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SUBGROUPS OF FINITE INDEX IN GROUPS WITH FINITE COMPLETE REWRITING SYSTEMS

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Abstract We show that if a group G has a finite complete rewriting system, and if H is a subgroup of G with $|G : H| = n$, then $H * F_{n-1}$ also has a finite complete rewriting system (where F_{n-1} is the free group of rank $n - 1$).

Keywords: complete rewriting system; directed 2-complex; subgroup

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1. Introduction

Groves and Smith [**4**,**5**] have proved that if a group G has a subgroup of finite index with a finite complete rewriting system, then G also has such a system. However, the converse of this result still remains open: [**5**, Question 2], [**1**, p. 41]. We will prove the following result.

Theorem 1.1. *Let H be a subgroup of finite index n in a group G. If G has a finite complete rewriting system, then so does the free product $H * F_{n-1}$, where F_{n-1} is the free group of rank $n - 1$.*

Remark 1.2. Theorem 1.1 provides a link with another open question [**7**] as follows. Let A and B be finitely presented groups (or monoids). If the free product $A*B$ has a finite complete rewriting system, do A and B also have finite complete rewriting systems? (We remark that we have obtained a sufficient condition for this to be true in [**10**, Theorem D]; however, this condition is not applicable in the setting of Theorem 1.1.)

A finitely presented group G can be represented geometrically as the fundamental group of a finite 2-complex $\mathcal{K}$ with a single 0-cell. A subgroup H of finite index n is then represented by an n-fold covering $\tilde{\mathcal{K}}$ of $\mathcal{K}$. If $\mathcal{K}'$ is the 2-complex obtained from $\tilde{\mathcal{K}}$ by identifying all 0-cells to a point, then $\pi_1(\mathcal{K}') \cong H * F_{n-1}$. Geometric properties of $\mathcal{K}$ can be lifted to $\tilde{\mathcal{K}}$ and $\mathcal{K}'$. This provides a geometric link between G and $H * F_{n-1}$. Such ideas have been exploited in [**2**, **3**, **9**] for example.

However, for Theorem 1.1, we need to view G as a monoid rather than a group (rewriting systems are a monoid concept). We thus need to work with *directed* 2-complexes. We obtain some results concerning these in § 2, and complete the proof of Theorem 1.1 in § 3.

The referee has kindly pointed out that another approach to our work is provided by [**6**].

2. Directed 2-complexes

Let $\Gamma = (V, E, \iota, \tau)$ be a directed graph, where V is the set of vertices, E the set of edges, ι the initial function, τ the terminal function. We let $P(\Gamma)$ denote the set of all paths in Γ, and let

$$P^{(2)}(\Gamma) := \{(p, q) : p, q \in P(\Gamma),\ \iota(p) = \iota(q),\ \tau(p) = \tau(q)\}.$$

A *rewriting system* R on Γ is a subset of $P^{(2)}(\Gamma)$. Its elements are referred to as rewriting rules and they are written, sometimes, in the form $r : r_{+1} = r_{-1}$ for $(r_{+1}, r_{-1}) \in R$. The single-step reduction relation $\longrightarrow_R$ is the following relation on $P(\Gamma)$: $p \longrightarrow_R q$ if and only if $p = p_1 r_{+1} p_2$ and $q = p_1 r_{-1} p_2$ for some $(r_{+1}, r_{-1}) \in R$ and $p_1, p_2 \in P(\Gamma)$. Its reflexive, transitive closure is denoted by $\longrightarrow_R^*$, and its reflexive, symmetric and transitive closure is denoted by $\longleftrightarrow_R^*$. For any $p \in P(\Gamma)$, let $[p]_R$ denote the equivalence class $\{q \in P(\Gamma) : q \longleftrightarrow_R^* p\}$. The empty path at v will be denoted by $\emptyset_v$ for any $v \in V$.

It is clear that if $p \longleftrightarrow_R^* p'$ and $q \longleftrightarrow_R^* q'$, then $pq \longleftrightarrow_R^* p'q'$, provided $\tau(p) = \iota(q)$. This enables us to define a partial multiplication of equivalence classes by

$$[p]_R[q]_R = [pq]_R \quad (\text{if } \tau(p) = \iota(q)).$$

In particular, if we fix a vertex v and consider the set $\{[p]_R : p \in P(\Gamma),\ \iota(p) = \tau(p) = v\}$, then we have a multiplication on this set, and it is a monoid (with identity $[\emptyset_v]$) under this multiplication. We call the pair $[\Gamma; R]$, a *directed 2-complex* $\mathcal{K}$, and call the above monoid the *fundamental monoid* of $\mathcal{K}$ at v, denoting it by $\pi_1^+(\mathcal{K}, v)$.

We say that R is *Noetherian* if there is no infinite reduction sequence

$$p_1 \longrightarrow_R p_2 \longrightarrow_R p_3 \longrightarrow_R \cdots.$$

We say that R is *locally confluent* if, whenever $p \longrightarrow_R q_1$ and $p \longrightarrow_R q_2$, there is a $q \in P(\Gamma)$ with $q_1 \longrightarrow_R^* q$ and $q_2 \longrightarrow_R^* q$. Also, R is called *confluent* if, whenever $p \longrightarrow_R^* q_1$ and $p \longrightarrow_R^* q_2$, there is a $q \in P(\Gamma)$ with $q_1 \longrightarrow_R^* q$ and $q_2 \longrightarrow_R^* q$. If R is both Noetherian and confluent, we say that R is *complete*. It is easy to prove (by Noetherian induction) that if R is Noetherian and locally confluent, then R is complete.

A monoid presentation $\mathcal{P} = [X; R]$ can be considered as a directed 2-complex $\mathcal{K} = [\Gamma; R]$, where Γ is a graph with one vertex o and an edge x ($\iota(x) = \tau(x) = o$) for each $x \in X$, and a word on X is considered as a path in Γ. It is clear that the monoid presented by $\mathcal{P}$ is isomorphic to the fundamental monoid of $\mathcal{K}$ at o. Then R is a complete rewriting system on X if and only if R is a complete rewriting system on Γ.

For any $v \in V$, we let $\mathrm{Star}(v) := \{e \in E : \iota(e) = v\}$.

Let Γ' be another directed graph with vertex set V' and edge set E'. A *mapping*

$$\phi : \Gamma \longrightarrow \Gamma'$$

is a function from $V \cup E$ to $V' \cup E'$ with $\phi(V) \subseteq V'$, $\phi(E) \subseteq E'$ and such that

$$\phi(\iota(e)) = \iota(\phi(e)), \qquad \phi(\tau(e)) = \tau(\phi(e)),$$

for all $e \in E$. Clearly, $\phi(\mathrm{Star}(v)) \subseteq \mathrm{Star}(\phi(v))$ for any $v \in V$. We say that ϕ is *locally bijective* if

$$\phi|_{\mathrm{Star}(v)} : \mathrm{Star}(v) \longrightarrow \mathrm{Star}(\phi(v))$$

is bijective for every $v \in V$.

It is easy to prove the following lemma.

Lemma 2.1. *Let $\phi : \Gamma \longrightarrow \Gamma'$ be a locally bijective mapping of directed graphs. For any path p' in Γ', if $\iota(p') = \phi(v)$ for some vertex v in Γ, then there is a unique path p in Γ such that $\iota(p) = v$ and $\phi(p) = p'$.*

We call p the *lift* of p' at v.

Let $\mathcal{K} = [\Gamma; R]$, $\mathcal{K}' = [\Gamma'; R']$ be directed 2-complexes. A mapping from $\mathcal{K}$ to $\mathcal{K}'$ is a mapping ϕ of directed graphs from Γ to Γ' such that $(\phi(r_{+1}), \phi(r_{-1})) \in R'$ for each $r \in R$. It is clear that if $p \longleftrightarrow_R^* q$ then $\phi(p) \longleftrightarrow_{R'}^* \phi(q)$. Thus, we get an induced homomorphism

$$\phi_* : \pi_1^+(\mathcal{K}, v) \longrightarrow \pi_1^+(\mathcal{K}', \phi(v)), \qquad \phi_*([p]_R) = [\phi(p)]_{R'}.$$

A mapping ϕ from $\mathcal{K}$ to $\mathcal{K}'$ will be called *locally bijective* if the underlying mapping of directed graphs is locally bijective and if $\phi^{-1}(R') = R$ (that is, if $r' = (r'_{+1}, r'_{-1}) \in R'$ and r_{+1}, r_{-1} are the unique lifts of r'_{+1}, r'_{-1} at some vertex v of $\mathcal{K}$, then $\tau(r_{+1}) = \tau(r_{-1})$ and $(r_{+1}, r_{-1}) \in R$).

Lemma 2.2. *Let $\phi : \mathcal{K} \longrightarrow \mathcal{K}'$ be locally bijective, and let p, q be paths in $\mathcal{K}$ with $\iota(p) = \iota(q)$. Then $\phi(p) \longrightarrow_{R'}^* \phi(q)$ if and only if $p \longrightarrow_R^* q$, and $\phi(p) \longleftrightarrow_{R'}^* \phi(q)$ if and only if $p \longleftrightarrow_R^* q$. In particular, for any vertex v of $\mathcal{K}$ the induced homomorphism*

$$\phi_* : \pi_1^+(\mathcal{K}, v) \longrightarrow \pi_1^+(\mathcal{K}', \phi(v))$$

is injective.

Proof. Similar to (1.2) of [**9**]. □

Lemma 2.3. *Let $\phi : \mathcal{K} \longrightarrow \mathcal{K}'$ be locally bijective. Suppose R' is a complete rewriting system on Γ'. Then R is a complete rewriting system on Γ.*

Proof. There can be no infinite reduction sequence in $\mathcal{K}$, for otherwise, applying ϕ would give an infinite reduction sequence in $\mathcal{K}'$. Thus R is Noetherian.

To show confluence of R, suppose that $p \longrightarrow_R^* q_1$ and $p \longrightarrow_R^* q_2$. Then $\phi(p) \longrightarrow_{R'}^* \phi(q_1)$ and $\phi(p) \longrightarrow_{R'}^* \phi(q_2)$. Since R' is confluent, there exists a path q' in $\mathcal{K}'$ such that $\phi(q_1) \longrightarrow_{R'}^* q'$ and $\phi(q_2) \longrightarrow_{R'}^* q'$. Let q be the lift of q' at $\iota(q_1)$ (note that $\iota(q') = \iota(\phi(q_1)) = \phi(\iota(q_1))$. Then $\phi(q_1) \longrightarrow_{R'}^* \phi(q)$ and $\phi(q_2) \longrightarrow_{R'}^* \phi(q)$. So $q_1 \longrightarrow_R^* q$ and $q_2 \longrightarrow_R^* q$ (by Lemma 2.2). □

Lemma 2.4. *Let Γ be a directed graph and let o be a vertex of Γ. Suppose that there is a path in Γ from o to v for every vertex v in Γ. Then there exists a subgraph T of Γ such that $V(T) = V(\Gamma)$, and there is a unique path in T from o to v for every $v \in V(\Gamma)$.*

We will call T a *maximal tree* of Γ at o.

Proof. Use Zorn's lemma. □

Let $\mathcal{K} = [\Gamma; R]$ be a directed 2-complex. Let T be a maximal tree of Γ at o, and, for each vertex v, let γ_v be the unique path in T from o to v. Let us say that T is *complemented* in $\mathcal{K}$ if, for each vertex v, there is a path γ'_v in Γ from v to o such that $\gamma_v\gamma'_v \longleftrightarrow_R^* \emptyset_o$ and $\gamma'_v\gamma_v \longleftrightarrow_R^* \emptyset_v$.

Every path p in Γ can be considered as a word on the edge set E.

Proposition 2.5. *If $\mathcal{K}$ has a complemented maximal tree T at o, then*

$$\mathcal{P}_{\mathcal{K},T} = [E; R \cup \{(e, \emptyset) : e \in E \cap T\}]$$

is a monoid presentation for $\pi_1^+(\mathcal{K}, o)$, where $\emptyset$ denotes the empty word on E.

Proof. For any closed path $p = e_1e_2 \cdots e_l$ in Γ at o, we have

$$\begin{aligned}[p]_R &= [\emptyset_o e_1 \gamma'_{\tau(e_1)}\gamma_{\tau(e_1)} e_2 \gamma'_{\tau(e_2)}\gamma_{\tau(e_2)} \cdots \gamma'_{\tau(e_{l-1})}\gamma_{\tau(e_{l-1})} e_l \emptyset_o]_R \\ &= [\gamma_{\iota(e_1)} e_1 \gamma'_{\tau(e_1)}]_R [\gamma_{\iota(e_2)} e_2 \gamma'_{\tau(e_2)}]_R \cdots [\gamma_{\iota(e_l)} e_l \gamma'_{\tau(e_l)}]_R.\end{aligned}$$

Thus, $\pi_1^+(\mathcal{K}, o)$ can be generated by $[\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R$ $(e \in E)$. If $e \in T$, then $\gamma_{\iota(e)} e = \gamma_{\tau(e)}$ (by uniqueness), so $[\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R = [\gamma_{\tau(e)}\gamma'_{\tau(e)}]_R = 1$.

Let M be the monoid defined by $\mathcal{P}_{\mathcal{K},T}$. Let ϕ_1 be the homomorphism from the free monoid on E to $\pi_1^+(\mathcal{K}, o)$, defined by

$$e \longmapsto [\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R \quad (e \in E).$$

If $e \in T$, then $\phi_1(e) = 1 = \phi_1(\emptyset)$. Also, if $r \in R$ with $r_{+1} = e_1e_2 \cdots e_m$, $r_{-1} = e'_1e'_2 \cdots e'_l$ $(e_i, e'_j \in E)$ say, then

$$\phi_1(r_{+1}) = [\gamma_{\iota(e_1)} e_1 \gamma'_{\tau(e_1)} \gamma_{\iota(e_2)} e_2 \gamma'_{\tau(e_2)} \cdots \gamma_{\iota(e_m)} e_m \gamma'_{\tau(e_m)}]_R = [\gamma_{\iota(r_{+1})} r_{+1} \gamma'_{\tau(r_{+1})}]_R.$$

Similarly, $\phi_1(r_{-1}) = [\gamma_{\iota(r_{-1})} r_{-1} \gamma'_{\tau(r_{-1})}]_R$. So $\phi_1(r_{+1}) = \phi_1(r_{-1})$. Thus we have an induced homomorphism

$$\phi_{1*} : M \longrightarrow \pi_1^+(\mathcal{K}, o), \qquad [e]_{R'} \longmapsto [\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R,$$

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where $R' = R \cup \{(e, \emptyset) : e \in E \cap T\}$.

Now regard $\mathcal{P}_{\mathcal{K},T}$ as a directed 2-complex with one vertex c, and consider the following mapping of directed 2-complexes

$$\phi_2 : \mathcal{K} \longrightarrow \mathcal{P}_{\mathcal{K},T}, \quad v \longmapsto c, \quad e \longmapsto e \quad (v \in V,\ e \in E).$$

We have an induced homomorphism

$$\phi_{2*} : \pi_1^+(\mathcal{K}, o) \longrightarrow M, \qquad \phi_{2*}([p]_R) = [\phi_2(p)]_{R'} = [p]_{R'}.$$

Since $[\gamma_v]_{R'} = 1$ and $[\gamma_v \gamma'_v]_R = 1$, we have $[\gamma'_v]_{R'} = [\gamma_v]_{R'}[\gamma'_v]_{R'} = [\gamma_v\gamma'_v]_{R'} = 1$ for any $v \in V$. Thus,

$$\phi_{2*}\phi_{1*}([e]_{R'}) = \phi_{2*}([\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R) = [\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_{R'} = [e]_{R'},$$
$$\phi_{1*}\phi_{2*}([\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R = \phi_{1*}([e]_{R'}) = [\gamma_{\iota(e)} e \gamma'_{\tau(e)}]_R.$$

So $M \cong \pi_1^+(\mathcal{K}, o)$. □

Let $\mathcal{K}^{\text{cone}}$ be the directed 2-complex obtained from $\mathcal{K}$ by adjoining a new vertex a, adjoining edges y_v, y_v^{-1} $(v \in V)$ with $\iota(y_v) = \tau(y_v^{-1}) = a, \tau(y_v) = \iota(y_v^{-1}) = v$, and adjoining additional rewriting rules $(y_v y_v^{-1}, \emptyset_a), (y_v^{-1} y_v, \emptyset_v)$.

Proposition 2.6. *If $\mathcal{K}$ has a complemented maximal tree T at o, then*

$$\pi_1^+(\mathcal{K}^{\text{cone}}, o) \cong \pi_1^+(\mathcal{K}, o) * F_{n-1},$$

where $n = |V|$, F_{n-1} is the free group of rank $n-1$. Also, $\mathcal{P} = [E; R]$ is a monoid presentation of $\pi_1^+(\mathcal{K}^{\text{cone}}, o)$.

Proof. Let $\mathcal{K}^{\text{cone}} = [\Gamma'; R']$, where $\Gamma' = (V', E', \iota, \tau)$. Then $V' = V \cup \{a\}$, $E' = E \cup \{y_v, y_v^{-1} : v \in V\}$ and $R' = R \cup \{(y_v y_v^{-1}, \emptyset_a), (y_v^{-1} y_v, \emptyset_v) : v \in V\}$. It is clear that $\mathcal{K}^{\text{cone}}$ has a complemented maximal tree $T_1 = T \cup \{y_o^{-1}\} \cup \{a\}$ at o. By Proposition 2.5, $\pi_1^+(\mathcal{K}, o)$ has a monoid presentation $\mathcal{P}_{\mathcal{K},T}$, and $\pi_1^+(\mathcal{K}^{\text{cone}}, o)$ has a monoid presentation

$$[E \cup \{y_v, y_v^{-1} : v \in V\};\ R \cup \{(y_v y_v^{-1}, \emptyset), (y_v^{-1} y_v, \emptyset) : v \in V\} \cup \{(e, \emptyset), (y_o^{-1}, \emptyset) : e \in E \cap T\}].$$

It is clear that the monoid defined by

$$[y_v, y_v^{-1}\ (v \in V);\ (y_v y_v^{-1}, \emptyset), (y_v^{-1} y_v, \emptyset), (y_o^{-1}, \emptyset)\ (v \in V)]$$

is the free group F_{n-1} of rank $n-1$. Thus, $\pi_1^+(\mathcal{K}^{\text{cone}}, o) \cong \pi_1^+(\mathcal{K}, o) * F_{n-1}$.

It is clear that $\mathcal{K}^{\text{cone}}$ also has a complemented maximal tree $T_2 = \{y_v : v \in V\} \cup V \cup \{a\}$ at a. So, by Proposition 2.5, $\pi_1^+(\mathcal{K}^{\text{cone}}, a)$ has a monoid presentation

$$[E \cup \{y_v, y_v^{-1} : v \in V\};\ R \cup \{(y_v y_v^{-1}, \emptyset), (y_v^{-1} y_v, \emptyset) : v \in V\} \cup \{(y_v, \emptyset) : v \in V\}].$$

The above presentation is Tietze equivalent [**8**] to the presentation $[E; R]$. It is easy to check that the mapping

$$\pi_1^+(\mathcal{K}^{\text{cone}}, a) \longrightarrow \pi_1^+(\mathcal{K}^{\text{cone}}, o), \qquad [p]_{R'} \longmapsto [y_o^{-1} p y_o]_{R'}$$

is an isomorphism. Thus $\mathcal{P} = [E; R]$ is a monoid presentation of $\pi_1^+(\mathcal{K}^{\text{cone}}, o)$. □

Proposition 2.7. *Let $\mathcal{K} = [\Gamma; R]$ be a finite directed 2-complex and let $n = |V|$. If R is complete on Γ and $\mathcal{K}$ has a complemented maximal tree T at o, then $\pi_1^+(\mathcal{K}, o) * F_{n-1}$ can be presented by a finite complete rewriting system.*

Proof. By Proposition 2.6, $\pi_1^+(\mathcal{K}, o) * F_{n-1}$ has a monoid presentation $\mathcal{P} = [E; R]$. We just need to show that R is a complete rewriting system on E.

Any word w on E can be written as $w = p_1 p_2 \cdots p_k$, where $p_i \in P(\Gamma)$ and $\tau(p_i) \neq \iota(p_{i+1})$. It is clear that $w \longrightarrow_R w'$ in $\mathcal{P}$ for some word w' on E if and only if $w' = p_1 \cdots p_{i-1} q_i p_{i+1} \cdots p_k$ and $p_i \longrightarrow_R q_i$ in $\mathcal{K}$. Since R is complete on Γ, it is easy to prove that R is Noetherian and locally confluent on E. So R is a finite complete rewriting system on E. □

3. Proof of Theorem 1.1

To complete the proof of Theorem 1.1, let G have a finite complete monoid presentation, which we can regard as a directed 2-complex $\mathcal{K}_1 = [\Gamma_1; R_1]$, where Γ_1 has a single vertex c and a finite edge set X, and where R_1 is a finite complete rewriting system on Γ_1. We will construct a finite directed 2-complex $\mathcal{K} = [\Gamma; R]$ satisfying the conditions of Proposition 2.7, with $\pi_1^+(\mathcal{K}, o) \cong H$. The construction is an adaptation of the standard method of constructing covering complexes of (undirected) 2-complexes.

Take Γ to have vertex set $V = \{Hg : g \in G\}$ (right cosets), edge set $\{(Hg, x)\colon g \in G,\ x \in X\}$ with $\iota((Hg, x)) = Hg$, $\tau((Hg, x)) = Hg[x]_{R_1}$, and $R = \{r^{(v)} : v \in V, r \in R_1\}$, where, for any $v = Hg \in V$, $r \in R_1$, with $r_{+1} = x_1 x_2 \cdots x_k$, $r_{-1} = x'_1 x'_2 \cdots x'_l$ say, let

$$r_{+1}^{(v)} = (Hg, x_1)(Hg[x_1]_{R_1}, x_2) \cdots (Hg[x_1 x_2 \cdots x_{k-1}]_{R_1}, x_k),$$
$$r_{-1}^{(v)} = (Hg, x'_1)(Hg[x'_1]_{R_1}, x'_2) \cdots (Hg[x'_1 x'_2 \cdots x'_{l-1}]_{R_1}, x'_l).$$

We have the locally bijective mapping

$$\phi : \mathcal{K} \longrightarrow \mathcal{K}_1, \quad Hg \longmapsto c, \quad (Hg, x) \longmapsto x \quad (g \in G,\ x \in X).$$

Thus R is complete on Γ by Lemma 2.3.

Let $v = Hg \in V$, with $g = [x_1 x_2 \cdots x_m]_{R_1}$ $(x_i \in X)$. Then

$$(H1, x_1)(H[x_1]_{R_1}, x_2) \cdots (H[x_1 x_2 \cdots x_{m-1}]_{R_1}, x_m)$$

is a path in Γ from $o = H1$ to v. Thus, by Lemma 2.4, there exists a maximal tree T of Γ at o. Let γ_v denote the unique path in T from o to v. Let $\phi(\gamma_v) = w$ and $g_1 = [w]_{R_1}$. Then $Hg_1 = H[\phi(\gamma_v)]_{R_1} = \tau(\gamma_v) = Hg$. Let $g_1^{-1} = [w']_{R_1}$, and let $w' = x_1'x_2'\cdots x_k'$ ($x_i' \in X$). Since $[w]_{R_1}[w']_{R_1} = g_1g_1^{-1} = 1$, there is a path $\gamma_v' = (Hg_1, x_1')(Hg_1[x_1']_{R_1}, x_2')\cdots(Hg_1[x_1'x_2'\cdots x_{k-1}']_{R_1}, x_k')$ in Γ from v to o. Because

$$\phi(\gamma_v\gamma_v') = \phi(\gamma_v)\phi(\gamma_v') = ww' \longleftrightarrow^*_{R_1} \emptyset = \phi(\emptyset_o),$$

by Lemma 2.2 we have $\gamma_v\gamma_v' \longleftrightarrow^*_R \emptyset_o$. Similarly, we have $\gamma_v'\gamma_v \longleftrightarrow^*_R \emptyset_v$. Thus T is complemented.

Let

$$\phi_* : \pi_1^+(\mathcal{K}, o) \longrightarrow \pi_1^+(\mathcal{K}_1, c) = G$$

be the homomorphism induced by ϕ. By Lemma 2.2, ϕ_* is injective. Let $[p]_R \in \pi_1^+(\mathcal{K}, o)$ (p a closed path in Γ at o). Then $H1 = \tau(p) = H[\phi(p)]_{R_1}$, so we have $\phi_*([p]_R) = [\phi(p)]_{R_1} \in H$. Conversely, if $[p']_{R_1} \in H$, and if p is the lift of p' at o, then $\tau(p) = H[p']_{R_1} = H1 = o$, so $\phi_*([p]_R) = [\phi(p)]_{R_1} = [p']_{R_1}$ with $[p]_R \in \pi_1^+(\mathcal{K}, o)$. Thus $\phi_* : \pi_1^+(\mathcal{K}, o) \longrightarrow H$ is an isomorphism.

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