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Local Theory of Primitive Theta Functions

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Abstract. We study primitive theta functions, which were first introduced by Shintani, in a purely local setting. We investigate a metaplectic representation of U(1) acting on the space of local primitive theta functions and give its explicit irreducible decomposition. As a by-product, we give a new proof of epsilon dichotomy for (U(1), U(1)).

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0. Introduction

0.1. The primitive theta functions studied in this paper were first introduced by Shintani in his unpublished work ([Sh]) on automorphic forms on $U(2,1)$, a quasi-split unitary group of degree 3. One of his main results says that a holomorphic automorphic form f on $U(2,1)$ can be expressed as a linear combination of primitive theta functions and their shifts (*refined Fourier–Jacobi expansion*). Furthermore, he showed that suitable quotients of coefficients in the expansion of a Hecke eigenform f are described in terms of the Satake parameters attached to f. This strongly suggests that primitive theta functions will play an important role in the arithmetic of automorphic forms on $U(2,1)$.

The main object of this paper is to study primitive theta functions in a purely local setting. We are mainly concerned with a metaplectic representation of U(1) acting on the space of local primitive theta functions. In particular, we give an explicit irreducible decomposition of the space of primitive theta functions. The key ingredient in the proof is a trace formula for the metaplectic representation.

In a forthcoming paper, we will give an application of our local results to the theory of Shintani's refined Fourier–Jacobi expansion.

0.2. We now explain the content of the paper and summarize the main results. The first three sections are of a preliminary nature. In Section 1, we define a Heisenberg group H and construct a smooth irreducible representation of H on a lattice model. Let F be a finite extension of the p-adic number field $\mathbf{Q}_p$. Let K be either a direct sum $F \oplus F$ or a quadratic extension of F, and fix an element $\kappa \in K^\times$ satisfying $\overline{\kappa} = -\kappa$, where $z \mapsto \overline{z}$ is the nontrivial automorphism of K/F. We define a nondegenerate alternating form $\langle\ ,\ \rangle : K \times K \to F$ by $\langle w, w' \rangle = Tr_{K/F}(\kappa \overline{w} w')$. Let H be $K \times F$ with multiplication law $(w, t)(w', t') = (w + w', t + t' + \langle w, w' \rangle / 2)$. Fix a nontrivial additive character ψ of F and take a lattice $\mathcal{L}$ of K that is self-dual with respect to the pairing $(w, w') \mapsto \psi(\langle w, w' \rangle)$ and satisfies $\frac{1}{2}(l + \overline{l}) \in \mathcal{L}$ for $l \in \mathcal{L}$. We realize an irreducible smooth representation ρ of H with central character $(0, t) \mapsto \psi(t)$ on the lattice model

$$V = \{\Phi \in \mathcal{S}(K) \mid \Phi(z + l) = \psi\left(\frac{1}{2}\langle z, l \rangle + \frac{1}{4}\langle l, \overline{l} \rangle\right) \Phi(z) \ \ (z \in K, l \in \mathcal{L})\}$$

by

$$(\rho(w, t)\,\Phi)(z) = \psi\left(\frac{1}{2}\langle z, w \rangle + t\right) \Phi(z + w) \qquad (z \in K, (w, t) \in H, \Phi \in V).$$

In Section 2, we define a metaplectic representation M of $K^1 = \{u \in K^\times \mid u\overline{u} = 1\}$ on V after [MVW]. For $u \in K^1$ and $\Phi \in V$, we put $M(u)\Phi = \Phi$ if $u = 1$ and

$$M(u)\Phi = \left|N_{K/F}(1 - u)\right|_F^{1/2} \int_K \psi\left(\frac{1}{2}\langle w, uw \rangle\right) \rho((1 - u)w, 0)\Phi \, \mathrm{d}w$$

if $u \neq 1$, where $|\cdot|_F$ denotes the normalized valuation of F and $\mathrm{d}w$ the Haar measure on K self-dual with respect to the pairing $(w, w') \mapsto \psi(\langle w, w' \rangle)$.

In Section 3, we recall definitions and basic properties of several local constants (Weil constants, Gauss sums and epsilon factors), which are frequently used in later discussions.

The object of Section 4 is to give a splitting of $M(u)$. For $z \in K^\times$, we put

$$\gamma(z) = \begin{cases} \omega(z) & \cdots \quad z \in F^\times, \\ \lambda_K(\psi)^{-1}\, \omega\left(\dfrac{z - \overline{z}}{\kappa}\right) & \cdots \quad z \in K^\times - F^\times, \end{cases}$$

where ω is the quadratic character of $F^\times$ associated with K/F and $\lambda_K(\psi)$ the Weil constant associated with $z \mapsto \psi(z\overline{z})$ (cf. Section 3.2). For $z \in K^\times$ and $\Phi \in V$, we set

$$\mathcal{M}(z)\,\Phi = \gamma(z) \cdot M(\overline{z}/z)\,\Phi.$$

We show that $z \mapsto \mathcal{M}(z)$ is a smooth representation of $K^\times$ on V (Theorem 4.5) by calculating the cocycle of M explicitly.

In Section 5, we define the space of local primitive theta functions. Let $\mathfrak{a} = \alpha\mathcal{O}_K$ $(\alpha \in K^\times)$ be a nonzero fractional ideal of K. Let $H(\mathfrak{a}) = \{(w, t) \in H \mid w \in \mathfrak{a}, t + \kappa w\overline{w}/2 \in \left(\kappa/(\theta - \overline{\theta})\right) N_{K/F}(\mathfrak{a}) \cdot \mathcal{O}_K\}$, where $\{1, \theta\}$ is an $\mathcal{O}_F$-basis of $\mathcal{O}_K$. Let $V(\mathfrak{a})$ be the $H(\mathfrak{a})$-invariant subspace of V. Then $V(\mathfrak{a}) = \{0\}$ unless $\mu_{\mathfrak{a}} := n_\psi + \mathrm{ord}_F(\kappa/(\theta - \overline{\theta})) + \mathrm{ord}_F N(\alpha) \geqslant 0$, where $n_\psi \in \mathbf{Z}$ is defined in Section 1.1. Denote by $\mathcal{P}_{\mathfrak{a}} \in \mathrm{End}(V)$ the projection operator of $V(\mathfrak{a})$. Suppose that $\mu_{\mathfrak{a}} \geqslant 0$.

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We call

$$V_{\rm prim}(\mathfrak{a}) = \{\Phi \in V(\mathfrak{a}) \mid \mathcal{P}_{\mathfrak{b}}\Phi = 0 \ \textit{for any} \ \mathfrak{b} \in R(\mathfrak{a}), \mathfrak{b} \neq \mathfrak{a}\}$$

the space of *primitive theta functions* in $V(\mathfrak{a})$, where $R(\mathfrak{a})$ is the set of fractional ideals $\mathfrak{b}$ of K satisfying $\mathfrak{a} \subset \mathfrak{b}$ and $\mu_{\mathfrak{b}} \geqslant 0$. (In the case where $\delta_{K/F} > 0$ and $\mu_{\mathfrak{a}} = 0$, the above condition is replaced by $\mathcal{Q}\Phi = 0$, where $\mathcal{Q}$ is defined by (5.6)).

Let Γ_K be $F^\times \cdot \mathcal{O}_K^\times$ if $K = F \oplus F$, and $K^\times$ if K is a field. Then $z \mapsto \mathcal{M}(z)|V_{\rm prim}(\mathfrak{a})$ defines a unitary representation $\mathcal{M}_{\mathfrak{a},{\rm prim}}$ of Γ_K on $V_{\rm prim}(\mathfrak{a})$. A unitary representation $\mathcal{M}_{\mathfrak{a}}$ of Γ_K on $V(\mathfrak{a})$ is similary defined. The representation $\mathcal{M}_{a,{\rm prim}}$ decomposes as a direct sum of unitary characters of Γ_K. Shintani ([Sh]) gave the irreducible decomposition of $\mathcal{M}_{\mathfrak{a},{\rm prim}}$ in the case where $F = \mathbf{Q}_p$ and $K = \mathbf{Q}(\sqrt{-1}) \otimes_{\mathbf{Q}} \mathbf{Q}_p$ by calculating the trace of $\mathcal{M}_{\mathfrak{a},{\rm prim}}$. Later Glauberman and Rogawski ([GR]) showed that $\mathcal{M}_{\mathfrak{a},{\rm prim}}$ is multiplicity-free in the general case and gave its irreducible decomposition *up to a character* except in the ramified case. A basis of V consisting of eigenfunctions of the metaplectic representation was given by Y. Kato ([Ka]) in a special case and by T. Yang ([Ya]) in the odd residual characteristic case.

The main purpose of the paper is to give an explicit irreducible decomposition of $\mathcal{M}_{\mathfrak{a},prim}$ and $\mathcal{M}_{\mathfrak{a}}$ by calculating their traces explicitly. In Section 6, we state the main results of the paper (Theorem 6.4 and Theorem 6.6). In the case where K is a field, they are summarized as follows. For a unitary character χ of $K^\times$, let $a(\chi)$ be the smallest nonnegative integer a such that χ is trivial on $(1 + \mathfrak{P}_K^a) \cap \mathcal{O}_K^\times$, where $\mathfrak{P}_K$ is the maximal ideal of $\mathcal{O}_K$. Put $\delta_{K/F} = {\rm ord}_F\, N(\theta - \overline{\theta})$ and $\psi_K = \psi \circ {\rm Tr}_{K/F}$. Let $\varepsilon(\chi, \psi_K)$ be the epsilon factor defined by Tate (cf. Section 3.6).

THEOREM. *Suppose that K is a field.*

(i) *A unitary character χ of $K^\times$ appears in V if and only if $\chi|_{F^\times} = \omega$ and $\varepsilon(\chi, \psi_K) = \chi(\kappa^{-1})$, and its multiplicity is equal to one.*
(ii) *Suppose that χ satisfies the conditions in* (i). *Then χ appears in $V_{prim}(\mathfrak{a})$* (*resp. $V(\mathfrak{a})$*) *if and only if $a(\chi) = \mu'_{\mathfrak{a}}$ (resp. $a(\chi) \leqslant \mu'_{\mathfrak{a}}$), where*

$$\mu'_{\mathfrak{a}} = \begin{cases} \mu_{\mathfrak{a}} & \cdots \quad \delta_{K/F} = 0\,, \\ 2(\mu_{\mathfrak{a}} + \delta_{K/F}) & \cdots \quad \delta_{K/F} > 0\,. \end{cases}$$

Remark. The first assertion of the theorem is known as 'epsilon dichotomy' for the dual pair (U(1), U(1)). This result was first proven by Moen ([Mo]) in the case where the residual characteristic of F is odd. Later Rogawski ([Ro]) completed the proof in the even residual characteristic case by combining Moen's result and a global method (the trace formula for U(3)). For epsilon dichotomy in a more general situation, we refer to [HKS], where the first assertion of the theorem is proved by a local method entirely different from ours. We also note that the second assertion of the theorem is important in an application to the theory of Fourier–Jacobi expansions of automorphic forms on U(2, 1).

In Sections 7–9, we prove Theorem 6.4 and Theorem 6.6 by employing Shintani's idea of calculating the trace of the metaplectic representation. In Section 7, we calculate $X_{\mathfrak{a}}(z) = \mathrm{Tr}(\mathcal{M}_{\mathfrak{a}}(z))$ and $X_{\mathfrak{a},\mathrm{prim}}(z) = Tr(\mathcal{M}_{\mathfrak{a},\mathrm{prim}}(z))$ in an explicit form. One of the advantages of realizing $\mathcal{M}$ on a lattice model is that we have a convenient trace formula for $\mathcal{M}_{\mathfrak{b}}(z)$ ($\mathfrak{b} \in R(\mathfrak{a})$) (cf. Proposition 7.3). The proofs of Theorem 6.4 and Theorem 6.6 are done by comparing the explicit formulas for $X_{\mathfrak{a}}$ and $X_{\mathfrak{a},\mathrm{prim}}$ with certain summation formulas for characters proved in Section 8 (in the unramified case) and in Section 9 (in the ramified case).

In Section 10, we reformulate Shintani's result on the inner structure of $V(\mathfrak{a})$ in our local setting. Namely, we show that each element of $V(\mathfrak{a})$ can be written as a sum of a primitive theta function and 'shifts' of primitive ones of 'lower index'.

NOTATION

For a vector space V over $\mathbf{C}$, Id_V denotes the identity operator on V. The cardinality of a finite set X is denoted by $\#(X)$. For a locally compact Abelian group G, let $G^\wedge$ stand for the group of unitary characters of G.

1. Heisenberg Group and Lattice Model

1.1. Let F be a finite extension of $\mathbf{Q}_p$ and π a prime element of F. Denote by $\mathcal{O}_F$ and $\mathfrak{p}_F = \pi\mathcal{O}_F$ the integer ring of F and the maximal ideal of $\mathcal{O}_F$, respectively. Put $q = \#(\mathcal{O}_F/\mathfrak{p}_F)$. For $a \in F^\times$, we put $|a|_F = q^{-\mathrm{ord}_F(a)}$, where $\mathrm{ord}_F\colon F^\times \to \mathbf{Z}$ is the additive valuation normalized by $\mathrm{ord}_F(\pi) = 1$. We normalize the Haar measure dx on F so that the volume of $\mathcal{O}_F$ is equal to 1. Throughout the paper, we fix a nontrivial additive character ψ of F. Let n_ψ be the largest integer n such that $\psi|_{\mathfrak{p}_F^{-n}} = 1$.

1.2. Let K be a commutative semisimple algebra over F with $\dim_F K = 2$. Then K is either isomorphic to $F \oplus F$ or a quadratic extension of F. We henceforth identify K with $F \oplus F$ in the former case. Let

$$\mathcal{O}_K = \begin{cases} \mathcal{O}_F \oplus \mathcal{O}_F & \cdots \quad K = F \oplus F, \\ \text{the integer ring of } K & \cdots \quad K \text{ is a field}. \end{cases}$$

Let $z \mapsto \bar{z}$ be the unique nontrivial automorphism of K/F. For $z \in K$, we put $Tr_{K/F}(z) = z + \bar{z}$, $N_{K/F}(z) = z\bar{z}$ and $|z|_K = |N_{K/F}(z)|_F$. If there is no fear of confusion, we write $N(z)$ for $N_{K/F}(z)$.

1.3. Fix an element κ of $K^\times$ with $\bar{\kappa} = -\kappa$ and define a nondegenerate alternating form $\langle\, ,\, \rangle\colon K \times K \to F$ by

$$\langle w, w' \rangle = Tr_{K/F}(\kappa \bar{w} w').$$

Let H be the Heisenberg group associated with the symplectic space $(K, \langle\ ,\ \rangle)$. By definition, the underlying set of H is $K \times F$ and the multiplication law is given by

$$(w, t)\,(w', t') = (w + w', t + t' + \frac{1}{2}\langle w, w'\rangle).$$

The center of H is $\{(0, t) \mid t \in F\}$.

1.4. Throughout the paper, by a lattice of K we always mean an $\mathcal{O}_F$-lattice of K. For a lattice L of K, the dual L^* of L (with respect to $\psi(\langle\ ,\ \rangle)$) is defined by $L^* = \{z \in K \mid \psi(\langle z, w\rangle) = 1$ for any $w \in L\}$. We say that L is self-dual with respect to $\psi(\langle\ ,\ \rangle)$ if $L = L^*$. Let dz be the Haar measure on K self-dual with respect to the pairing $(z, w) \mapsto \psi(\langle z, w\rangle)$. Then $\int_L \mathrm{d}z = 1$ for any lattice L self-dual with respect to $\psi(\langle\ ,\ \rangle)$.

1.5. We take a lattice $\mathcal{L}$ of K satisfying the following two conditions:

(i) $\mathcal{L}$ is self-dual with respect to $\psi(\langle 1, 2\rangle)$.
(ii) $l \in \mathcal{L} \Longrightarrow \frac{1}{2}(l + \bar{l}) \in \mathcal{L}$.

For example, $\mathcal{L} = \kappa^{-2}\mathfrak{p}_F^{-n_\psi} + (\kappa/2)\,\mathcal{O}_F$ satisfies these conditions. Define

$$\widetilde{\psi}((l, t)) = \psi\left(\frac{1}{4}\langle l, \bar{l}\rangle + t\right)$$

for $(l, t) \in H(\mathcal{L}) = \mathcal{L} \times F$. Then $\widetilde{\psi}$ is a character of $H(\mathcal{L})$. By general theory (cf. [MVW, Ch.2, I.3]), $\mathrm{Ind}_{H(\mathcal{L})}^{H}\widetilde{\psi}$ is a smooth irreducible representation of H with central character $(0, t) \mapsto \psi(t)$. Recall that such a representation is unique up to equivalence. The representation $\mathrm{Ind}_{H(\mathcal{L})}^{H}\widetilde{\psi}$ is identified with the *lattice model* (V, ρ) with respect to $\mathcal{L}$ given by

$$V = \left\{\Phi \in \mathcal{S}(K) \mid \Phi(z + l) = \psi\left(\frac{1}{2}\langle z, l\rangle + \frac{1}{4}\langle l, \bar{l}\rangle\right)\Phi(z) \quad (z \in K, l \in \mathcal{L})\right\},$$

$$(\rho(h)\Phi)(z) = \psi\left(\frac{1}{2}\langle z, w\rangle + t\right)\Phi(z + w) \qquad (h = (w, t) \in H, z \in K, \Phi \in V),$$

where $\mathcal{S}(K)$ denotes the space of locally constant compactly supported functions on K. Note that $\psi(\langle l, \bar{l}\rangle/4) = 1\,(\text{resp.}\ \pm 1)$ for $l \in \mathcal{L}$ if the residual characteristic of F is odd (resp. even). We define an inner product on V by

$$(\Phi, \Phi') = \int_K \Phi(z)\,\overline{\Phi'(z)}\,\mathrm{d}z \qquad (\Phi, \Phi' \in V).$$

It is easy to see that $\rho(h)$ $(h \in H)$ is a unitary operator with respect to the inner product. In what follows, we write $\rho(w, t)$ for $\rho((w, t))$ if there is no fear of confusion.

Remark. In [MVW, Ch.2, II.8], the lattice model is defined in the odd residual characteristic case. To define the lattice model in the even residual characteristic case, we need the factor $\langle l, \bar{l}\rangle/4$ in the definition of V.

1.6. Let Φ^0 be an element of V such that the support of Φ^0 is contained in $\mathcal{L}$ and $\Phi^0(0) = 1$. Then

$$\Phi^0(z) = \begin{cases} \psi\left(\frac{1}{4}\langle z, \bar{z}\rangle\right) & \cdots \quad z \in \mathcal{L}, \\ 0 & \cdots \quad z \in K - \mathcal{L}. \end{cases}$$

If $\Phi \in V$ satisfies $\rho(l, 0)\,\Phi = \psi(\langle l, \bar{l}\rangle/4)\,\Phi$ for any $l \in \mathcal{L}$, then Φ is a scalar multiple of Φ^0. The following elementary fact is crucial for the trace formula in Section 7.

1.7. LEMMA *For* $\Phi \in V$, *we have* $\Phi(z) = (\rho(z, 0)\Phi, \Phi^0)$ $(z \in K)$.

2. Metaplectic Representation

2.1. Let $K^1 = \{u \in K^\times \mid u\bar{u} = 1\}$ act on H by $u \cdot (w, t) = (uw, t)$ $(u \in K^1$, $(w, t) \in H)$. For $u \in K^1$, we define $M(u) \in \mathrm{End}(V)$ as follows (cf. [MVW, Ch.2, II.2]). If $u = 1$, we put $M(u) = \mathrm{Id}_V$. If $u \neq 1$, we put

$$\begin{aligned}(M(u)\Phi)(z) &= |1 - u|_K^{1/2} \int_K \psi\left(\frac{1}{2}\langle w, uw\rangle\right)(\rho((1 - u)w, 0)\Phi)(z)\,\mathrm{d}w \\ &= |1 - u|_K^{-1/2} \int_K \psi(\tau_u w\bar{w} + \frac{1}{2}\langle z, w\rangle)\,\Phi(z + w)\,\mathrm{d}w \qquad (\Phi \in V, z \in K),\end{aligned}$$

where

$$\tau_u = \frac{1}{2}\mathrm{Tr}_{K/F}\left(\frac{\kappa}{1 - u}\right) = \frac{\kappa}{2}\frac{1 + u}{1 - u}.$$

2.2. LEMMA. *Let* $u \in K^1$.

(i) $M(u) \circ \rho(h) = \rho(u \cdot h) \circ M(u)$ $(h \in H)$.
(ii) $(M(u)\Phi, \Phi') = (\Phi, M(u^{-1})\Phi')$ $(\Phi, \Phi' \in V)$.
(iii) $M(u) \circ M(u^{-1}) = \mathrm{Id}_V$.

Proof. The first and second assertions are easily verified. The third one is obvious when $u = 1$. Let $u \in K^1 - \{1\}$, and fix $z \in K$ and $\Phi \in V$. Take sufficiently large lattices L, L' of K, such that

$$uL \subset L', \quad \frac{u}{u^{-1} - 1}L^* \subset L', \quad \mathrm{Supp}(M(u^{-1})\Phi) \subset z + (1 - u)L,$$
$$\mathrm{Supp}(\Phi) \subset z + (1 - u)L + (1 - u^{-1})L'.$$

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Then we have

$$\begin{aligned}
&|1-u|_K^{-1}\, M(u)M(u^{-1})\Phi(z)\\
&= \int_L dw \int_{L'} dw'\ \psi\Big(\frac{1}{2}\langle w, uw\rangle + \frac{1}{2}\langle z, (1-u)w\rangle + \frac{1}{2}\langle w', u^{-1}w'\rangle + \\
&\qquad + \frac{1}{2}\langle z+(1-u)w, (1-u^{-1})w'\rangle\Big)\, \Phi(z+(1-u)w+(1-u^{-1})w')\\
&= \int_{L'} \Big(\int_L \psi(\langle w, u^{-1}(u^{-1}-1)w'\rangle)\, dw\Big)\, \psi\Big(\frac{1}{2}\langle w', u^{-1}w'\rangle + \frac{1}{2}\langle z, (1-u^{-1})w'\rangle\Big) \times\\
&\qquad \times \Phi(z+(1-u^{-1})w')\, dw'\\
&= \mathrm{vol}(L) \int_{u(u^{-1}-1)^{-1}L^*} \psi\Big(\frac{1}{2}\langle w', u^{-1}w'\rangle + \frac{1}{2}\langle z, (1-u^{-1})w'\rangle\Big) \times\\
&\qquad \times \Phi(z+(1-u^{-1})w')\, \mathrm{d}w',
\end{aligned}$$

where $\mathrm{vol}(L) = \int_L \mathrm{d}w$. Since we can take L^* sufficiently small, the above is equal to

$$\mathrm{vol}(L)\cdot \mathrm{vol}\Big(\frac{u}{u^{-1}-1}L^*\Big)\, \Phi(z) = |1-u|_K^{-1}\, \Phi(z).$$

This completes the proof of the lemma. □

2.3. In view of Lemma 2.2 and the Stone–von Neumann theorem (cf. [MVW, Ch.2, I.2]), the mapping $u \mapsto M(u)$ defines a projective representation of K^1 on V. Namely, we have

$$M(u)\circ M(u') = c(u,u')\, M(uu') \qquad (u, u' \in K^1)$$

with $c(u,u') \in \mathbf{C}^\times$.

Remark. For $u \in K^1$ and $\Phi \in V$, put

$$M'(u)\Phi(z) = \int_{\mathcal{L}} \psi\Big(\frac{1}{2}\langle l, z\rangle + \frac{1}{4}\langle l, \bar{l}\rangle\Big)\, \Phi(u^{-1}(z+l))\, \mathrm{d}l \quad (z \in K).$$

Then $M'(u)$ defines an endomorphism of V satisfying the property of Lemma 2.2 (i). If the residual characteristic of F is odd, it is known that $M'(u) \in GL(V)$ and hence $u \mapsto M'(u)$ defines a metaplectic representation of K^1 on V (cf. [MVW, Ch.2, II.8]). On the other hand, in the even residual characteristic case, there exists $u \in K^1$ such that $M'(u) = 0$.

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3. Local Constants

3.1. In this section, we recall the definition of several local constants and their basic properties. Let $\{1, \theta\}$ be an $\mathcal{O}_F$-basis of $\mathcal{O}_K$ and put

$$\delta_{K/F} = \mathrm{ord}_F\, N(\theta - \overline{\theta})\,. \tag{3.1}$$

Then $\delta_{K/F} > 0$ if and only if K/F is a ramified quadratic extension.

3.2. We first recall the definition of Weil constants. Let

$$\widehat{f}(z) = \int_K f(w)\,\psi(\langle w, z\rangle)\,\mathrm{d}w$$

be the Fourier transform of $f \in \mathcal{S}(K)$. Due to Weil ([We]), there exists a nonzero complex number $\lambda_K(\psi)$ such that

$$\int_K f(z)\,\psi(az\overline{z})\,\mathrm{d}z = \lambda_K(\psi)\cdot\omega(a)|a|_F^{-1}\,|\kappa|_K^{1/2}\int_K \widehat{f}(z)\,\psi(a^{-1}\kappa^2 z\overline{z})\,\mathrm{d}z \tag{3.2}$$

holds for any $f \in \mathcal{S}(K)$ and $a \in F^\times$. Here ω denotes the quadratic character of $F^\times$ associated with K/F by local class field theory. Then $\lambda_K(\psi)^2 = \omega(-1)$. The following fact is well-known.

3.3. LEMMA. *Let $a \in F^\times$.*

(i) *Suppose that $\delta_{K/F} = 0$. Then $\lambda_K(\psi) = \omega(\pi^{n_\psi})$.*

(ii) *Suppose that $\delta_{K/F} > 0$. Then*

$$\int_{\mathcal{O}_F^\times} \omega(t)\,\psi(at)\,dt = \begin{cases} q^{-\delta_{K/F}/2}\,\omega(a)\,\lambda_K(\psi) & \cdots \quad \mathrm{ord}_F(a) = -n_\psi - \delta_{K/F}, \\ 0 & \cdots \quad \text{otherwise.} \end{cases}$$

(iii) *There exists a lattice L_0 of K such that, for any lattice L of K containing L_0, the integral $\int_L \psi(az\overline{z})\,\mathrm{d}z$ takes the value $\omega(a)|a|_F^{-1}\,|\kappa|_K^{1/2}\,\lambda_K(\psi)$.*

3.4. We next define a Gauss sum. Throughout the paper, by an ideal $\mathfrak{a}$ of K, we always mean a nonzero fractional ideal of K (namely, $\mathfrak{a} = \alpha\mathcal{O}_K$ for some $\alpha \in K^\times$). We put $\mathrm{ord}_F\, N(\mathfrak{a}) = \mathrm{ord}_F\, N(\alpha)$ and $|N(\mathfrak{a})|_F = |N(\alpha)|_F$. Let $d_{\mathfrak{a}}z$ be the Haar measure on K normalized by $\int_{\mathfrak{a}} \mathrm{d}_{\mathfrak{a}}z = 1$. Note that

$$\mathrm{d}_{\mathfrak{a}}z = q^{\mu_{\mathfrak{a}} + \delta_{K/F}}\,\mathrm{d}z\,, \tag{3.3}$$

where we put

$$\mu_{\mathfrak{a}} = n_\psi + \mathrm{ord}_F \frac{\kappa}{\theta - \overline{\theta}} + \mathrm{ord}_F\, N(\mathfrak{a})\,. \tag{3.4}$$

For $a \in F^\times$, we define a Gauss sum

$$S_{\mathfrak{a}}(a) = \int_{\mathfrak{a}} \psi(az\bar{z})\,\mathrm{d}_{\mathfrak{a}}z\,. \tag{3.5}$$

By (3.2), we obtain

3.5. LEMMA.

$$S_{\mathfrak{a}}(a) = \begin{cases} 1 & \cdots \quad \mathrm{ord}_F(a) \geqslant -\mathrm{ord}_F N(\mathfrak{a}) - n_\psi\,, \\ q^{n_\psi+\delta_{K/F}/2}\,|N(\mathfrak{a})|_F^{-1}\,\omega(a)|a|_F^{-1}\,\lambda_K(\psi) & \cdots \quad \mathrm{ord}_F(a) \leqslant -\mathrm{ord}_F N(\mathfrak{a}) - n_\psi - \delta_{K/F}\,, \\ 0 & \cdots \quad \text{otherwise}\,. \end{cases}$$

3.6 Let $\mathfrak{P}_K$ be $\pi\mathcal{O}_K$ if $K = F \oplus F$ and the maximal ideal of $\mathcal{O}_K$ otherwise, and put $q_K = \#(\mathcal{O}_K/\mathfrak{P}_K)$. When K is a field, let $\mathrm{ord}_K\colon K^\times \to \mathbf{Z}$ be the additive valuation of K normalized by $q_K^{-\mathrm{ord}_K(z)} = |N_{K/F}(z)|_F$ for $z \in K^\times$. Set $\psi_K = \psi \circ \mathrm{Tr}_{K/F}$ and denote by n_{ψ_K} the largest integer n such that ψ_K is trivial on $\mathfrak{P}_K^{-n}$. Then

$$n_{\psi_K} = \begin{cases} n_\psi & \cdots \quad \delta_{K/F} = 0\,, \\ 2n_\psi + \delta_{K/F} & \cdots \quad \delta_{K/F} > 0\,. \end{cases} \tag{3.6}$$

We put

$$\Gamma_K = \begin{cases} F^\times \cdot \mathcal{O}_K^\times & \cdots \quad K = F \oplus F\,, \\ K^\times & \cdots \quad K \text{ is a field}\,. \end{cases} \tag{3.7}$$

For $\chi \in (\Gamma_K)^\wedge$, let $a(\chi)$ be the smallest nonnegative integer a such that χ is trivial on $(1 + \mathfrak{P}_K^a) \cap \mathcal{O}_K^\times$. We set

$$\varepsilon(\chi, \psi_K) = \chi(c)\frac{S}{|S|}, \quad S = \int_{\mathcal{O}_K^\times} \chi^{-1}(u)\,\psi_K\Big(\frac{u}{c}\Big)\,\mathrm{d}_{\mathcal{O}_K}u\,, \tag{3.8}$$

where c is an element of $K^\times$ with $\mathrm{ord}_K(c) = a(\chi) + n_{\psi_K}$ if K is a field, and $c = \pi^{a(\chi)+n_\psi}(\theta - \bar{\theta})$ if $K = F \oplus F$. Note that $\varepsilon(\chi, \psi_K)$ is the epsilon factor defined by Tate (cf. [Ta]) when K is a field. It is known that $|S| = q_K^{-a(\chi)/2}$ if $a(\chi) > 0$ and that

$$\varepsilon(\chi, \psi_K)\,\varepsilon(\chi^{-1}, \psi_K) = \chi(-1)\,. \tag{3.9}$$

3.7. PROPOSITION. *Let $\chi \in (\Gamma_K)^\wedge$ and suppose that $\chi|_{F^\times} = \omega$.*

(i) *We have $\varepsilon(\chi, \psi_K) = \pm\chi(\kappa^{-1})$.*

(ii) *If $K = F \oplus F$, we have $\varepsilon(\chi, \psi_K) = \chi(\kappa^{-1})$.*

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(iii) *If K/F is an unramified quadratic extension, we have*

$$\varepsilon(\chi, \psi_K) = (-1)^{a(\chi)+n_\psi+\mathrm{ord}_F(\kappa/(\theta-\bar{\theta}))}\,\chi(\kappa^{-1})$$

Proof. By the assumption $\chi|_{F^\times} = \omega$, we have $\varepsilon(\chi, \psi_K) = \varepsilon(\chi^{-1}, \psi_K)$. By (3.9), we obtain $\varepsilon(\chi, \psi_K)^2 = \chi(-1) = \chi(\kappa^{-1})^2$, which proves (i). The second assertion is easily verified. Though the third assertion can be proved by the argument in [Ro, Remark to Proposition 4.1] using a formula of Fröhlich and Queyrut ([FQ]), we give its elementary proof for completeness. Suppose that K/F is an unramified quadratic extension and put $l = a(\chi)$. We can take $c = \pi^{l+n_\psi}(\theta - \bar{\theta})$. Since $\chi(c) = (-1)^{l+n_\psi+\mathrm{ord}_F(\kappa/(\theta-\bar{\theta}))}\chi(\kappa^{-1})$, we only have to show that $S > 0$. Observe that $S = I + J$, where

$$I = \int_{\mathcal{O}_F^\times} \mathrm{d}x \int_{\mathcal{O}_F} \mathrm{d}y\, \chi^{-1}(x+\theta y)\,\psi\Big(\frac{y}{\pi^{l+n_\psi}}\Big),$$
$$J = \int_{\mathfrak{p}_F} \mathrm{d}x \int_{\mathcal{O}_F^\times} \mathrm{d}y\, \chi^{-1}(x+\theta y)\,\psi\Big(\frac{y}{\pi^{l+n_\psi}}\Big).$$

First suppose that $l = 0$. Then we have $I = 1 - q^{-1}$ and $J = q^{-1}(1 - q^{-1})$, and hence $S = 1 - q^{-2} > 0$. Next suppose that $l \geqslant 1$. Then

$$I = \int_{\mathcal{O}_F} \chi^{-1}(1+\theta y)\,\mathrm{d}y \int_{\mathcal{O}_F^\times} \psi\Big(\frac{xy}{\pi^{l+n_\psi}}\Big)\,\mathrm{d}x = \begin{cases} q^{-l} & \cdots\quad l \geqslant 2, \\ q^{-l} - q^{-1}\displaystyle\int_{\mathcal{O}_F} \chi^{-1}(1+\theta y)\,\mathrm{d}y & \cdots\quad l = 1. \end{cases}$$

On the other hand,

$$J = \int_{\mathfrak{p}_F} \chi^{-1}(x+\theta)\,dx \int_{\mathcal{O}_F^\times} \psi\Big(\frac{y}{\pi^{l+n_\psi}}\Big)\,\mathrm{d}y = \begin{cases} 0 & \cdots\quad l \geqslant 2, \\ -q^{-1}\displaystyle\int_{\mathfrak{p}_F} \chi^{-1}(x+\theta)\,dx & \cdots\quad l = 1. \end{cases}$$

Since

$$\int_{\mathcal{O}_F} \chi^{-1}(1+\theta y)\,dy + \int_{\mathfrak{p}_F} \chi^{-1}(x+\theta)\,\mathrm{d}x = (1-q^{-1})^{-1}\int_{\mathcal{O}_K^\times} \chi^{-1}(u)\,\mathrm{d}u = 0,$$

we obtain $S = q^{-l} > 0$ and the proposition has been proved. □

4. Splitting of Metaplectic Repesentation

4.1. In this section, we calculate explicitly the cocycle $c(u, u')$ for $u, u' \in K^1$ defined in Section 2.3 and give a splitting of M. For $u, u' \in K^1 - \{1\}$, put

$$\sigma_{u,u'} = \tau_u + \tau_{u'} = \kappa \frac{1 - uu'}{(1-u)(1-u')} \in F. \tag{4.1}$$

4.2. PROPOSITION. *For $u, u' \in K^1$, we have*

$$c(u, u') = \begin{cases} 1 & \cdots \quad u = 1 \text{ or } u' = 1 \text{ or } uu' = 1, \\ \lambda_K(\psi)\,\omega(\sigma_{u,u'}) & \cdots \quad \text{otherwise}. \end{cases}$$

Proof. The assertion is easily verified if one of u, u', uu' is equal to 1. Suppose that $u \neq 1, u' \neq 1$ and $uu' \neq 1$. Let $z \in K^\times$ and $\Phi \in V$. We set

$$I(u, u') = |(1-u)(1-u')|_K^{-1/2}\,(M(u)M(u')\Phi)(z).$$

Taking sufficiently large lattices L, L' of K, we have

$$I(u, u') = \int_L \mathrm{d}w \int_{L'} \mathrm{d}w'\, \psi\left(\frac{1}{2}\langle w, uw\rangle + \frac{1}{2}\langle w', u'w'\rangle + \frac{1}{2}\langle (1-u)w, (1-u')w'\rangle\right) \times$$
$$\times\, \rho((1-u)w + (1-u')w', 0)\Phi(z).$$

We may (and do) suppose that $u'L' \subset L$. Changing the variable w into $w + u'w'$, we get

$$I(u, u') = \int_L \mathrm{d}w \int_{L'} \mathrm{d}w'\, \psi\left(\frac{1}{2}\langle w, uw\rangle + \frac{1}{2}\langle w, (1-u^{-1})(1+uu')w'\rangle + \frac{1}{2}\langle w', uu'w'\rangle\right) \times$$
$$\times\, \rho((1-u)w + (1-uu')w', 0)\Phi(z).$$

Replacing the integral $\int_{L'} \mathrm{d}w'$ by $\int_K \mathrm{d}w'$ and changing the variable w' into $w' - (1-u)/(1-uu')\,w$, we obtain

$$\begin{aligned} I(u, u') &= \int_L \psi(-\kappa^2 \sigma_{u,u'}^{-1}\, w\overline{w}) dw \cdot |1 - uu'|_K^{-1/2} M(uu')\Phi(z) \\ &= \lambda_K(\psi)\,\omega(-\kappa^2 \sigma_{u,u'}^{-1})| - \kappa^2 \sigma_{u,u'}^{-1}|_F^{-1} |\kappa|_K^{1/2} \cdot |1 - uu'|_K^{-1/2} M(uu')\Phi(z) \\ &= \lambda_K(\psi)\,\omega(\sigma_{u,u'})\,|(1-u)(1-u')|_K^{-1/2}\, M(uu')\Phi(z) \end{aligned}$$

in view of Lemma 3.3 (iii). The proposition has been proved. □

4.3. For $z \in K^\times$, we define $\gamma(z) \in \mathbf{C}^\times$ and $\mathcal{M}(z) \in GL(V)$ by

$$\gamma(z) = \begin{cases} \omega(z) & \cdots \quad z \in F^\times, \\ \lambda_K(\psi)^{-1}\,\omega\left(\dfrac{z - \overline{z}}{\kappa}\right) & \cdots \quad z \in K^\times - F^\times \end{cases} \tag{4.2}$$

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and

$$\mathcal{M}(z) = \gamma(z)\, M(\overline{z}/z)\,. \tag{4.3}$$

4.4. LEMMA. *For* $z \in K^\times$ *and* $\Phi, \Phi' \in V$, *we have*

$$(\mathcal{M}(z)\,\Phi, \Phi') = (\Phi, \mathcal{M}(z^{-1})\,\Phi')\,.$$

Proof. This follows from Lemma 2.2 (ii) and the fact that $\overline{\gamma(z^{-1})} = \gamma(z)$ for $z \in K^\times$. □

4.5. THEOREM. *The mapping* $z \mapsto \mathcal{M}(z)$ *gives rise to a smooth representation of* $K^\times$ *on* V.

Proof. By Proposition 4.2, we have $c(\overline{z}/z, \overline{z'}/z') = \gamma(zz')/\gamma(z)\gamma(z')$ for $z, z' \in K^\times$, which implies that $z \mapsto \mathcal{M}(z)$ is a homomorphism. The smoothness is easily verified. □

Remark. T. Yang ([Ya]) gives a splitting of M' (cf. Remark to Section 2.3) in the odd residual characteristic case.

5. The Space of Primitive Theta Functions

5.1. In what follows, we fix an $\mathcal{O}_F$-basis $\{1, \theta\}$ of $\mathcal{O}_K$. When K/F is a ramified quadratic extension, we assume that $\mathrm{ord}_F\, N(\theta) = 1$ and fix a prime element Π of K. We put $v = -(\overline{\theta}\,/\,(\theta - \overline{\theta}))$. Then $v + \overline{v} = 1$ and $v \in \mathcal{D}_{K/F}^{-1} = \{z \in K \mid \mathrm{Tr}_{K/F}(zw) \in \mathcal{O}_F$ for any $w \in \mathcal{O}_K\}$. Note that $v \in \Pi\mathcal{D}_{K/F}^{-1}$ if $\delta_{K/F} > 0$.

5.2. Let $\mathfrak{a}$ be an ideal of K. We set

$$H(\mathfrak{a}) = \left\{(w, t) \in H \mid w \in \mathfrak{a},\, t + \frac{\kappa}{2} w\overline{w} \in \kappa N(\mathfrak{a})\mathcal{D}_{K/F}^{-1}\right\}. \tag{5.1}$$

Then $H(\mathfrak{a})$ is an open compact subgroup of H. It is easily seen that

$$H(\mathfrak{a}) = \left\{(w, t + t_w) \mid w \in \mathfrak{a},\, t \in \frac{\kappa}{\theta - \overline{\theta}} N(\mathfrak{a})\right\}$$

where

$$t_w = \frac{\kappa}{2}(v - \overline{v})w\overline{w}\,. \tag{5.2}$$

We put

$$V(\mathfrak{a}) = \{\Phi \in V \mid \rho(h_0)\,\Phi = \Phi \ \text{ for any } h_0 \in H(\mathfrak{a})\}. \tag{5.3}$$

Since

$$\mathrm{ord}_F\left(\frac{\kappa}{\theta-\bar{\theta}}N(\mathfrak{a})\right)=\mu_{\mathfrak{a}}-n_\psi\,,$$

we have $V(\mathfrak{a})=\{0\}$ unless $\mu_{\mathfrak{a}}\geqslant 0$.

5.3. Let $d_{\mathfrak{a}}h$ be the Haar measure on H normalized by $\int_{H(\mathfrak{a})}\mathrm{d}_{\mathfrak{a}}h=1$. We define an operator $\mathcal{P}_{\mathfrak{a}}\in\mathrm{End}(V)$ by

$$\mathcal{P}_{\mathfrak{a}}\,\Phi=\int_{H(\mathfrak{a})}\rho(h)\,\Phi\,\mathrm{d}_{\mathfrak{a}}h\,. \tag{5.4}$$

Let

$$R(\mathfrak{a})=\{\mathfrak{b}\mid \text{an ideal of } K,\,\mathfrak{a}\subset\mathfrak{b},\,\mu_{\mathfrak{b}}\geqslant 0\}\,. \tag{5.5}$$

The following fact is easily verified.

5.4. LEMMA.

(i) *We have* $V(\mathfrak{a})=\{\Phi\in V\mid \mathcal{P}_{\mathfrak{a}}\,\Phi=\Phi\}$.

(ii) *For* $\Phi\in V$, *we have*

$$\mathcal{P}_{\mathfrak{a}}\Phi=\begin{cases}\displaystyle\int_{\mathfrak{a}}\rho(w,t_w)\,\Phi\,\mathrm{d}_{\mathfrak{a}}w & \cdots\quad \mu_{\mathfrak{a}}\geqslant 0\,,\\ 0 & \cdots\quad \mu_{\mathfrak{a}}<0\,.\end{cases}$$

(Recall that $d_{\mathfrak{a}}w$ is defined in Section 3.4.)

(iii) *For* $\Phi,\Phi'\in V$, *we have* $(\mathcal{P}_{\mathfrak{a}}\,\Phi,\Phi')=(\Phi,\mathcal{P}_{\mathfrak{a}}\Phi')$.

(iv) *For* $\mathfrak{b}\in R(\mathfrak{a})$, *we have* $\mathcal{P}_{\mathfrak{a}}\mathcal{P}_{\mathfrak{b}}=\mathcal{P}_{\mathfrak{b}}\mathcal{P}_{\mathfrak{a}}=\mathcal{P}_{\mathfrak{b}}$, *and hence* $\mathcal{P}_{\mathfrak{b}}V\subset V(\mathfrak{a})$ *and* $\mathcal{P}_{\mathfrak{a}}V=V(\mathfrak{a})$.

5.5. Suppose that $\delta_{K/F}>0$ and $\mu_{\mathfrak{a}}=0$. We define an operator $\mathcal{Q}\in\mathrm{End}(V)$ by

$$\mathcal{Q}\Phi=\int_{\mathfrak{a}_1}\rho(w,t_w)\,\Phi\,\mathrm{d}_{\mathfrak{a}_1}w, \tag{5.6}$$

where we put $\mathfrak{a}_1=\Pi^{-1}\mathfrak{a}$. Note that $\mathcal{Q}\neq\mathcal{P}_{\mathfrak{a}_1}=0$, since $\mathcal{Q}\neq 0$ (cf. Proposition 5.8 (ii)). The following fact is proved similarly as Lemma 5.4.

5.6. LEMMA.

(i) *We have* $\mathcal{Q}\mathcal{P}_{\mathfrak{a}}=\mathcal{P}_{\mathfrak{a}}\mathcal{Q}=\mathcal{Q}$ *and hence* $\mathcal{Q}\,V\subset V(\mathfrak{a})$.

(ii) *We have* $\mathcal{Q}^2=\mathcal{Q}$.

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5.7. We henceforth fix an ideal $\mathfrak{a}$ of K with $\mu_{\mathfrak{a}} \geqslant 0$. First suppose that $\delta_{K/F} = 0$ or $\mu_{\mathfrak{a}} > 0$. We set

$$V_{\mathrm{prim}}(\mathfrak{a}) = \{\Phi \in V(\mathfrak{a}) \mid \mathcal{P}_{\mathfrak{b}}\Phi = 0 \ \text{ for any } \ \mathfrak{b}\in \mathrm{R}(\mathfrak{a}),\ \mathfrak{b} \neq \mathfrak{a}\}\,. \tag{5.7}$$

Next suppose that $\delta_{K/F} > 0$ and $\mu_{\mathfrak{a}} = 0$. Set

$$V_{\mathrm{prim}}(\mathfrak{a}) = \{\Phi \in V(\mathfrak{a}) \mid \mathcal{Q}\Phi = 0\} \tag{5.8}$$

and

$$V^1(\mathfrak{a}) = \{\Phi \in V(\mathfrak{a}) \mid \mathcal{Q}\Phi = \Phi\}\,. \tag{5.9}$$

Then $V(\mathfrak{a}) = V_{\mathrm{prim}}(\mathfrak{a}) \oplus V^1(\mathfrak{a})$ in view of Lemma 5.6. In both cases, we call $V_{\mathrm{prim}}(\mathfrak{a})$ the space of *primitive theta functions* in $V(\mathfrak{a})$. Denote by $\mathcal{P}_{\mathfrak{a},\mathrm{prim}} \in \mathrm{End}(V)$ the projection operator of $V_{\mathrm{prim}}(\mathfrak{a})$.

5.8. PROPOSITION.

(i) *For* $\mathfrak{b} \in R(\mathfrak{a})$ *we have* $Tr(\mathcal{P}_{\mathfrak{b}} \mid V(\mathfrak{a})) = q^{\mu_{\mathfrak{b}}+\delta_{K/F}}$.
(ii) *We have* $\mathrm{Tr}(\mathcal{Q}|V(\mathfrak{a})) = q^{\delta_{K/F}-1}$ *when* $\delta_{K/F} > 0$ *and* $\mu_{\mathfrak{a}} = 0$.

Proof. By Lemma 1.7, we have

$$\mathcal{P}_{\mathfrak{b}}\,\Phi(z) = \int_K k_{\mathfrak{b}}(z, w)\,\Phi(w)\,\mathrm{d}w\,,$$

where

$$k_{\mathfrak{b}}(z, w) = \int_{\mathfrak{b}} \psi(t_{w'} + \frac{1}{2}\langle w, w' + z\rangle + \frac{1}{2}\langle z, w'\rangle)\,\overline{\Phi^0(w - w' - z)}\,\mathrm{d}_{\mathfrak{b}}w'\,.$$

Observe that $z \mapsto k_{\mathfrak{b}}(z, w)$ and $w \mapsto \overline{k_{\mathfrak{b}}(z, w)}$ are in $V(\mathfrak{a})$. It follows that, for a sufficiently large lattice L of K, we have

$$\begin{aligned}
\mathrm{Tr}(\mathcal{P}_{\mathfrak{b}} \mid V(\mathfrak{a})) &= \int_L k_{\mathfrak{b}}(z, z)\,\mathrm{d}z \\
&= \int_{\mathfrak{b}} \left(\int_L \psi(\langle z, w'\rangle)\,\mathrm{d}z\right) \psi(t_{w'})\overline{\Phi^0(-w')}\,\mathrm{d}_{\mathfrak{b}}w' \\
&= \mathrm{vol}(L) \int_{\mathfrak{b}\cap L^*} \psi(t_{w'})\overline{\Phi^0(-w')}\,\mathrm{d}_{\mathfrak{b}}w'\,.
\end{aligned}$$

Since we can take L^* sufficiently small, we have

$$\mathrm{Tr}(\mathcal{P}_{\mathfrak{b}} \mid V(\mathfrak{a})) = \mathrm{vol}(L)\,\mathrm{vol}(L^*) \times \frac{\mathrm{d}_{\mathfrak{b}}w'}{\mathrm{d}w'} = q^{\mu_{\mathfrak{b}}+\delta_{K/F}}\,,$$

which proves (i). The second assertion is proved similarly. □

5.9. LEMMA.

(a) *If* $K = F \oplus F$, *we have*

$$\mathcal{P}_{\mathfrak{a},\mathrm{prim}} = \begin{cases} \mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\mathfrak{a}_1} - \mathcal{P}_{\mathfrak{a}_2} + \mathcal{P}_{\mathfrak{a}_{12}} & \cdots \quad \mu_{\mathfrak{a}} \geqslant 2\,, \\ \mathcal{P}_{\mathfrak{a}} + \dfrac{1}{1-q^{-1}}(-\mathcal{P}_{\mathfrak{a}_1} - \mathcal{P}_{\mathfrak{a}_2} + \mathcal{P}_{\mathfrak{a}_1}\mathcal{P}_{\mathfrak{a}_2} + \mathcal{P}_{\mathfrak{a}_2}\mathcal{P}_{\mathfrak{a}_1}) & \cdots \quad \mu_{\mathfrak{a}} = 1\,, \\ \mathcal{P}_{\mathfrak{a}} & \cdots \quad \mu_{\mathfrak{a}} = 0\,, \end{cases}$$

where $\mathfrak{a}_1 = (\pi^{-1}, 1)\mathfrak{a}$, $\mathfrak{a}_2 = (1, \pi^{-1})\mathfrak{a}$ *and* $\mathfrak{a}_{12} = (\pi^{-1}, \pi^{-1})\mathfrak{a}$.

(b) *If* K/F *is an unramified quadratic extension, we have*

$$\mathcal{P}_{\mathfrak{a},\mathrm{prim}} = \begin{cases} \mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\pi^{-1}\mathfrak{a}} & \cdots \quad \mu_{\mathfrak{a}} \geqslant 2\,, \\ \mathcal{P}_{\mathfrak{a}} & \cdots \quad \mu_{\mathfrak{a}} = 0, 1\,. \end{cases}$$

(c) *If* K/F *is a ramified quadratic extension, we have*

$$\mathcal{P}_{\mathfrak{a},\mathrm{prim}} = \begin{cases} \mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\Pi^{-1}\mathfrak{a}} & \cdots \quad \mu_{\mathfrak{a}} \geqslant 1\,, \\ \mathcal{P}_{\mathfrak{a}} - \mathcal{Q} & \cdots \quad \mu_{\mathfrak{a}} = 0\,. \end{cases}$$

Proof. We prove the lemma only in the case where $K = F \oplus F$ and $\mu_{\mathfrak{a}} = 1$, since the assertions in the other cases easily follow from Lemma 5.4 and Lemma 5.6. By Lemma 5.4 and Proposition 5.8, we have

$$\mathcal{P}_{\mathfrak{a}}\mathcal{P}_{\mathfrak{a}_i} = \mathcal{P}_{\mathfrak{a}_i}, \quad (\mathcal{P}_{\mathfrak{a}_i})^2 = \mathcal{P}_{\mathfrak{a}_i}, \quad Tr(\mathcal{P}_{\mathfrak{a}_i} | V(\mathfrak{a})) = 1 \qquad (i = 1, 2).$$

By using an argument similar to that of the proof of Proposition 5.8, we obtain

$$\mathrm{Tr}(\mathcal{P}_{\mathfrak{a}_1}\mathcal{P}_{\mathfrak{a}_2} | V(\mathfrak{a})) = \mathrm{Tr}(\mathcal{P}_{\mathfrak{a}_2}\mathcal{P}_{\mathfrak{a}_1} | V(\mathfrak{a})) = q^{-1}\,. \tag{5.10}$$

It follows that there exist $\Phi_1, \Phi_2 \in V(\mathfrak{a}) - \{0\}$ (unique up to scalar multiples) satisfying $\mathcal{P}_{\mathfrak{a}_i}\Phi_i = \Phi_i$ $(i = 1, 2)$, and that $V(\mathfrak{a}) = (\mathbf{C}\Phi_1 + \mathbf{C}\Phi_2) \oplus V_{\mathrm{prim}}(\mathfrak{a})$. Since $\mathcal{P}_{\mathfrak{a}_2}(\mathcal{P}_{\mathfrak{a}_2}\Phi_1) = \mathcal{P}_{\mathfrak{a}_2}\Phi_1$, we have $\mathcal{P}_{\mathfrak{a}_2}\Phi_1 = c\Phi_2$ with $c \in \mathbf{C}$. Similarly we have $\mathcal{P}_{\mathfrak{a}_1}\Phi_2 = c'\Phi_1$ with $c' \in \mathbf{C}$. By (5.10), we have $cc' = q^{-1}$. Put

$$\mathcal{P}' = \mathcal{P}_{\mathfrak{a}} + \frac{1}{1-q^{-1}}(-\mathcal{P}_{\mathfrak{a}_1} - \mathcal{P}_{\mathfrak{a}_2} + \mathcal{P}_{\mathfrak{a}_1}\mathcal{P}_{\mathfrak{a}_2} + \mathcal{P}_{\mathfrak{a}_2}\mathcal{P}_{\mathfrak{a}_1}).$$

Then $\mathcal{P}'V \subset V(\mathfrak{a})$, $\mathcal{P}'$ is the identity on $V_{\mathrm{prim}}(\mathfrak{a})$ and $\mathcal{P}'\Phi_i = 0$ $(i = 1, 2)$, which implies $\mathcal{P}' = \mathcal{P}_{\mathfrak{a},\mathrm{prim}}$. □

5.10 PROPOSITION.

(i) $\dim_{\mathbf{C}} V(\mathfrak{a}) = q^{\mu_{\mathfrak{a}} + \delta_{K/F}}$.

(ii) (a) *If* $K = F \oplus F$, *we have*

$$\dim_{\mathbf{C}} V_{\mathrm{prim}}(\mathfrak{a}) = \begin{cases} q^{\mu_{\mathfrak{a}}}(1-q^{-1})^2 & \cdots \quad \mu_{\mathfrak{a}} \geqslant 2\,, \\ q - 2 & \cdots \quad \mu_{\mathfrak{a}} = 1\,, \\ 1 & \cdots \quad \mu_{\mathfrak{a}} = 0\,. \end{cases}$$

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(b) *If K/F is an unramified quadratic extension, we have*

$$\dim_{\mathbf{C}} V_{\mathrm{prim}}(\mathfrak{a}) = \begin{cases} q^{\mu_{\mathfrak{a}}}(1-q^{-2}) & \cdots \quad \mu_{\mathfrak{a}} \geqslant 2\,, \\ q & \cdots \quad \mu_{\mathfrak{a}} = 1\,, \\ 1 & \cdots \quad \mu_{\mathfrak{a}} = 0\,. \end{cases}$$

(c) *If K/F is a ramified quadratic extension, we have*

$$\dim_{\mathbf{C}} V_{\mathrm{prim}}(\mathfrak{a}) = q^{\mu_{\mathfrak{a}}+\delta_{K/F}}(1-q^{-1})\,.$$

Moreover if $\mu_{\mathfrak{a}} = 0$, we have $\dim_{\mathbf{C}} V^1(\mathfrak{a}) = q^{\delta_{K/F}-1}$.

Proof. Observe that $\dim_{\mathbf{C}} V(\mathfrak{a}) = \mathrm{Tr}(\mathcal{P}_{\mathfrak{a}}|V(\mathfrak{a}))$ and $\dim_{\mathbf{C}} V_{\mathrm{prim}}(\mathfrak{a}) = Tr(\mathcal{P}_{\mathfrak{a},\mathrm{prim}}| V(\mathfrak{a}))$. The proposition now follows from Proposition 5.8, Lemma 5.9 and (5.10). □

6. Main Results

6.1. In the remaining part of the paper, we fix an ideal $\mathfrak{a}$ of K satisfying $\mu_{\mathfrak{a}} \geqslant 0$. Let $z \in \Gamma_K$ (recall that Γ_K is defined by (3.7)). For $\mathfrak{b} \in R(\mathfrak{a})$, we set

$$X_{\mathfrak{b}}(z) = \mathrm{Tr}(\mathcal{P}_{\mathfrak{b}}\mathcal{M}(z)|V(\mathfrak{a})) = \mathrm{Tr}(\mathcal{P}_{\mathfrak{b}}\mathcal{M}(z)|V(\mathfrak{b}))\,. \tag{6.1}$$

Note that $X_{\mathfrak{b}}(tz) = \omega(t)\,X_{\mathfrak{b}}(z)$ for $t \in F^{\times}$. Define

$$X_{\mathfrak{a},\mathrm{prim}}(z) = \mathrm{Tr}(\mathcal{P}_{\mathfrak{a},\mathrm{prim}}\mathcal{M}(z)|V(\mathfrak{a}))\,. \tag{6.2}$$

When $\delta_{K/F} > 0$ and $\mu_{\mathfrak{a}} = 0$, we set

$$X^1_{\mathfrak{a}}(z) = \mathrm{Tr}(\mathcal{Q}\mathcal{M}(z)|V(\mathfrak{a}))\,. \tag{6.3}$$

6.2. Let $z \in \Gamma_K$. Since $\bar{z}/z \in \mathcal{O}_K^{\times}$, we have $\mathcal{M}(z)\mathcal{P}_{\mathfrak{b}} = \mathcal{P}_{\mathfrak{b}}\mathcal{M}(z)$ for $\mathfrak{b} \in R(\mathfrak{a})$. This implies that $\mathcal{M}(z)\,V(\mathfrak{a}) \subset V(\mathfrak{a})$ and $\mathcal{M}(z)\,V_{\mathrm{prim}}(\mathfrak{a}) \subset V_{\mathrm{prim}}(\mathfrak{a})$, and hence $\mathcal{M}$ induces a representation $\mathcal{M}_{\mathfrak{a}}$ (resp. $\mathcal{M}_{\mathfrak{a},\mathrm{prim}}$) of Γ_K on $V(\mathfrak{a})$ (resp. $V_{\mathrm{prim}}(\mathfrak{a})$). Then

$$X_{\mathfrak{a}}(z) = \mathrm{Tr}(\mathcal{M}_{\mathfrak{a}}(z)), \quad X_{\mathfrak{a},\mathrm{prim}}(z) = \mathrm{Tr}(\mathcal{M}_{\mathfrak{a},\mathrm{prim}}(z))\,. \tag{6.4}$$

We also have, when $\delta_{K/F} > 0$ and $\mu_{\mathfrak{a}} = 0$,

$$X^1_{\mathfrak{a}}(z) = \mathrm{Tr}(\mathcal{M}(z)|V^1(\mathfrak{a}))\,. \tag{6.5}$$

6.3. Let $\mathcal{X}(\omega) = \{\chi \in (\Gamma_K)^{\wedge} \mid \chi|_{F^{\times}} = \omega\}$. For $l \in \mathbf{Z}, l \geqslant 0$, we set

$$\mathcal{X}(\omega; l) = \{\chi \in \mathcal{X}(\omega) \mid a(\chi) \leqslant l\} \tag{6.6}$$

and

$$\mathcal{X}_{\mathrm{prim}}(\omega; l) = \{\chi \in \mathcal{X}(\omega) \mid a(\chi) = l\} \tag{6.7}$$

(recall that $a(\chi)$ is defined in Section 3.6). Set

$$\mathcal{X}^{+}(\omega; l) = \{\chi \in \mathcal{X}(\omega; l) \mid \varepsilon(\chi, \psi_K) = \chi(\kappa^{-1})\} \tag{6.8}$$

and

$$\mathcal{X}^{+}_{\mathrm{prim}}(\omega; l) = \{\chi \in \mathcal{X}_{\mathrm{prim}}(\omega; l) \mid \varepsilon(\chi, \psi_K) = \chi(\kappa^{-1})\}\,. \tag{6.9}$$

If $K = F \oplus F$, we have $\mathcal{X}^{+}_{\mathrm{prim}}(\omega; l) = \mathcal{X}_{\mathrm{prim}}(\omega; l)$ for $l \geqslant 0$. If K/F is an unramified quadratic extension,

$$\mathcal{X}^{+}_{\mathrm{prim}}(\omega; l) = \begin{cases} \mathcal{X}_{\mathrm{prim}}(\omega; l) & \cdots \quad l \equiv n_\psi + \mathrm{ord}_F(\kappa/(\theta - \overline{\theta})) \pmod 2 \\ \emptyset & \cdots \quad \text{otherwise} \end{cases}$$

in view of Proposition 3.7 (iii). We now state the main results of the paper.

6.4. THEOREM.

(i) *If* $\delta_{K/F} = 0$, *we have*

$$X_{\mathfrak{a},\mathrm{prim}} = \sum_{\chi \in \mathcal{X}^{+}_{\mathrm{prim}}(\omega;\, \mu_{\mathfrak{a}})} \chi = \sum_{\chi \in \mathcal{X}_{\mathrm{prim}}(\omega;\, \mu_{\mathfrak{a}})} \chi\,.$$

(ii) *If* $\delta_{K/F} > 0$, *we have*

$$X_{\mathfrak{a},\mathrm{prim}} = \sum_{\chi \in \mathcal{X}^{+}_{\mathrm{prim}}(\omega;\, 2(\mu_{\mathfrak{a}} + \delta_{K/F}))} \chi\,.$$

(iii) *If* $\delta_{K/F} > 0$ *and* $\mu_{\mathfrak{a}} = 0$, *we have*

$$X^{1}_{\mathfrak{a}} = \sum_{\chi \in \mathcal{X}^{+}_{\mathrm{prim}}(\omega;\, 2\delta_{K/F} - 1)} \chi\,.$$

The following fact is proved in [Sh] in the case where $F = \mathbf{Q}_p$ and $K = \mathbf{Q}(\sqrt{-1}) \otimes_{\mathbf{Q}} \mathbf{Q}_p$, and in [GR] in the general case.

6.5. COROLLARY. *The representation* $\mathcal{M}_{\mathfrak{a},\mathrm{prim}}$ *of* Γ_K *is multiplicity-free.*

6.6. THEOREM.

(i) *If* $K = F \oplus F$, *we have*

$$\begin{aligned} X_{\mathfrak{a}} &= \sum_{0 \leqslant k \leqslant \mu_{\mathfrak{a}}} \sum_{\chi \in \mathcal{X}^{+}_{\mathrm{prim}}(\omega;\, \mu_{\mathfrak{a}} - k)} (k+1)\,\chi \\ &= \sum_{\chi \in \mathcal{X}^{+}(\omega; \mu_{\mathfrak{a}})} (\mu_{\mathfrak{a}} - a(\chi) + 1)\,\chi\,. \end{aligned}$$

(ii) *If K/F is an unramified quadratic extension, we have*

$$\begin{aligned} X_{\mathfrak{a}} &= \sum_{k \geqslant 0, \mu_{\mathfrak{a}}-2k \geqslant 0} \ \sum_{\chi \in \mathcal{X}^+_{\mathrm{prim}}(\omega;\, \mu_{\mathfrak{a}}-2k)} \chi \\ &= \sum_{\chi \in \mathcal{X}^+(\omega; \mu_{\mathfrak{a}})} \chi . \end{aligned}$$

(iii) *If K/F is a ramified quadratic extension, we have*

$$\begin{aligned} X_{\mathfrak{a}} &= \sum_{0 \leqslant k \leqslant \mu_{\mathfrak{a}}} \ \sum_{\chi \in \mathcal{X}^+_{\mathrm{prim}}(\omega;\, 2(k+\delta_{K/F}))} \chi + \sum_{\chi \in \mathcal{X}^+_{\mathrm{prim}}(\omega;\, 2\delta_{K/F}-1)} \chi \\ &= \sum_{\chi \in \mathcal{X}^+(\omega; 2(\mu_{\mathfrak{a}}+\delta_{K/F}))} \chi . \end{aligned}$$

As a direct consequence of Theorem 6.6, we have proved the following result due to Moen [Mo] and Rogawski [Ro], which is known as 'epsilon dichotomy' for (U(1), U(1)) (cf. Section 0.2).

6.7. COROLLARY. *Suppose that K is a field.*

(i) *The metaplectic representation $\mathcal{M}$ of $K^\times$ on V is multiplicity-free.*
(ii) *A unitary character χ of $K^\times$ appears in $\mathcal{M}$ if and only if $\chi|_{F^\times} = \omega$ and $\varepsilon(\chi, \psi_K) = \chi(\kappa^{-1})$.*

7. Trace Formula

7.1. The object of this section is to calculate $X_{\mathfrak{a}}(z)$, $X_{\mathfrak{a},\mathrm{prim}}(z)$ and $X^1_{\mathfrak{a}}(z)$ for $z \in \Gamma_K$. We first give the kernel function of the operator $\mathcal{P}_{\mathfrak{b}} M(u)$ for $u \in K^1$. By Lemma 1.7, we have

$$(M(u)\,\Phi)(z) = \int_K \eta_u(z, z')\,\Phi(z')\,\mathrm{d}z' \qquad (\Phi \in V, u \in K^1, z \in K),$$

where $\eta_u(z, z') = \overline{(M(u^{-1})\,\rho(-z, 0)\,\Phi^0)(z')}$. It follows that

$$(\mathcal{P}_{\mathfrak{b}}\, M(u)\,\Phi)(z) = \int_K \eta_{u,\mathfrak{b}}(z, z')\,\Phi(z')\,\mathrm{d}z' \qquad (\Phi \in V, z \in K),$$

where

$$\eta_{u,\mathfrak{b}}(z, z') = \int_{\mathfrak{b}} \psi\left(\frac{1}{2}\langle z, w\rangle + t_w\right) \eta_u(z + w, z')\,\mathrm{d}_{\mathfrak{b}} w$$

Observe that, if $u \neq 1$, we have

$$\eta_u(z, z') = |1-u|_K^{-1/2} \int_K \psi\left(\tau_u w\overline{w} + \frac{1}{2}\langle z', z-w\rangle + \frac{1}{2}\langle w, z\rangle\right) \overline{\Phi^0(z'-z+w)}\, \mathrm{d}w$$

in view of the definition of $M(u)$ (cf. Section 2.1).

7.2. For $z = x + \theta y \in K^\times$, we put

$$\upsilon(z) = \begin{cases} \mathrm{ord}_F(y/x) & \cdots \quad x \neq 0, \\ -\infty & \cdots \quad x = 0. \end{cases} \tag{7.1}$$

7.3. PROPOSITION. *For* $z = x + \theta y \in \Gamma_K$, *we have*

$$X_{\mathfrak{b}}(z) = |N(z)|_F^{1/2} \times \begin{cases} q^{\mu_{\mathfrak{b}}+\delta_{K/F}}\, \omega(x)\, |x|_F^{-1} & \cdots \quad \upsilon(z) \geqslant \mu_{\mathfrak{b}} + \delta_{K/F}, \\ q^{\delta_{K/F}/2}\, \lambda_K(\psi)^{-1}\, \omega\left(\frac{\theta-\overline{\theta}}{\kappa} y\right) |y|_F^{-1} & \cdots \quad \upsilon(z) \leqslant \mu_{\mathfrak{b}}, \\ 0 & \cdots \quad \text{otherwise}. \end{cases}$$

Proof. If $z = x \in F^\times$, we have $X_{\mathfrak{b}}(z) = q^{\mu_{\mathfrak{b}}+\delta_{K/F}}\, \omega(x)$ and the assertion is clear in this case. Suppose that $z \in K^\times - F^\times$. By an argument similar to that in the proof of Proposition 5.8, we have, for a sufficiently large lattice L of K,

$$\begin{aligned} Tr(\mathcal{P}_{\mathfrak{b}} M(\overline{z}/z) \mid V(\mathfrak{a})) &= \int_L \eta_{\overline{z}/z, \mathfrak{b}}(w', w')\, \mathrm{d}w' \\ &= \left|1 - \overline{z}/z\right|_K^{-1/2} \int_{\mathfrak{b}} \psi(t_w + \tau_{\overline{z}/z}\, w\overline{w})\, \mathrm{d}_{\mathfrak{b}} w \\ &= \left|1 - \overline{z}/z\right|_K^{-1/2} S_{\mathfrak{b}}\left(\tau_{\overline{z}/z} + \frac{\kappa}{2}(v - \overline{v})\right) \\ &= \left|1 - \overline{z}/z\right|_K^{-1/2} S_{\mathfrak{b}}\left(\frac{\kappa}{\theta - \overline{\theta}} \frac{x}{y}\right). \end{aligned}$$

Since

$$\left|1 - \overline{z}/z\right|_K^{-1/2} = |N(z)|_F^{1/2}\, q^{\delta_{K/F}/2}\, |y|_F^{-1} \quad \text{and} \quad \gamma(z) = \lambda_K(\psi)^{-1}\, \omega\left(\frac{\theta - \overline{\theta}}{\kappa} y\right),$$

we obtain

$$\begin{aligned} X_{\mathfrak{b}}(z) &= \gamma(z)\, \mathrm{Tr}(\mathcal{P}_{\mathfrak{b}} M(\overline{z}/z) | V(\mathfrak{a})) \\ &= |N(z)|_F^{1/2}\, q^{\delta_{K/F}/2}\, \lambda_K(\psi)^{-1}\, \omega\left(\frac{\theta - \overline{\theta}}{\kappa} y\right) |y|_F^{-1}\, S_{\mathfrak{b}}\left(\frac{\kappa}{\theta - \overline{\theta}} \frac{x}{y}\right). \end{aligned}$$

The proposition now follows from Lemma 3.5. □

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7.4. LEMMA. *If* $z \in (1+\pi^{\mu_{\mathfrak{a}}+\delta_{K/F}}\mathcal{O}_K) \cap \mathcal{O}_K^\times$, *we have* $\mathcal{M}_{\mathfrak{a}}(z) = \mathrm{Id}_{V(\mathfrak{a})}$.

Proof. Since $\mathcal{M}_{\mathfrak{a}}(z)$ is a unitary operator on $V(\mathfrak{a})$, it is sufficient to show that $X_{\mathfrak{a}}(z) = \dim_{\mathbf{C}} V(\mathfrak{a})$ for $z \in (1+\pi^{\mu_{\mathfrak{a}}+\delta_{K/F}}\mathcal{O}_K) \cap \mathcal{O}_K^\times$. This fact is a straightforward consequence of Proposition 7.3 and Proposition 5.10 (i). □

7.5. PROPOSITION. *Assume that* $\delta_{K/F} = 0$ *and let* $z = x+\theta y \in \mathcal{O}_K^\times$. *If* $\mu_{\mathfrak{a}} \geqslant 2$, *we have*

$$X_{\mathfrak{a},\mathrm{prim}}(z) = \begin{cases} q^{\mu_{\mathfrak{a}}}(1-q^{-1})(1-\omega(\pi)q^{-1}) & \cdots \quad \mathrm{ord}_F(y) \geqslant \mu_{\mathfrak{a}}\,, \\ -q^{\mu_{\mathfrak{a}}-1}(1-\omega(\pi)q^{-1}) & \cdots \quad \mathrm{ord}_F(y) = \mu_{\mathfrak{a}}-1\,, \\ 0 & \cdots \quad 0 \leqslant \mathrm{ord}_F(y) \leqslant \mu_{\mathfrak{a}}-2\,. \end{cases}$$

If $\mu_{\mathfrak{a}} = 1$, *we have*

$$X_{\mathfrak{a},\mathrm{prim}}(z) = \begin{cases} q-1-\omega(\pi) & \cdots \quad \mathrm{ord}_F(y) \geqslant 1\,, \\ -1 & \cdots \quad \mathrm{ord}_F(y) = 0\,. \end{cases}$$

If $\mu_{\mathfrak{a}} = 0$, *we have* $X_{\mathfrak{a},\mathrm{prim}}(z) = 1$.

Proof. The proposition is a direct consequence of Lemma 5.9 and Proposition 7.3 except in the case where $K = F \oplus F$ and $\mu_{\mathfrak{a}} = 1$. In this case, we further need

$$\mathrm{Tr}(\mathcal{P}_{\mathfrak{a}_1}\mathcal{P}_{\mathfrak{a}_2}\mathcal{M}(z) \mid V(\mathfrak{a})) = Tr(\mathcal{P}_{\mathfrak{a}_2}\mathcal{P}_{\mathfrak{a}_1}\mathcal{M}(z) \mid V(\mathfrak{a})) = q^{-1} \quad (z \in \mathcal{O}_K^\times),$$

which is easily verified (for the definition of $\mathfrak{a}_i$, see Lemma 5.9). □

The following two results are proved similarly as in the case $\delta_{K/F} = 0$.

7.6. PROPOSITION. *Assume that* $\delta_{K/F} > 0$ *and let* $z = x+\theta y \in K^\times$. *If* $\delta_{K/F} = 1$, *we have*

$$X_{\mathfrak{a},\mathrm{prim}}(z) = \begin{cases} q^{\mu_{\mathfrak{a}}+1}(1-q^{-1})\omega(x) & \cdots \quad \upsilon(z) \geqslant \mu_{\mathfrak{a}}+1\,, \\ q^{\mu_{\mathfrak{a}}}\{q^{1/2}\lambda_K(\psi)^{-1}\omega\left(\dfrac{\theta-\bar{\theta}}{\kappa}y\right) - \omega(x)\} & \cdots \quad \upsilon(z) = \mu_{\mathfrak{a}}\,, \\ 0 & \cdots \quad \upsilon(z) \leqslant \mu_{\mathfrak{a}}-1\,. \end{cases}$$

If $\delta_{K/F} \geqslant 2$, *we have*

$$X_{\mathfrak{a},\mathrm{prim}}(z) = \begin{cases} q^{\mu_{\mathfrak{a}}+\delta_{K/F}}(1-q^{-1})\omega(x) & \cdots \quad \upsilon(z) \geqslant \mu_{\mathfrak{a}}+\delta_{K/F}\,, \\ -q^{\mu_{\mathfrak{a}}+\delta_{K/F}-1}\omega(x) & \cdots \quad \upsilon(z) = \mu_{\mathfrak{a}}+\delta_{K/F}-1\,, \\ 0 & \cdots \quad \mu_{\mathfrak{a}} < \upsilon(z) < \mu_{\mathfrak{a}}+\delta_{K/F}-1\,, \\ q^{\mu_{\mathfrak{a}}+\delta_{K/F}/2}\lambda_K(\psi)^{-1}\omega\left(\dfrac{\theta-\bar{\theta}}{\kappa}y\right) & \cdots \quad \upsilon(z) = \mu_{\mathfrak{a}}\,, \\ 0 & \cdots \quad \upsilon(z) \leqslant \mu_{\mathfrak{a}}-1\,. \end{cases}$$

(For the definition of $\upsilon(z)$, see (7.1).)

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7.7. PROPOSITION. *Assume that* $\delta_{K/F} > 0$ *and* $\mu_{\mathfrak{a}} = 0$. *For* $z = x + \theta y \in K^\times$, *we have*

$$X^1_{\mathfrak{a}}(z) = \begin{cases} q^{\delta_{K/F}-1}\omega(x) & \cdots \quad \upsilon(z) \geqslant \delta_{K/F} - 1\,, \\ 0 & \cdots \quad 0 \leqslant \upsilon(z) \leqslant \delta_{K/F} - 2\,, \\ q^{(\delta_{K/F}-1)/2}\lambda_K(\psi)^{-1}\omega\left(\dfrac{\theta - \overline{\theta}}{\kappa} y\right) & \cdots \quad \upsilon(z) \leqslant -1\,. \end{cases}$$

7.8. *Remark*. Suppose that K is a field. Letting $\mathfrak{b}$ be sufficiently small in Proposition 7.3, we see that the character of $\mathcal{M}(z)$ at $z \in K^\times - F^\times$ is equal to

$$\lambda_K(\psi)^{-1}\,\omega\left(\frac{z - \overline{z}}{\kappa}\right)\left|1 - \frac{\overline{z}}{z}\right|_K^{-1/2}$$

This character formula has been proved in [Pr] in the odd residual characteristic case (see also [Ho] for the $Sp_n(\mathbf{F}_q)$ case).

8. Irreducible Decomposition: The Case $\boldsymbol{\delta_{K/F} = 0}$

8.1. In this section, we suppose that $\delta_{K/F} = 0$ and prove Theorem 6.4 and Theorem 6.6 in this case. Recall that $\Gamma_K = F^\times \cdot \mathcal{O}_K^\times$.

Let l be a nonnegative integer. Put

$$\Gamma_K(l) = F^\times \cdot \begin{cases} 1 + \pi^l\mathcal{O}_K & \cdots \quad l > 0\,, \\ \mathcal{O}_K^\times & \cdots \quad l = 0\,. \end{cases}$$

Note that $\mathcal{O}_K^\times \cap \Gamma_K(l) = \{z = x + \theta y \in \mathcal{O}_K^\times \mid \mathrm{ord}_F(y) \geqslant l\}$ for $l \geqslant 0$, and that

$$\mathcal{X}(\omega; l) = \{\chi \in (\Gamma_K)^\wedge \mid \chi|_{F^\times} = \omega, \chi|_{\mathcal{O}_K^\times \cap \Gamma_K(l)} = 1\}\,.$$

The following fact is easily verified.

8.2. LEMMA. *We have*

$$\begin{aligned} \#(\mathcal{X}(\omega; l)) &= \#(\Gamma_K/\Gamma_K(l)) \\ &= \begin{cases} q^l(1 - \omega(\pi)q^{-1}) & \cdots \quad l \geqslant 1\,, \\ 1 & \cdots \quad l = 0 \end{cases} \end{aligned}$$

and

$$\#(\mathcal{X}_{\mathrm{prim}}(\omega; l)) = \begin{cases} q^l(1 - q^{-1})(1 - \omega(\pi)q^{-1}) & \cdots \quad l \geqslant 2\,, \\ q - 1 - \omega(\pi) & \cdots \quad l = 1\,, \\ 1 & \cdots \quad l = 0\,. \end{cases}$$

We now quote the following elementary fact from the representation theory of finite Abelian groups.

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8.3. LEMMA *Let G be a finite Abelian group and H a subgroup of G. Then we have*

$$\sum_{\chi\in G^\wedge, \chi|_H\neq 1} \chi(g) = \begin{cases} \#(G)(1-1/\#(H)) & \cdots \quad g = e\,, \\ -\#(G)/\#(H) & \cdots \quad g \in H - \{e\}\,, \\ 0 & \cdots \quad g \in G - H\,, \end{cases}$$

where e denotes the unit element of G.

8.4. PROPOSITION. *Let $z = x + \theta y \in \mathcal{O}_K^\times$ and μ be a nonnegative integer.*

(i) *If $\mu \geqslant 2$, we have*

$$\sum_{\chi\in\mathcal{X}_{\mathrm{prim}}(\omega;\,\mu)} \chi(z) = \begin{cases} q^\mu(1-q^{-1})(1-\omega(\pi)q^{-1}) & \cdots \quad \mathrm{ord}_F(y) \geqslant \mu\,, \\ -q^{\mu-1}(1-\omega(\pi)q^{-1}) & \cdots \quad \mathrm{ord}_F(y) = \mu-1\,, \\ 0 & \cdots \quad \mathrm{ord}_F(y) \leqslant \mu-2\,. \end{cases}$$

(ii) *If $\mu = 1$, we have*

$$\sum_{\chi\in\mathcal{X}_{\mathrm{prim}}(\omega;\,\mu)} \chi(z) = \begin{cases} q-1-\omega(\pi) & \cdots \quad \mathrm{ord}_F(y) \geqslant 1 \\ -1 & \cdots \quad \mathrm{ord}_F(y) = 0\,. \end{cases}$$

(iii) *If $\mu = 0$, we have*

$$\sum_{\chi\in\mathcal{X}_{\mathrm{prim}}(\omega;\,\mu)} \chi(z) = 1\,.$$

Proof. The assertion (iii) is obvious. The other assertions are immediate consequences of Section 8.1 and Lemma 8.3. □

8.5. Combining Proposition 7.5 and Proposition 8.4, we have proved Theorem 6.4. We can prove Theorem 6.6 in a similar manner.

9. Irreducible Decomposition: The Case $\delta_{K/F} > 0$

9.1. In this section, we suppose that $\delta_{K/F} > 0$ and prove Theorem 6.4 and Theorem 6.6 in this case. For $\mu \geqslant 0$ and $z \in K^\times$, we put

$$I_\mu(z) = \frac{1}{2} \sum_{\chi\in\mathcal{X}_{\mathrm{prim}}(\omega;\,2(\mu+\delta_{K/F}))} \chi(z) \tag{9.1}$$

and

$$J_\mu(z) = \frac{1}{2} \sum_{\chi\in\mathcal{X}_{\mathrm{prim}}(\omega;\,2(\mu+\delta_{K/F}))} \chi(\kappa)\,\varepsilon(\chi, \psi_K)\cdot\chi(z)\,. \tag{9.2}$$

We also put

$$I'(z) = \frac{1}{2} \sum_{\chi \in \mathcal{X}_{prim}(\omega;\, 2\delta_{K/F}-1)} \chi(z) \tag{9.3}$$

and

$$J'(z) = \frac{1}{2} \sum_{\chi \in \mathcal{X}_{prim}(\omega;\, 2\delta_{K/F}-1)} \chi(\kappa)\, \varepsilon(\chi, \psi_K) \cdot \chi(z)\,. \tag{9.4}$$

Then we have

$$\sum_{\chi \in \mathcal{X}^{+}_{prim}(\omega;\, 2(\mu+\delta_{K/F}))} \chi(z) = I_\mu(z) + J_\mu(z)$$

and

$$\sum_{\chi \in \mathcal{X}^{+}_{prim}(\omega;\, 2\delta_{K/F}-1)} \chi(z) = I'(z) + J'(z)\,.$$

In view of Proposition 7.6, Proposition 7.7 and the equalities above, the proofs of Theorem 6.4 and Theorem 6.6 are reduced to the following result.

9.2. PROPOSITION. *Let* $\mu \geqslant 0$ *and* $z = x + \theta y \in K^\times$.

(i) *We have*

$$I_\mu(z) = \begin{cases} q^{\mu+\delta_{K/F}}\,(1-q^{-1})\,\omega(x) & \cdots \quad \upsilon(z) \geqslant \mu + \delta_{K/F}\,, \\ -q^{\mu+\delta_{K/F}-1}\,\omega(x) & \cdots \quad \upsilon(z) = \mu + \delta_{K/F} - 1\,, \\ 0 & \cdots \quad \upsilon(z) \leqslant \mu + \delta_{K/F} - 2 \end{cases} \tag{9.5}$$

and

$$J_\mu(z) = \begin{cases} q^{\mu+\delta_{K/F}/2}\,\lambda_K(\psi)^{-1}\,\omega\!\left(\dfrac{\theta-\bar\theta}{\kappa} y\right) & \cdots \quad \upsilon(z) = \mu\,, \\ 0 & \cdots \quad \upsilon(z) \neq \mu\,. \end{cases} \tag{9.6}$$

(ii) *We have*

$$I'(z) = \begin{cases} q^{\delta_{K/F}-1}\,\omega(x) & \cdots \quad \upsilon(z) \geqslant \delta_{K/F} - 1\,, \\ 0 & \cdots \quad \upsilon(z) < \delta_{K/F} - 1 \end{cases} \tag{9.7}$$

and

$$J'(z) = \begin{cases} 0 & \cdots \quad \upsilon(z) \geqslant 0\,, \\ q^{(\delta_{K/F}-1)/2}\,\lambda_K(\psi)^{-1}\,\omega\!\left(\dfrac{\theta-\bar\theta}{\kappa} y\right) & \cdots \quad \upsilon(z) < 0\,. \end{cases} \tag{9.8}$$

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9.3. For $l \in \mathbf{Z}, l > 0$, we put

$$\Gamma_K(l) = F^\times (1 + \mathfrak{P}_K^l). \tag{9.9}$$

To prove Proposition 9.2, we need the following elementary fact, the proof of which is omitted.

9.4. LEMMA. *Let* $z \in K^\times$ *and* $l > 0$.

(i) *We have* $\upsilon(z) = l \Longleftrightarrow z \in \Gamma_K(2l+1) - \Gamma_K(2l+2)$.

(ii) *We have*

$$\#(K^\times/\Gamma_K(l)) = 2 \times \begin{cases} q^{l/2} & \cdots \quad 1 \ \text{ is even}, \\ q^{(l-1)/2} & \cdots \quad 1 \ \text{ is odd}, \end{cases}$$

$$\#(\Gamma_K(l)/\Gamma_K(l+1)) = \begin{cases} 1 & \cdots \quad 1 \ \text{ is even}, \\ q & \cdots \quad 1 \ \text{ is odd}. \end{cases}$$

9.5. *Proof of* (9.5) *and* (9.7). First suppose that $\mu > 0$. Take $\chi_0 \in \mathcal{X}(\omega; 2(\mu + \delta_{K/F} - 1))$. Then we have

$$\mathcal{X}_{\text{prim}}(\omega; 2(\mu + \delta_{K/F})) = \{\chi_0\, \xi \mid \xi \in G^\wedge, \xi|_H \neq 1\},$$

where $G = K^\times/\Gamma_K(2(\mu + \delta_{K/F}))$ and $H = \Gamma_K(2(\mu + \delta_{K/F} - 1))/\Gamma_K(2(\mu + \delta_{K/F}))$. The equality (9.5) now follows from Lemma 8.3, Lemma 9.4 and the fact that $\chi_0(z) = \omega(x)$ if $\upsilon(z) \geqslant \mu + \delta_{K/F} - 1$. Next suppose that $\mu = 0$. In this case, we have

$$\mathcal{X}_{\text{prim}}(\omega; 2\delta_{K/F}) = \{\chi_0'\, \xi \mid \xi \in G^\wedge, \xi|_H \neq 1\},$$

where χ_0' is an element of $\mathcal{X}(\omega; 2\delta_{K/F} - 1)$, $G = K^\times/\Gamma_K(2\delta_{K/F})$ and $H = \Gamma_K(2\delta_{K/F} - 1)/\Gamma_K(2\delta_{K/F})$. Then (9.5) is proved similarly as in the case $\mu > 0$. The equality (9.7) is proved similarly as (9.5). □

9.6. *Proof of* (9.6) *and* (9.8). Put $\alpha = \mathrm{Tr}_{K/F}(\theta)$ and $\beta = N_{K/F}(\theta)$. Then $\mathrm{ord}_F\, \alpha \geqslant 1$ and $\mathrm{ord}_F\, \beta = 1$ (cf. Section 5.1). We take a prime element π of F with $\omega(\pi) = 1$ and put $c = \pi^{\mu + \delta_{K/F} + n_\psi}(\theta - \overline{\theta})$. Then, for $\chi \in \mathcal{X}_{prim}(\omega; 2(\mu + \delta_{K/F}))$, we have

$$\varepsilon(\chi, \psi_K) = q^{\mu + \delta_{K/F}}\, \chi(c) \int_{\mathcal{O}_K^\times} \chi^{-1}(u)\, \psi_K\!\left(\frac{u}{c}\right) \mathrm{d}_{\mathcal{O}_K} u .$$

It follows that, for $z = x + \theta y \in K^\times$, we have

$$J_\mu(z) = q^{\mu + \delta_{K/F}}\, \omega\!\left(-\frac{\theta - \overline{\theta}}{\kappa}\right) \int_{\mathcal{O}_K^\times} I_\mu(z\overline{u})\, \psi_K\!\left(\frac{u}{c}\right) \mathrm{d}_{\mathcal{O}_K} u \tag{9.10}$$

(note that $I_\mu(N_{K/F}(w)z) = I_\mu(z)$ for $w \in K^\times$). We first show that $J_\mu(z) = 0$ if

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$\upsilon(z) = \mathrm{ord}_F(y/x) < 0$. For $u = s + \theta t \in \mathcal{O}_K^\times$, we have

$$\upsilon(z\overline{u}) = \mathrm{ord}_F \frac{s - tx/y}{\beta t + (s + \alpha t)\, x/y} \leqslant -1,$$

since $s \in \mathcal{O}_F^\times$, $t \in \mathcal{O}_F$. In view of (9.5), we see that $I_\mu(z\overline{u}) = 0$, which proves our claim. Suppose now that $\upsilon(z) \geqslant 0$. Then $x^{-1}z = 1 + \theta y/x \in \mathcal{O}_K^\times$. Changing the variable u into $(1 + \theta y/x)\overline{u}$ in the integral in (9.10), we have

$$\begin{aligned} J_\mu(z) &= q^{\mu+\delta_{K/F}}\, \omega\Big(-\frac{\theta - \overline{\theta}}{\kappa} x\Big) \int_{\mathcal{O}_K^\times} I_\mu(u)\, \psi_K\Big(\frac{(1 + \theta y/x)\overline{u}}{c}\Big)\, \mathrm{d}_{\mathcal{O}_K} u \\ &= q^{\mu+\delta_{K/F}}\, \omega\Big(-\frac{\theta - \overline{\theta}}{\kappa} x\Big) \int_{\mathcal{O}_F^\times} \mathrm{d}s \int_{\mathcal{O}_F} \mathrm{d}t\, I_\mu(s + \theta t)\, \psi_K\Big(\frac{(1 + \theta y/x)(s + \overline{\theta} t)}{c}\Big). \end{aligned}$$

Changing the variable t into st and using Lemma 3.3 (ii), we obtain

$$\begin{aligned} J_\mu(z) &= q^{\mu+\delta_{K/F}}\, \omega\Big(-\frac{\theta - \overline{\theta}}{\kappa} x\Big) \int_{\mathcal{O}_F} \left\{ \int_{\mathcal{O}_F^\times} \omega(s)\, \psi\Big(s \frac{y/x - t}{\pi^{\mu+\delta_{K/F}+n_\psi}}\Big)\, \mathrm{d}s \right\} I_\mu(1 + \theta t)\, \mathrm{d}t \\ &= q^{\mu+\delta_{K/F}/2}\, \lambda_K(\psi)\, \omega\Big(-\frac{\theta - \overline{\theta}}{\kappa} x\Big) J'_\mu(z), \end{aligned}$$

where

$$J'_\mu(z) = \int_{\mathcal{O}_F,\, \mathrm{ord}_F(t - y/x) = \mu} \omega\Big(\frac{y}{x} - t\Big)\, I_\mu(1 + t\theta)\, \mathrm{d}t.$$

It follows from (9.5) that, if $\upsilon(z) \neq \mu$, we have $J'_\mu(z) = 0$ and hence $J_\mu(z) = 0$. Finally suppose that $\upsilon(z) = \mu$ and put $\eta = -y/\pi^\mu x \in \mathcal{O}_F^\times$. Then we have

$$J'_\mu(z) = q^{-\mu} \int_{\mathcal{O}_F^\times} \omega(-t')\, I_\mu(1 + \pi^\mu(t' - \eta)\theta)\, \mathrm{d}t'.$$

By (9.5), we obtain

$$\begin{aligned} J'_\mu(z) &= q^{-\mu}\omega(-1) \left\{ \int_{\mathcal{O}_F^\times,\, \mathrm{ord}_F(t' - \eta) = \delta_{K/F} - 1} (-q^{\mu+\delta_{K/F}-1})\, \omega(t')\, \mathrm{d}t' \right. \\ &\qquad \left. + \int_{\mathcal{O}_F^\times,\, \mathrm{ord}_F(t' - \eta) \geqslant \delta_{K/F}} (1 - q^{-1}) q^{\mu+\delta_{K/F}}\, \omega(t')\, \mathrm{d}t' \right\} \\ &= q^{-\mu}\omega(-1)\{(-q^{\mu+\delta_{K/F}-1})(-q^{-\delta_{K/F}}\omega(\eta)) + (1 - q^{-1})\, q^{\mu+\delta_{K/F}} \cdot q^{-\delta_{K/F}}\omega(\eta)\} \\ &= \omega\Big(\frac{y}{x}\Big). \end{aligned}$$

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We thus have

$$J_\mu(z) = q^{\mu+\delta_{K/F}/2}\,\lambda_K(\psi)\,\omega\Big(-\frac{\theta-\overline{\theta}}{\kappa}y\Big) = q^{\mu+\delta_{K/F}/2}\,\lambda_K(\psi)^{-1}\,\omega\Big(\frac{\theta-\overline{\theta}}{\kappa}y\Big),$$

which proves (9.6). The equality (9.8) is proved similarly (in this case, we take $c = \pi^{\delta_{K/F}+n_\psi}(\theta-\overline{\theta})/\overline{\theta}$). $\square$

10. The Inner Structure of $V(\mathfrak{a})$

10.1. The object of this section is to show that each element of $V(\mathfrak{a})$ can be written as a sum of a primitive theta function and 'shifts' of primitive ones of 'lower index'. Note that the results of this section were essentially proven in [Sh].

For $r \in F^\times$, put $\psi_r(t) = \psi(rt)$. Let $\mathcal{L}^{(r)}$ be a lattice of K self-dual with respect to $\psi_r(\langle\,,\rangle)$ satisfying $(l+\overline{l})/2 \in \mathcal{L}^{(r)}$ for $l \in \mathcal{L}^{(r)}$. Let $(V^{(r)}, \rho^{(r)})$ be the lattice model with respect to $\mathcal{L}^{(r)}$ (cf. Section 1.5). The following fact, which was proven in [Sh] in the global setting, is crucial in the later discussion. We postpone its proof until the end of this section.

10.2. PROPOSITION. *For $\beta \in K^\times$, there exists a linear isomorphism $\mathcal{I}_\beta^{(r)}$ of $V^{(r/N(\beta))}$ and $V^{(r)}$ satisfying*

$$\mathcal{I}_\beta^{(r)} \circ \rho^{(r/N(\beta))}(\beta w, N(\beta)t) = \rho^{(r)}(w,t) \circ \mathcal{I}_\beta^{(r)} \qquad ((w,t) \in H)\,. \tag{10.1}$$

Remark. By the Stone–von Neumann theorem, $\mathcal{I}_\beta^{(r)}$ is uniquely determined by (10.1) up to scalar multiples. This implies that, for $\beta, \beta' \in K^\times$, we have $\mathcal{I}_\beta^{(r)} \circ \mathcal{I}_{\beta'}^{(r/N(\beta))} = c^{(r)}(\beta,\beta') \cdot \mathcal{I}_{\beta\beta'}^{(r)}$ with $c^{(r)}(\beta,\beta') \in \mathbf{C}^\times$.

10.3. For an ideal $\mathfrak{a}$ of K, we let $V^{(r)}(\mathfrak{a}) \subset V^{(r)}$ and $\mathcal{P}_\mathfrak{a}^{(r)} \in \mathrm{End}(V^{(r)})$ as in Section 5. Note that $V^{(r)}(\mathfrak{a}) = \{0\}$ unless $\mu_\mathfrak{a}^{(r)} := n_\psi + \mathrm{ord}_F(r\kappa/(\theta-\overline{\theta})) + \mathrm{ord}_F N(\mathfrak{a}) \geqslant 0$. When $\mu_\mathfrak{a}^{(r)} \geqslant 0$, we define the primitive part $V_{\mathrm{prim}}^{(r)}(\mathfrak{a})$ of $V^{(r)}(\mathfrak{a})$ as in Section 5.7. By (10.1), we have

$$\mathcal{I}_\beta^{(r)} \circ \mathcal{P}_{\beta\mathfrak{a}}^{(r/N(\beta))} = \mathcal{P}_\mathfrak{a}^{(r)} \circ \mathcal{I}_\beta^{(r)}\,. \tag{10.2}$$

In what follows, we often write $V(\mathfrak{a}), \mathcal{P}_\mathfrak{a}, \mu_\mathfrak{a}$ and $\mathcal{I}_\beta$ for $V^{(1)}(\mathfrak{a}), \mathcal{P}_\mathfrak{a}^{(1)}, \mu_\mathfrak{a}^{(1)}$ and $\mathcal{I}_\beta^{(1)}$ respectively.

10.4. LEMMA. *Let β be a nonzero element of $\mathcal{O}_K$ with $\mu_\mathfrak{a} - \mathrm{ord}_F N(\beta) \geqslant 0$.*

(i) *We have $\mathcal{I}_\beta(V^{(1/N(\beta))}(\mathfrak{a})) \subset V(\mathfrak{a})$.*
(ii) *If $\beta' \in \mathcal{O}_K$ and $\beta/\beta' \in \mathcal{O}_K^\times$, we have $\mathcal{I}_\beta(V_{\mathrm{prim}}^{(1/N(\beta))}(\mathfrak{a})) = \mathcal{I}_{\beta'}(V_{\mathrm{prim}}^{(1/N(\beta'))}(\mathfrak{a}))$.*

Proof. The lemma follows from (10.2) and Lemma 5.4 (i). $\square$

10.5. For an integral ideal $\mathfrak{b}$ of K with $\mu_{\mathfrak{a}} - \mathrm{ord}_F\, N(\mathfrak{b}) \geqslant 0$, we define a subspace $V_{\mathfrak{b}}(\mathfrak{a})$ of $V(\mathfrak{a})$ by

$$V_{\mathfrak{b}}(\mathfrak{a}) = \mathcal{I}_\beta(V_{prim}^{(1/N(\beta))}(\mathfrak{a})), \tag{10.3}$$

where $\mathfrak{b} = \beta\,\mathcal{O}_K$. Note that the right-hand side of (10.3) is independent of the choice of β by Lemma 10.4 (ii). Observe that $V_{\mathcal{O}_K}(\mathfrak{a}) = V_{prim}(\mathfrak{a})$ and that

$$\dim V_{\mathfrak{b}}(\mathfrak{a}) = \dim V_{prim}^{(1/N(\beta))}(\mathfrak{a})\,. \tag{10.4}$$

10.6. LEMMA. *Let* $\mathfrak{b}, \mathfrak{b}'$ *be integral ideals of* K *with*

$$\mathfrak{b} \neq \mathfrak{b}', \qquad \mu_{\mathfrak{a}} - \mathrm{ord}_F\, N(\mathfrak{b}) \geqslant 0 \quad \textit{and} \quad \mu_{\mathfrak{a}} - \mathrm{ord}_F\, N(\mathfrak{b}') \geqslant 0\,.$$

If $\mathrm{ord}_F\, N(\mathfrak{b}) \neq \mathrm{ord}_F\, N(\mathfrak{b}')$ *or* $\mathrm{ord}_F\, N(\mathfrak{b}) = \mathrm{ord}_F\, N(\mathfrak{b}') < \mu_{\mathfrak{a}}$, *then we have* $V_{\mathfrak{b}}(\mathfrak{a}) \perp V_{\mathfrak{b}'}(\mathfrak{a})$.

Proof. Let $\mathfrak{b} = \beta\mathcal{O}_K$ and $\mathfrak{b}' = \beta'\mathcal{O}_K$ with $\beta, \beta' \in \mathcal{O}_K$, and let $\Phi \in V_{\mathrm{prim}}^{(1/N(\beta))}(\mathfrak{a})$, $\Phi' \in V_{\mathrm{prim}}^{(1/N(\beta'))}(\mathfrak{a})$. First consider the case where K is a field. We may (and do) suppose that $\beta'/\beta \in \mathcal{O}_K - \mathcal{O}_K^\times$. By (10.2), we have

$$\begin{aligned}(\mathcal{I}_\beta\Phi, \mathcal{I}_{\beta'}\Phi') &= (\mathcal{I}_\beta\Phi, \mathcal{I}_{\beta'}\,\mathcal{P}_{\mathfrak{a}}^{(1/N(\beta'))}\Phi') = (\mathcal{I}_\beta\Phi, \mathcal{P}_{\beta'^{-1}\mathfrak{a}}\mathcal{I}_{\beta'}\Phi')\\ &= (\mathcal{P}_{\beta'^{-1}\mathfrak{a}}\mathcal{I}_\beta\Phi, \mathcal{I}_{\beta'}\Phi') = (\mathcal{I}_\beta\mathcal{P}_{\beta'^{-1}\beta\mathfrak{a}}^{(1/N(\beta))}\Phi, \mathcal{I}_{\beta'}\Phi')\,,\end{aligned}$$

which vanishes by the primitivity of Φ. Next consider the case where $K = F \oplus F$ and put $\Pi_1 = (\pi, 1), \Pi_2 = (1, \pi)$. Observe that, for $j \in \mathbf{Z}$ with $\mu_{\mathfrak{a}} - \mathrm{ord}_F\, N(\beta) - (j+1) \geqslant 0$, we have

$$\mathcal{P}_{\Pi_1^{-1}\Pi_2^{-j}\mathfrak{a}}^{(1/N(\beta))}\Phi = \mathcal{P}_{\Pi_1^{-j}\Pi_2^{-1}\mathfrak{a}}^{(1/N(\beta))}\Phi = 0 \qquad (\Phi \in V_{\mathrm{prim}}^{(1/N(\beta))}(\mathfrak{a}))\,. \tag{10.5}$$

This fact is easily verified and we omit its proof. We may (and do) suppose that $\beta = \Pi_1^k\Pi_2^l$ and $\beta' = \Pi_1^{k'}\Pi_2^{l'}$ $(k, l, k', l' \geqslant 0)$ with $k + l < k' + l' \leqslant \mu_{\mathfrak{a}}$ or $k + l = k' + l' < \mu_{\mathfrak{a}}$. First consider the case $k' > k$. By (10.5), we have

$$(\mathcal{I}_\beta\Phi, \mathcal{I}_{\beta'}\Phi') = (\mathcal{I}_\beta\Phi, \mathcal{I}_{\beta'}\mathcal{P}_{\Pi_1^{k'-k-1}\mathfrak{a}}^{(1/N(\beta'))}\Phi') = (\mathcal{I}_\beta\mathcal{P}_{\Pi_1^{-1}\Pi_2^{l-l'}\mathfrak{a}}^{(1/N(\beta))}\Phi, \mathcal{I}_{\beta'}\Phi') = 0\,,$$

since $(\mu_{\mathfrak{a}} - k - l) - (l' - l + 1) \geqslant 0$. The assertion in the case $k' \leqslant k$ is proved similarly as above. □

10.7. LEMMA. *Suppose that* $K = F \oplus F$ *and put* $\mathfrak{P}_1 = \Pi_1\mathcal{O}_K$, $\mathfrak{P}_2 = \Pi_2\mathcal{O}_K$. *Then*

$$W = \sum_{k=0}^{\mu_{\mathfrak{a}}} V_{\mathfrak{P}_1^k\,\mathfrak{P}_2^{\mu_{\mathfrak{a}}-k}}(\mathfrak{a}) \tag{10.6}$$

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is a direct sum and hence

$$\dim W = \mu_{\mathfrak{a}} + 1\,. \tag{10.7}$$

Proof. For simplicity, we write μ for $\mu_{\mathfrak{a}}$. Observe that $V^{(1/\pi^{\mu})}(\mathfrak{a}) = V_{prim}^{(1/\pi^{\mu})}(\mathfrak{a}) = \mathbf{C}\cdot\Phi_0$ $(\Phi_0 \neq 0)$. We now show that $\{\mathcal{I}_{\Pi_1^k\Pi_2^{\mu-k}}\Phi_0\}_{0\leqslant k\leqslant\mu}$ is linearly independent, which implies the lemma. Since the claim is trivial when $\mu = 0$, we assume $\mu > 0$. Suppose that there exists a nontrivial linear relation

$$\sum_{k=0}^{\mu} a_k\,\mathcal{I}_{\Pi_1^k\Pi_2^{\mu-k}}\Phi_0 = 0$$

with $a_k \in \mathbf{C}, a_0 = \cdots = a_{l-1} = 0, a_l \neq 0$. For $\beta \in K^\times$, we write $\mathcal{I}'_\beta$ for $\mathcal{I}_\beta^{(1/\pi^{l+1})}$ to simlify the notation. Apply $\mathcal{I}'_{\Pi_1^{-l-1}}$ to both sides of the above equality. In view of the Remark to Proposition 10.2, we obtain

$$\mathcal{I}'_{\Pi_1^{-1}\Pi_2^{\mu-l}}\Phi_0 = \sum_{k=l+1}^{\mu} b_k\,\mathcal{I}'_{\Pi_1^{k-l-1}\Pi_2^{\mu-k}}\Phi_0 \quad (b_k \in \mathbf{C})\,. \tag{10.8}$$

Since the right-hand side of (10.8) is $H(\mathfrak{a})$-invariant, we have

$$\mathcal{I}'_{\Pi_1^{-1}\Pi_2^{\mu-l}}\Phi_0 = \mathcal{P}_{\mathfrak{a}}^{(1/\pi^{l+1})}\,\mathcal{I}'_{\Pi_1^{-1}\Pi_2^{\mu-l}}\Phi_0 = \mathcal{I}'_{\Pi_1^{-1}\Pi_2^{\mu-l}}\,\mathcal{P}^{(1/\pi^{\mu})}_{\Pi_1^{-1}\Pi_2^{\mu-l}\mathfrak{a}}\Phi_0.$$

Since $\mathcal{I}'_{\Pi_1^{-1}\Pi_2^{\mu-l}}$ is a bijection, we have $\Phi_0 = \mathcal{P}^{(1/\pi^{\mu})}_{\Pi_1^{-1}\Pi_2^{\mu-l}\mathfrak{a}}\Phi_0 = 0$, which is a contradiction. □

Remark. The sum (10.6) is not necessarily an orthogonal sum.

We now state the main result of this section.

10.8. THEOREM ([Sh]).

(i) *If* $K = F \oplus F$, *we have an orthogonal decomposition* $V(\mathfrak{a}) = \bigoplus_{\mathfrak{b}} V_{\mathfrak{b}}(\mathfrak{a}) \oplus W$, *where* $\mathfrak{b}$ *runs over the integral ideals of* K *with* $\mu_{\mathfrak{a}} - \mathrm{ord}_F N(\mathfrak{b}) > 0$.

(ii) *If* K *is a field, we have an orthogonal decomposition* $V(\mathfrak{a}) = \bigoplus_{\mathfrak{b}} V_{\mathfrak{b}}(\mathfrak{a})$, *where* $\mathfrak{b}$ *runs over the integral ideals of* K *with* $\mu_{\mathfrak{a}} - \mathrm{ord}_F N(\mathfrak{b}) \geqslant 0$.

Proof. By Lemma 10.6, the sum is an orthogonal direct sum in both cases. By Proposition 5.10, (10.4) and (10.7), we see that $\dim V(\mathfrak{a})$ is equal to the dimension of the direct sum in both cases, which proves the theorem. □

10.9. From now on, we fix $r \in F^\times$ and $\beta \in K^\times$, and show the existence of $\mathcal{I}_\beta^{(r)}$ satisfying the condition of Proposition 10.2. For simplicity, we write $\psi', \mathcal{L}'$ and (V', ρ') for $\psi_{r/N(\beta)}, \mathcal{L}^{(r/N(\beta))}$ and $(V^{(r/N(\beta))}, \rho^{(r/N(\beta))})$, respectively. Define

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$\mathcal{R} \in \mathrm{End}(V')$ by

$$\mathcal{R} = \int_{\mathcal{L}^{(r)}} \psi_r\left(\frac{1}{4}\langle l, \bar{l}\rangle\right) \rho'(\beta l, 0)\, \mathrm{d}l\,.$$

The following lemma can be proved similarly as in Section 5 and we omit its proof.

10.10. LEMMA.

(i) *We have* $\mathcal{R}^2 = \mathcal{R}$.
(ii) *There exists an ideal* $\mathfrak{a}_0$ *of* K *such that, for any ideal* $\mathfrak{a} \subset \mathfrak{a}_0$, *we have* $\mathcal{R}\, V'(\mathfrak{a}) \subset V'(\mathfrak{a})$ *and* $\mathrm{Tr}(\mathcal{R} | V'(\mathfrak{a})) = 1$.

10.11. COROLLARY. *There exists a nonzero element* Φ'_β *of* V' *(unique up to scalar multiples) satisfying*

$$\rho'(\beta l, 0)\, \Phi'_\beta = \psi\left(-\frac{1}{4}\langle l, \bar{l}\rangle\right) \Phi'_\beta \qquad (l \in \mathcal{L}^{(r)})\,.$$

10.12. *Proof of Proposition 10.2*. Take Φ'_β as in Corollary 10.11. For $\Phi' \in V'$, we put

$$\mathcal{I}^{(r)}_\beta\, \Phi'(z) = (\rho'(\beta z, 0)\, \Phi', \Phi'_\beta) \qquad (z \in K)\,. \tag{10.9}$$

It is easily verified that $\mathcal{I}^{(r)}_\beta\, \Phi' \in V^{(r)}$ and that

$$\mathcal{I}^{(r)}_\beta \circ \rho'(\beta w, N(\beta) t) = \rho^{(r)}(w, t) \circ \mathcal{I}^{(r)}_\beta \qquad ((w, t) \in H)\,. \tag{10.10}$$

To prove the bijectivity of $\mathcal{I}^{(r)}_\beta$, we have only to show that $\mathcal{I}^{(r)}_\beta \neq 0$ in view of (10.10) and the Stone–von Neumann theorem. Let Φ'^0 be the element of V' given by

$$\Phi'^0(z) = \begin{cases} \psi'\left(\frac{1}{4}\langle z, \bar{z}\rangle\right) & \cdots \quad z \in \mathcal{L}' \\ 0 & \cdots \quad \text{otherwise}\,. \end{cases}$$

Then $\mathcal{I}^{(r)}_\beta\, \Phi'^0(z) = (\rho'(\beta z, 0)\, \Phi'^0, \Phi'_\beta) = \overline{\Phi'_\beta(-\beta z)}$ (cf. Lemma 1.7) and hence $\mathcal{I}^{(r)}_\beta\, \Phi'^0 \neq 0$, which completes the proof of Proposition 10.2.

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References

[FQ] Fröhlich, A. and Queyrut, J.: On the functional equation of the Artin L-function for characters of real representations, *Invent. Math.* **20** (1973), 125–138.

[GR] Glauberman, G.I. and Rogawski, J.D.: On theta functions with complex multiplication, *J. reine angew. Math.* **395** (1989), 68–101.

[HKS] Harris, M., Kudla, S.S. and Sweet, W.J.: Theta dichotomy for unitary groups, *J. Amer. Math. Soc.* **9** (1996), 941–1004.

[Ho] Howe, R.: On the character of Weil's representation, *Trans. Amer. Math. Soc.* **177** (1973), 287–298.

[Ka] Kato, Y.: On theta functions occuring in the Fourier expansion of automorphic forms on unitary groups of order 3 (in Japanese), Master thesis (1980), Univ. of Tokyo.

[Mo] Moen, C.: The dual pair $(\mathrm{U}(1), \mathrm{U}(1))$ over a p-adic field, *Pacific J. Math.* **158** (1991), 365–386.

[MVW] Mœglin, C. Vignéras, M.-F. and Waldspurger, J.-L.: *Correspondences de Howe sur un corps p-adique*, Lecture Notes in Math. 1291, Springer–Verlag, New York, 1987.

[Pr] Prasad, D.: A brief survey on the theta correspondence, *Contemp. Math.* **210** (1998), 171–193.

[Ro] Rogawski, J.D.: The multiplicity formula for A-packets, In: R.P. Langlands and D. Ramakrishnan (eds), *The Zeta Functions of Picard Modular Surfaces*, 1992, pp. 395–419.

[Sh] Shintani, T.: On automorphic forms on unitary groups of order 3, unpublished manuscript (1979).

[Ta] Tate, J.: Number theoretic background, *Proc. Sympos. Pure Math.* **33**, part 2 (1979), 3–26.

[We] Weil, A.: Sur certains groupes d'opérateurs unitaires, *Acta Math.* **111** (1964), 143–211.

[Ya] Yang, T.: Eigenfunctions of the Weil representation of unitary groups of one variable, *Trans. Amer. Math. Soc.* **350** (1998), 2393–2407.