathcal{U} \longrightarrow \mathcal{U}$ by choosing $\rho(U)$ such that $f(\bar{U}) \subset \rho(U)$. Since f is onto, all $\rho^n(\mathcal{U})$'s are covers of X. We can find $l < m$ such that $\rho^l(\mathcal{U}) = \rho^m(\mathcal{U})$. Let $\mathcal{U}'$ be the cover $\rho^l(\mathcal{U})$ and note that $\rho^{m-l}(\mathcal{U}') = \mathcal{U}'$. Thus ρ^{m-l} is an onto function on a finite set and thus a bijection. The orbits under ρ^{m-l} are thus cycles. Choose p to be a common multiple of the lengths of all these orbits. Let $n = p(m - l)$.

We now prove the main result that onto J-contractions do not exist. We use the standard notation $\mathrm{st}(A, \mathcal{V})$ to abbreviate $\bigcup\{V \in \mathcal{V} : A \cap V \neq \emptyset\}$.

https://doi.org/10.4153/CMB-1994-081-0 Published online by Cambridge University Press

PROPOSITION 4. *If $T: X \longrightarrow X$ is an onto J-contraction of a compact connected Hausdorff space, then $|X| = 1$.*

PROOF. Suppose otherwise. Let $\mathcal{U}$ be any finite J-contractive open cover for T which does not contain X. By Lemma 1, there is $n > 0$ such that, without loss of generality, $(\forall U \in \mathcal{U})\ T^n(\bar{U}) \subset U$. By Proposition 2, T^n is still an onto J-contraction. We shall assume, without loss of generality, that $(\forall U \in \mathcal{U})\ T(\bar{U}) \subset U$ where $\mathcal{U}$ is an open cover which does not contain X.

Let $U \in \mathcal{U}$. Choose $y_0 \in T(\bar{U} - U)$ which is possible since X is connected. Choose $y_1 \in \bar{U} - U$ such that $T(y_1) = y_0$. Choose y_{n+1} such that $T(y_{n+1}) = y_n$ for $n \geq 1$ which is possible since T is onto. If $y_{n+1} \in U$, then $y_n = T(y_{n+1}) \in T(U) \subset U$. Since $y_1 \notin U$, we get that $\{y_n : n \geq 1\} \subset X - U$. However $y_0 \in T(\bar{U}) \subset U$.

We claim that $\{y_n : n \in \omega\}$ are all distinct. Otherwise, there exists $m < n$ such that $y_n = y_m$. Since $T^i(y_j) = y_{j-i}$ whenever $j \geq i \geq 0$, we have $T^m(y_m) = T^m(y_n)$. Thus $y_0 = y_{n-m}$ but one is in U and the other one is not.

Let A be the closed set of all cluster points of $\{y_n : n \geq 1\}$. Now $\{y_n : n \geq 1\} \subset X - U$ and thus $y_0 \notin A$ since $A \subset X - U$.

We claim that $T(A) \subset A$. If $T(a) \notin A$ while $a \in A$, then let $T(a) \in W$ where W is an open set with $W \cap \{y_n : n \geq 1\}$ being finite. Suppose that $(\forall n \geq p)\ y_n \notin W$. Now $a \in T^{-1}(W)$. Since $a \in A$, there is $y_n \in T^{-1}(W)$ for some $n \geq 2$ and $n > p$. Thus $y_{n-1} = T(y_n) \in W$ which is a contradiction.

Find a J-contractive open cover $\mathcal{V}$ for T which contains no element intersecting both A and y_0.

We claim that $T\big(\mathrm{st}(A, \mathcal{V})\big) \subset \mathrm{st}(A, \mathcal{V})$. If $V \in \mathcal{V}$ and $a \in A \cap V$, then find $V' \in \mathcal{V}$ such that $T(V) \subset V'$. Now $T(a) \in T(V) \cap T(A) \subset V' \cap A$ so $V' \cap A \neq \emptyset$ as required.

Thus there is n such that $y_n \in \mathrm{st}(A, \mathcal{V})$ so that $y_0 = T^n(y_n) \subset \mathrm{st}(A, \mathcal{V})$ which is impossible.

THEOREM 4. *If T is a J-contraction of any connected compact Hausdorff space X, then T has a unique fixed point.*

PROOF. Let $\mathcal{A}$ be the family of all nonempty continua A of X such that $T(A) \subset A$. Let $\mathcal{A}'$ be a maximal decreasing subfamily of $\mathcal{A}$ such that every element of $\mathcal{A}'$ contains all fixed points of T. Let $B = \bigcap \mathcal{A}'$. Since the decreasing intersection of nonempty continua is a nonempty continuum, we know that B is a continuum. If $a \in B$, then $(\forall A \in \mathcal{A}')T(a) \in T(A) \subset A$ and thus $T(a) \in B$. Thus $B \in \mathcal{A}'$.

If $T(B)$ is a proper subset of B, then since $T\big(T(B)\big) \subset T(B)$, we can add $T(B)$ to $\mathcal{A}'$ which contradicts maximality. Thus $T(B) = B$, and Proposition 4 and Proposition 1 imply that $|B| = 1$.

THEOREM 5. *If T is a J-contraction of any connected compact Hausdorff space X, and $x \in X$, then $\{T^n(x) : n \in \omega\}$ converges to the unique fixed point.*

PROOF. First notice that if T is a J-contraction with fixed point p of any compact Hausdorff space X and U is an open neighborhood of p, then there is an open neighborhood V of p such that $V \subset U$ and $T(\bar{V}) \subset V$.

To see this, let $\mathcal{S}$ be a finite J-contractive open refinement of $\{U, X - \{p\}\}$ for T. Suppose $R, R' \in \mathcal{S}$. If $p \in R$ and $T(\bar{R}) \subset R'$, then $p \in R'$. Let $V = \bigcup\{R \in \mathcal{S} : p \in R\}$.

Now suppose $x \in X$ and U is an open neighborhood of the unique fixed point p. We shall show that

$$(\exists n \in \omega)\,(\forall m > n)\, T^m(x) \in U.$$

Let $V \subset U$ be such that $T(\bar{V}) \subset V$ and $p \in V$. Take a finite J-contractive open refinement $\mathcal{R}$ of $\{V, X - T(\bar{V})\}$ for T. Let $\mathcal{S} = \{R \in \mathcal{R} : R \not\subset V\} \cup \{V\}$. The open cover $\mathcal{S}$ is J-contractive as well. Note that $R \in \mathcal{S}$ and $R \neq V$ implies that $R \cap T(\bar{V}) = \emptyset$. List $\mathcal{S} = \{S_0, \ldots, S_{n-1}\}$ where $n \in \omega$ and $S_0 = V$. Define $\pi : n \longrightarrow n$ such that $T(\overline{S_i}) \subset S_{\pi(i)}$.

Next notice that if $S_i \cap S_0 \neq \emptyset$, then $\pi(i) = 0$. To see this, let $x \in S_i \cap S_0 = S_i \cap V$. Thus $T(x) \in T(S_i) \cap T(V) \subset S_{\pi(i)} \cap T(\bar{V})$ and so $S_{\pi(i)} = V$ and $\pi(i) = 0$.

In fact, if $n \in \omega$ and $|\operatorname{ran}(\pi^n)| > 1$, then there is a nonzero $j \in \operatorname{ran}(\pi^n)$ such that $S_j \cap S_0 \neq \emptyset$. To see this, note that $T^n(X) = \bigcup\{S_j : j \in \operatorname{ran}(\pi^n)\}$ is connected and is the union of the two open sets S_0 and $\bigcup\{S_j : j \in \operatorname{ran}(\pi^n); j \neq 0\}$. Thus these open sets are not disjoint.

We prove, by induction, that $(\forall i < n)\, |\operatorname{ran}(\pi^i)| \leq n - i$. Fix $i < n - 1$ and assume that $|\operatorname{ran}(\pi^i)| \leq n - i$. We may also assume that $|\operatorname{ran}(\pi^i)| > 1$ since $|\operatorname{ran}(\pi^{i+1})| \leq |\operatorname{ran}(\pi^i)|$. We can choose a nonzero $j \in \operatorname{ran}(\pi^i)$ such that $S_j \cap S_0 \neq \emptyset$. Thus $\pi(j) = 0$ and $|\operatorname{ran}(\pi^{i+1})| = \left|\pi\big(\operatorname{ran}(\pi^i - \{0,j\})\big) \cup \{\pi(j), \pi(0)\}\right| \leq (n - i - 2) + 1 = n - (i+1)$ as required.

Thus we deduce that $|\operatorname{ran}(\pi^{n-1})| = 1$ and thus that $T^n(X) = \bigcup\{T^n(S_j) : j \in n\} \subset \bigcup\{S_{\pi^n(j)} : j \in n\} = S_0 = V$. Now since $m \geq n \Rightarrow T^m(X) = T^{m-n}\big(T^n(X)\big) \subset T^{m-n}(V) \subset V \subset U$, the proof is complete.

We thank Jeff Connor for informing us of Janos' results.

PROBLEM 1. Can the definition of J-contraction be generalized to non-compact non-metrizable spaces (for example, paracompact Čech-complete spaces)?

REFERENCES

1. M. Edelstein, *A shorter proof of Janos' theorem*, Proc. Amer. Math. Soc. **20**(1969), 509–510.
2. L. Janos, *A converse to Banach's contraction mapping theorem*, Proc. Amer. Math. Soc. **18**(1967), 287–289.

Department of Mathematics
Ohio University
Athens, Ohio 45701-2979

Department of Mathematics
Ohio University
Athens, Ohio 45701-2979

Department of Mathematics
York University
North York, Ontario
M3J 1P3