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RAISING THE LEVEL FOR GL_n

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Abstract

We prove a simple level-raising result for regular algebraic, conjugate self-dual automorphic forms on GL_n. This gives a systematic way to construct irreducible Galois representations whose residual representation is reducible.

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1. Introduction

Let N be a positive integer, and let f be an elliptic modular newform of weight 2 and level $\Gamma_0(N)$. If l is a prime and ι is a choice of isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$, then there is an associated Galois representation

$$r_\iota(f) : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l),$$

unramified outside Nl, uniquely characterized by the requirement that the trace of Frobenius at a prime $p \nmid Nl$ equal the pth Fourier coefficient of f (or rather, its image in $\overline{\mathbb{Q}}_l$ under ι).

After possibly making a change of basis, we may assume that $r_\iota(f)$ takes its values in $\mathrm{GL}_2(\overline{\mathbb{Z}}_l)$. We may then consider the reduced representation $\overline{r_\iota(f)} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$, which we assume to be irreducible. Let p be a prime not dividing Nl, and let $\alpha_1, \alpha_2 \in \overline{\mathbb{F}}_l^\times$ denote the eigenvalues of Frob_p. If $\alpha_1 = p^{\pm 1}\alpha_2$, then there exists a lift of $\overline{r_\iota(f)}|_{G_{\mathbb{Q}_p}}$ to a representation

$$\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$$

such that $\rho \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l$ corresponds, under the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, to an unramified twist of the Steinberg representation, which has conductor p. It therefore makes sense to ask if there exists an elliptic modular newform g of weight 2 and level $\Gamma_0(Np)$ such that $\overline{r_\iota(g)} \cong \overline{r_\iota(f)}$, there being in this instance no obstruction from local–global compatibility.

This question was first posed and answered by Ribet [Rib84], and the theme of congruences between algebraic automorphic representations has been developed in many different directions since that work. In particular, an understanding of such congruences plays a fundamental role in the proofs of all known automorphy lifting theorems.

The aim of this work is to prove new level-raising theorems for automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_E)$, where E is a CM field satisfying some additional hypotheses. (By definition, a CM field is an imaginary quadratic extension of a totally real number field.) Suppose that π is regular algebraic and conjugate self-dual. In this case, it is known that there exists a Galois representation $r_\iota(\pi) : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$, and one can formulate the question of level raising in much the same way as we have done for elliptic modular forms above. Broadly speaking, there are two main approaches. The first is to try to understand directly the natural integral structures appearing in spaces of algebraic automorphic forms. In this case, one can attempt to generalize Ribet's original argument. For unitary groups, this rests on the still unproven 'Ihara lemma' of [CHT08]. This conjecture is a statement about the structure as a $\mathrm{GL}_n(\mathbb{Q}_p)$-representation of the space of automorphic forms in characteristic l, which generalizes the classical fact that the direct sum of the two degeneracy maps from level $\Gamma_0(N)$ to level $\Gamma_0(Np)$ is injective in characteristic l (at least, after localizing at a non-Eisenstein maximal ideal).

If the residual representation $\overline{r_\iota(\pi)}$ has large image (and, in particular, is irreducible), a second approach is possible. A trick due to Taylor [Tay08] allows one to use automorphy-lifting theorems to construct automorphic representations π' congruent to π modulo l, and such that π' has essentially any local behavior away from l not ruled out by the existence of a lifting ρ as above; see [Gee11].

In this work, we therefore restrict our focus to regular algebraic, conjugate self-dual automorphic representations π of the form $\pi = \pi_1 \boxplus \pi_2$, where the π_i are cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ and $n_1 + n_2 = n$. By the theory of endoscopy, these representations often admit a descent to discrete automorphic representations of unitary groups. In this paper, we exploit this fact to find congruences between representations of this form and cuspidal automorphic representations on $\mathrm{GL}_n(\mathbb{A}_E)$, by studying the integral structure of spaces of algebraic automorphic forms on unitary similitude groups.

We now give an example of a theorem that follows from our main result. Suppose that E is an imaginary CM field with maximal totally real subfield F, and let p be a rational prime which is inert in F. Let w_0 denote a place of E above p, and suppose that w_0 is split over F. We assume that $[F:\mathbb{Q}]$ is odd. Let n_1, n_2 be distinct even integers, and let π_1, π_2 be cuspidal, conjugate self-dual automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E)$ and $\mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively, such that $\pi_1 \boxplus \pi_2$ is regular algebraic of strictly regular weight (see (2.2) below).

THEOREM 1.1. *Suppose that π_{1,w_0} and π_{2,w_0} are isomorphic to unramified twists of the Steinberg representation. Then there exists a set $\mathcal{L}$ of rational primes l of Dirichlet density 1 such that, for all $l \in \mathcal{L}$ and for all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, there exists a finite order character $\psi : G_E \to \mathbb{C}^\times$ with $\psi\psi^c = 1$, a CM quadratic extension E_1/E, and an RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_{E_1})$ satisfying the following.*

- $\overline{r_\iota(\Pi)} \cong \overline{r_\iota(\pi_1 \boxplus (\pi_2 \otimes \psi))}|_{G_{E_1}}$.
- *If w_1 is a place of E_1 above w_0, then Π_{w_1} is an unramified twist of the Steinberg representation.*
- *Π has the same infinity type as the base change of $\pi_1 \boxplus \pi_2$ to E_1 and is unramified at the primes dividing l.*

*Moreover, if $\pi_1 \boxplus \pi_2$ is ι-ordinary in the sense of [**Ger**, Definition 5.1.2], then we can assume that Π is also ι-ordinary.*

For our main theorem, see Theorem 7.1 below. It is worth noting that, at the same time as proving our main result, we also establish the analogue of Ihara's lemma in the simplest possible nontrivial case. This is a new result even when we localize at a non-Eisenstein maximal ideal, and it would presumably allow one to establish the first nonminimal $R = \mathbb{T}$ theorems for Galois representations of unitary type, when our hypotheses are satisfied, although we have not pursued this here.

Our main interest in proving such theorems is the applications to automorphy-lifting theorems for RACSDC automorphic representations with residually reducible Galois representations. We note that for applications of this type it is essential to be able to find congruences to automorphic representations which have the same l-adic Hodge type at the primes dividing l. By combining the theorems of this paper with the main theorem of [**Tho**], one can often prove the automorphy of Galois representations $r : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ satisfying the following kinds of condition.

- r is ordinary, and there exists a place w of E not dividing l at which r looks like it corresponds to a twist the Steinberg representation; that is, there is an unramified character $\psi : G_{E_w} \to \overline{\mathbb{Q}}_l^\times$ and an isomorphism $r|_{G_{E_w}}^{ss} \cong \psi \oplus \psi\epsilon^{-1} \oplus \cdots \oplus \psi\epsilon^{1-n}$ (where ϵ denotes the cyclotomic character).

- The residual representation $\overline{r}$ is reducible, and the Jordan–Hölder factors of $\overline{r}$ are residually automorphic.

We now come to a description of main ideas of this paper. Let E be an imaginary CM field, and let π be an RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Let F denote the maximal totally real subfield of E. If I is a definite unitary group over F, split over E, then one expects (provided there is no local obstruction) to be able to descend π to an automorphic representation of $I(\mathbb{A}_F)$, which therefore admits a description in terms of algebraic modular forms. Since spaces of algebraic modular forms admit natural integral structures, they are a good place to look for congruences. (In the body of the paper, we use the notation I_1 for a definite unitary group, and I for a closely related group of unitary similitudes. In this introduction, we ignore this slight difference.)

Let v be a place of F, split in E, such that $I(F_v) \cong \mathrm{GL}_n(F_v)$. Our first main observation is that one can prove level-raising results for algebraic modular forms on I at the place v if one assumes that the cohomology groups of the $\mathcal{O}_F[1/v]$-arithmetic subgroups of $I(F_v)$ have no l-torsion. We thus reduce the problem of level raising to the problem of showing that these groups have no l-torsion.

In order to show such torsion vanishing, we compare the cohomology of these arithmetic groups with the cohomology of a PEL-type Shimura variety $S(G, U)$ obtained by 'switching primes', which is associated to an inner form G of I which has type $U(1, n-1) \times U(n)^{d-1}$ at infinity and which looks like a division algebra locally at the place v. According to a theorem of Rapoport and Zink [RZ96], these varieties admit a v-adic uniformization by the Drinfeld upper half plane. The link between the cohomology of these Shimura varieties and the group cohomology of the $\mathcal{O}_F[1/v]$-arithmetic groups of $I(F_v)$ comes from the weight spectral sequence, whose definition we recall in Section 5, and which computes the cohomology of these Shimura varieties. We show that the E_1-page of this spectral sequence can be written down in terms of algebraic modular forms on the definite unitary group I, and that the cohomology groups of the $\mathcal{O}_F[1/v]$-arithmetic groups of $I(F_v)$ appear as the terms in the first row of the E_2-page of this spectral sequence.

Lan and Suh [LS12] have proved torsion-vanishing results for the cohomology of local systems on Shimura varieties of sufficiently regular weight, using geometric methods. When the weight spectral sequence of $S(G, U)$ degenerates at E_2, we deduce from their results that the cohomology of our arithmetic groups with corresponding coefficient systems has no l-torsion. Tying everything together, this allows us to go back and prove a level-raising result for RACSDC automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$.

The assumption that l is a banal characteristic for $\mathrm{GL}_n(F_v)$ plays a key role at several points of the proof. First, our level-raising arguments use the fact that the representation $\mathrm{Ind}_B^{\mathrm{GL}_n} 1$ decomposes in banal characteristic in the same way as it does in characteristic 0, and that the cohomology groups $H^*(\mathrm{GL}_n(F_v), \pi)$ of the irreducible constituents have the same explicit description. (Here, Ind denotes the unnormalized induction, as in Section 4.) A related point is that the complex which appears as the first row of the E_1-page of the weight spectral sequence computes the desired cohomology groups only in banal characteristic. Finally, to show that the weight spectral sequence degenerates at E_2, even in characteristic l, we use a trick that mimics the use of weights in characteristic 0, and this again relies on the fact that the elements $1, q_v, \ldots, q_v^{n-1}$ are distinct modulo l. Our main theorem also contains the assumption that the weight of the initial automorphic representation is sufficiently regular, relative to the prime l. This is necessary in order to be able to apply the results of [**LS12**].

We now describe the organization of this paper. In Section 2, we introduce notation regarding the RACSDC automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$ and their relation to automorphic forms on unitary and unitary similitude groups. In Section 3, we introduce the Drinfeld upper half plane. In Section 4, we show how to prove level-raising results in banal characteristic, assuming that suitable cohomology groups are l-torsion free. The methods in this section are purely local.

In Section 5, we introduce the weight spectral sequence. In Section 6, we introduce the above-described Shimura varieties, their v-adic uniformization, and carry out the most important global steps in our argument: the comparison of cohomology of inner forms, and the proof of degeneration at E_2 of the weight spectral sequence in characteristic l, in our special case. We apply this in Section 6.5 to prove a level-raising result on the group I. Finally, in Section 7, we combine this with the results of Section 2 to deduce our main result, Theorem 7.1, which is a level-raising result for conjugate self-dual automorphic representations on GL_n.

1.1. Notation. If F is a number field, then we write G_F for its absolute Galois group. If v is a finite place of F, then we write G_{F_v} for a choice of decomposition group at v, and q_v for the cardinality of the residue field at v.

We fix for every prime l an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$. If $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation, then the semisimplification of the reduction modulo l of ρ with respect to some invariant lattice depends only on ρ, up to isomorphism, and we will write $\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ for this reduced representation.

If p is a prime and K is a finite extension of $\mathbb{Q}_p$, then there is a bijection

$$\mathrm{rec}_K : \mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K) \leftrightarrow \mathrm{WD}_{\mathbb{C}}^n W_K,$$

characterized by a certain equality of ϵ-factors and L-factors on either side; see [**HT01**, **Hen02**]. Here, we write (for $\Omega = \mathbb{C}$ or $\overline{\mathbb{Q}}_l$) $\mathrm{Adm}_\Omega \mathrm{GL}_n(K)$ for

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the set of isomorphism classes of irreducible admissible representations of this group over Ω, and $\mathrm{WD}^n_\Omega W_K$ for the set of Frobenius-semisimple Weil–Deligne representations (r, N) of W_K valued in $\mathrm{GL}_n(\Omega)$. We define $\mathrm{rec}^T_K(\pi) = \mathrm{rec}_K(\pi|\cdot|^{(1-n)/2})$. This is the normalization of the local Langlands correspondence with good rationality properties; in particular, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ and any $\pi \in \mathrm{Adm}_\mathbb{C}\,\mathrm{GL}_n(K)$, there is an isomorphism

$$\mathrm{rec}^T_K({}^\sigma\pi) \cong {}^\sigma\,\mathrm{rec}^T_K(\pi).$$

This can be seen using, for example, the characterization of rec_K and the description given in [Tat79, Section 3] of the Galois action on local ϵ-factors and L-factors. As a consequence, rec^T_K gives rise to a well-defined bijection

$$\mathrm{rec}^T_K : \mathrm{Adm}_\Omega\,\mathrm{GL}_n(K) \leftrightarrow \mathrm{WD}^n_\Omega W_K.$$

Suppose instead that K is a finite extension of $\mathbb{R}$. Then there is a bijection

$$\mathrm{rec}_K : \mathrm{Adm}_\mathbb{C}\,\mathrm{GL}_n(K) \leftrightarrow \mathrm{Rep}^n_\mathbb{C} W_K.$$

Here, we write $\mathrm{Adm}_\mathbb{C}\,\mathrm{GL}_n(K)$ for the set of infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(K)$, and $\mathrm{Rep}^n_\mathbb{C} W_K$ for the set of continuous representations of W_K into $\mathrm{GL}_n(\mathbb{C})$. We define $\mathrm{rec}^T_K(\pi) = \mathrm{rec}_K(\pi|\cdot|^{(1-n)/2})$.

2. Automorphic representations

2.1. GL_n. Let E be an imaginary CM field with totally real subfield F, and let $c \in \mathrm{Gal}(E/F)$ denote the nontrivial element. We say that an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ is RACSDC if it satisfies the following conditions.

- It is conjugate self-dual: $\pi^c \cong \pi^\vee$.
- It is cuspidal.
- It is regular algebraic. By definition, this means that, for each place $v|\infty$ of E, the representation $\mathrm{rec}^T_{E_v}(\pi_v)$ is a direct sum of pairwise distinct algebraic characters.

If π is a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, and π_∞ is tempered, then, for each embedding $\tau : E \hookrightarrow \mathbb{C}$, we are given a representation $r_\tau : \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$, induced by $\mathrm{rec}_{E_v}(\pi_v)$, where v is the infinite place induced by τ, and the isomorphism $E_v^\times \cong \mathbb{C}^\times$ induced by τ. This representation has the form

$$r_\tau(z) = ((z/\overline{z})^{a_{\tau,1}}, \ldots, (z/\overline{z})^{a_{\tau,n}}), \tag{2.1}$$

where $a_{\tau,i} \in (n-1)/2 + \mathbb{Z}$. We will refer to the tuple $\mathbf{a} = (a_{\tau,1}, \ldots, a_{\tau,n})_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$, where for each τ we have $a_{\tau,1} > a_{\tau,2} > \cdots > a_{\tau,n}$, as the infinity type of π. More

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generally, if π is any automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, and the parameters $r_\tau(z)$ associated to π are given by formula (2.1) for some real numbers $a_{\tau,i} \in \mathbb{R}$, we use the same formula to define the infinity type $\mathbf{a}$ of π. We will say that the infinity type of π is strictly regular if, for each embedding $\tau : E \hookrightarrow \mathbb{C}$, we have

$$a_{\tau,i} > a_{\tau,i+1} + 1 \tag{2.2}$$

for each i.

Suppose that π_1, π_2 are conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E)$, $\mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively, and that $\pi = \pi_1 \boxplus \pi_2$ is regular algebraic. Then the representations $\pi_i|\cdot|^{(n_i-n)/2}$ are regular algebraic. We call a representation π arising in this way an RACSD sum of cuspidal representations. In this case, define $\mathbf{a}^i = (a^i_\tau)_{\tau\in\mathrm{Hom}(E,\mathbb{C})}$ by the requirement that $(a^i_{\tau,1} + (n_i - n)/2, \ldots, a^i_{\tau,n_i} + (n_i - n)/2)$ equal the infinity type of $\pi_i|\cdot|^{(n_i-n)/2}$, and define $\mathbf{b} = (b_\tau)_{\tau\in\mathrm{Hom}(E,\mathbb{C})}$ by the formula

$$(b_{\tau,1}, \ldots, b_{\tau,n}) = (a^1_{\tau,1}, \ldots, a^1_{\tau,n_1}, a^2_{\tau,1}, \ldots, a^2_{\tau,n_2}).$$

Then there is a unique tuple $\mathbf{w} = (w_\tau)_{\tau\in\mathrm{Hom}(E,\mathbb{C})} \in S_n^{\mathrm{Hom}(E,\mathbb{C})}$ such that, for each $\tau \in \mathrm{Hom}(E, \mathbb{C})$, the infinity type of π is $(b_{\tau,w_\tau(1)}, \ldots, b_{\tau,w_\tau(n)})_{\tau\in\mathrm{Hom}(E,\mathbb{C})}$. We will say that $\pi = \pi_1 \boxplus \pi_2$ satisfies the sign condition if the following condition is satisfied. Choose for each place $v|\infty$ of F an embedding $\tau : E \hookrightarrow \mathbb{C}$ inducing v. Then

$$\prod_v \operatorname{sgn} w_{\tau(v)} = 1. \tag{2.3}$$

We remark that this condition is always satisfied if, for example, there is an imaginary CM subfield $E' \subset E$ such that $[E : E'] = 2$ and π arises as a base change from E'.

THEOREM 2.1. *Suppose that π_1, π_2 are cuspidal conjugate self-dual automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$, and that $\pi = \pi_1 \boxplus \pi_2$ is a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Then, for each isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, there is a continuous semisimple representation*

$$r_\iota(\pi) : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l),$$

uniquely characterized by the following local–global compatibility property at all primes w of E not dividing l:

$$\mathrm{WD}(r_\iota(\pi)|^{F\text{-}ss}_{G_{E_w}}) \cong \mathrm{rec}^T_{E_w}(\iota^{-1}\pi_w). \tag{2.4}$$

Proof. Arguing as in the proof of [Gue11, Theorem 2.3], we can find continuous characters $\psi_i : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ such that $\psi\psi^c = 1$ and the restriction of ψ_i to $(E \otimes_{E,\tau} \mathbb{C})^\times$ is given by $\psi_i(z) = (z/z^c)^{\delta_{i,\tau}}$, where $\delta_{i,\tau} = 0$ if $n - n_i$ is even and

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$\delta_{i,\tau} = 1/2$ if $n - n_i$ is odd. Then each $\pi_i \psi_i$ is RACSDC, and the representations $r_\iota(\pi_i \psi_i)$, characterized by a similar local–global compatibility condition, exist; see [Car12, Theorem 1.1]. We now simply take

$$r_\iota(\pi) = r_\iota(\pi_1\psi_1) \otimes r_\iota(\psi_1^{-1}|\cdot|^{(n_1-n)/2}) \oplus r_\iota(\pi_2\psi_2) \otimes r_\iota(\psi_2^{-1}|\cdot|^{(n_2-n)/2}). \qquad \square$$

If π is a regular algebraic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ of infinity type $\mathbf{a}$, we also define a tuple $\boldsymbol{\lambda} = (\lambda_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})} = (\lambda_{\tau,1}, \ldots, \lambda_{\tau,n})_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$, which we call the weight of π, by the formula $\lambda_{\tau,i} = -a_{\tau,n+1-i} + (n-1)/2 - (n-i)$. Then, for each $\tau : E \hookrightarrow \mathbb{C}$, we have $\lambda_{\tau,1} \geqslant \cdots \geqslant \lambda_{\tau,n}$, and the irreducible admissible representation of $\mathrm{GL}_n(\mathbb{C})$ corresponding to r_τ has the same infinitesimal character as the dual of the algebraic representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ_τ. The representation π is strictly regular if and only if for each τ we have $\lambda_{\tau,1} > \cdots > \lambda_{\tau,n}$.

2.2. Algebraic modular forms. Let E be an imaginary CM field with totally real subfield F. We suppose that $E = E_0 \cdot F$, where E_0 is a quadratic imaginary extension of $\mathbb{Q}$, and that E/F is everywhere unramified. Let $\dagger$ denote an involution of the second kind on the matrix algebra $M_n(E)$ corresponding to a Hermitian form on E^n. We define reductive groups I over $\mathbb{Q}$ and I_1 over F by their functors of points:

$$I(R) = \{g \in M_n(E) \otimes_{\mathbb{Q}} R \mid gg^\dagger = c(g) \in R^\times\}$$

and

$$I_1(R) = \{g \in M_n(E) \otimes_F R \mid gg^\dagger = 1\}.$$

We suppose that I is quasi-split at every finite place and that $I_1(\mathbb{R})$ is compact. (This can always be achieved. Indeed, there is an obstruction from the Hasse principle only if n is even and $[F : \mathbb{Q}]$ is odd. However, the assumption that E/F is everywhere unramified implies that $[F : \mathbb{Q}]$ is even, by [Gro03, Proposition 3.1].) If $v = ww^c$ is a place of F split in E and dividing the rational prime p, then there are isomorphisms

$$\iota_w : I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \prod_{w'|p} \mathrm{GL}_n(E_{w'}),$$
$$\iota_w : I_1(F_v) \cong \mathrm{GL}_n(E_w),$$

the product being over the primes w' of E above p with the same restriction to E_0 as w. We observe that $I(\mathbb{R})$ is not compact, but that the group I nevertheless satisfies the conditions of [Gro99, Proposition 1.4]. In particular, we can define spaces of automorphic forms on the groups I and I_1 with integral coefficients.

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Fix a prime l, and let K be a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$ and residue field k. Let $U_l \subset I(\mathbb{Q}_l)$ be an open compact subgroup, and suppose that M is a finite $\mathcal{O}$-module on which U_l acts continuously in the l-adic topology. In this case, we define $\mathcal{A}(M)$ to denote the set of locally constant functions $f : I(\mathbb{A}^\infty) \to M$ such that, for all $\gamma \in I(\mathbb{Q})$, $f(\gamma g) = f(g)$. We endow this space with an action of $I(\mathbb{A}^{l,\infty}) \times U_l$ by setting $(g \cdot f)(h) = g_l f(hg)$, where g_l denotes the projection to the l-component. If $U \subset I(\mathbb{A}^{l,\infty}) \times U_l$ is a subgroup, we set $\mathcal{A}(U, M) = \mathcal{A}(M)^U$.

LEMMA 2.2. *Let $p \neq l$ be a prime, and suppose that U^p is an open compact subgroup of $I(\mathbb{A}^{p,\infty})$ whose projection to $I(\mathbb{Q}_l)$ is contained in U_l. Then $\mathcal{A}(U^p, M)$ is an admissible representation of $I(\mathbb{Q}_p)$, in the following sense: for any open compact subgroup $U_p \subset I(\mathbb{Q}_p)$, $\mathcal{A}(U^p, M)^{U_p}$ is a finite $\mathcal{O}$-module.*

Proof. Let $U_p \subset I(\mathbb{Q}_p)$ be an open compact subgroup. By [Gro99, Proposition 1.4], $I(\mathbb{Q}) \subset I(\mathbb{A}^\infty)$ is a discrete cocompact subgroup, and the quotient $I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/U^p U_p$ is finite. Let $g_1, \ldots, g_s \in I(\mathbb{A}^\infty)$ be representatives. There is an isomorphism of $\mathcal{O}$-modules

$$\mathcal{A}(U^p U_p, M) \cong \oplus_{i=1}^s M^{\Gamma_i}, \quad f \mapsto (f(g_i))_{i=1,\ldots,s},$$

where $\Gamma_i = I(\mathbb{Q}) \cap g_i U^p U_p g_i^{-1}$. □

LEMMA 2.3. 1. *Let σ be an automorphic representation of $I(\mathbb{A})$ such that σ_∞ is isomorphic to the restriction of an algebraic representation of $I(\mathbb{C})$ to $I(\mathbb{R})$. Then there exists an automorphic representation σ_1 of $I_1(\mathbb{A}_F)$ satisfying the following.*

- *For each place p of $\mathbb{Q}$ split in E_0, $\sigma_{1,p}$ is isomorphic to the restriction of σ_p to the group $I_1(F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \subset I(\mathbb{Q}_p)$.*
- *$\sigma_{1,\infty}$ is isomorphic to the restriction of σ_∞ to $I_1(\mathbb{R})$.*

2. *Let σ_1 be an automorphic representation of $I_1(\mathbb{A}_F)$. Then there exists an automorphic representation σ of $I(\mathbb{A})$ satisfying the following.*

- *σ_∞ is isomorphic to the restriction of an algebraic representation of $I(\mathbb{C})$ to $I(\mathbb{R})$. The restriction of σ_∞ to $I_1(\mathbb{R})$ is isomorphic to $\sigma_{1,\infty}$.*
- *For each prime p split in E_0, the restriction of σ_p to the group $I_1(F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \subset I(\mathbb{Q}_p)$ is isomorphic to $\sigma_{1,p}$. If $\sigma_{1,p}$ is unramified, then σ_p is unramified. If $\sigma_{1,p}$ has an Iwahori-fixed vector, then σ_p has an Iwahori-fixed vector.*

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Proof. Let $T = \mathrm{Res}_{\mathbb{Q}}^{E_0} \mathbb{G}_m$, and let $T_1 \subset T$ denote the subtorus of elements of norm 1. Then there is an exact sequence of algebraic groups

$$1 \longrightarrow T_1 \longrightarrow T \times \mathrm{Res}_{\mathbb{Q}}^{F} I_1 \longrightarrow I \longrightarrow 1,$$

where T_1 is embedded diagonally. Let $\mathcal{A}$ denote the space of automorphic forms on I, a semisimple admissible representation of $I(\mathbb{A})$. Arguing as in the proof of [HT01, Theorem VI.2.1], we see that, given an automorphic representation σ of $I(\mathbb{A})$ appearing in $\mathcal{A}$, there is an element $g \in I(\mathbb{A})$ and an irreducible admissible constituent τ of the pullback of σ to $T(\mathbb{A}) \times I_1(\mathbb{A}_F)$ such that $\tau_1^g \cong \psi \otimes \sigma_1$ is automorphic. The representation σ_1 then has the desired properties.

Suppose conversely that σ_1 is as in the second part of the lemma. Arguing as in the proof of [HT01, Lemma VI.2.10], we can find an algebraic Hecke character $\psi : E_0^\times \backslash \mathbb{A}_{E_0}^\times$ such that the central character ω_{σ_1} of σ_1 satisfies the relation $\omega_{\sigma_1}(z) = \psi(z^{-1})$. If p is a prime split in E_0 and $\sigma_{1,p}$ is unramified or has an Iwahori-fixed vector, then ω_{σ_1} is unramified at p, and after multiplying ψ by a character of the form $\chi \circ \mathbb{N}_{E_0/\mathbb{Q}}$, χ a Dirichlet character, we can assume that ψ is unramified at all such primes.

Now $\psi \otimes \sigma_1$ is an automorphic representation of the group $T(\mathbb{A}) \times I_1(\mathbb{A}_F)$, and (see the proof of [HT01, Theorem VI.2.9]) it is a subrepresentation of the pullback to $T(\mathbb{A}) \times I_1(\mathbb{A}_F)$ of an automorphic representation σ of $I(\mathbb{A})$, which now satisfies the desired properties. □

PROPOSITION 2.4. 1. *Let π_1, π_2 be cuspidal, conjugate self-dual automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E)$, $\mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively, such that $\pi = \pi_1 \boxplus \pi_2$ is regular algebraic. Suppose that the following conditions are satisfied.*

- *If π_w is ramified, then w is split over F.*
- *$n_1 n_2$ is even.*
- *$\pi = \pi_1 \boxplus \pi_2$ satisfies the sign condition 2.3.*

Then there exists a cuspidal automorphic representation σ of $I_1(\mathbb{A}_F)$ of which π is the base change in the following sense: at every place of E at which π is unramified, the correspondence is given by the unramified base change. For every place $v = ww^c$ of F split in E, we have $\pi_w \cong \sigma_v \circ \iota_w$. The representation σ_∞ is the dual of the algebraic representation of $I_1(F \otimes_{\mathbb{Q}} \mathbb{R})$ of highest weight equal to the weight of π.

2. *Suppose conversely that σ is a cuspidal automorphic representation of $I_1(\mathbb{A}_F)$. Then there exists a partition $n = n_1 + \cdots + n_r$ and discrete automorphic representations π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ such that, at finite places,*

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$\pi_1 \boxplus \cdots \boxplus \pi_r$ is the base change of σ in the above sense. If we suppose furthermore that the π_i are cuspidal, then π_∞ is the base change of σ_∞.

Proof. The first part is proved in [CT, Proposition 2.9]. (The proof in [CT, Proposition 2.9] uses the results of [Mok], which are in turn conditional on the stabilization of the twisted trace formula. We refer the reader to [CT13, Introduction] for a more detailed discussion of this conditionality. One could presumably give a proof of the proposition in our case of interest using the results of [Lab11] instead of the results of [Mok], but we have chosen not to do this here.) The second part follows immediately from [Lab11, Corollaire 5.3]. □

3. Drinfeld's upper half plane

In this section, let F be a finite extension of $\mathbb{Q}_p$, and fix an integer $n \geqslant 2$. We write ϖ for a choice of uniformizer of F, and q for the cardinality of the residue field $\mathcal{O}_F/\varpi$. The Drinfeld p-adic upper half plane over F is a formal scheme over $\mathcal{O}_F$ whose rigid generic fiber can be identified with the open subspace of $\mathbb{P}^{n-1}_F$ obtained by deleting all F-rational hyperplanes. It receives a faithful action of the group $\mathrm{PGL}_n(F)$ and uniformizes certain Shimura varieties.

We first recall the Bruhat–Tits building BT of $\mathrm{PGL}_n(F)$. It is a simplicial complex with vertices the homothety classes of $\mathcal{O}_F$-lattices $M \subset F^n$. A set $\{M_1, \ldots, M_r\}$ of lattices up to homothety represents a simplex if we can choose representatives such that $\varpi M_r \subset M_1 \subset \cdots \subset M_r$. The maximal simplices have dimension $n-1$, and, for each k, $\mathrm{PGL}_n(F)$ acts transitively on the set of simplices of dimension k with a marked vertex. We write $\mathrm{BT}(i)$ for the set of simplices of BT of dimension i.

We write $\Omega_{\mathcal{O}_F}$ for the Drinfeld upper half plane over $\mathcal{O}_F$; see [RZ96, Section 3.71] or [Mus78]. It is a p-adic formal scheme, formally locally of finite type over $\mathrm{Spf}\,\mathcal{O}_F$, which receives a left action of $\mathrm{PGL}_n(F)$. The irreducible components of the special fiber of $\Omega_{\mathcal{O}_F}$ are geometrically irreducible, and in canonical bijection with the vertices in $\mathrm{BT}(0)$. Moreover, they are smooth, and the special fiber is a strict normal crossings divisor. In fact, BT can also be described as follows: it is the simplicial complex whose vertices are in bijection with the set of irreducible components of the special fiber of $\Omega_{\mathcal{O}_F}$. Vertices $v_1, \ldots, v_r$ give rise to a simplex if and only if the corresponding irreducible components have nontrivial intersection. If $v, w \in \mathrm{BT}(0)$, then we write $d(v, w)$ for the distance of the shortest path joining v and w; BT is connected, and $d(v, w)$ is always finite.

The irreducible component of the special fiber corresponding to the homothety class of the lattice M can be described as follows: let $Y_0 = \mathbb{P}(M) \otimes_{\mathcal{O}_F} (\mathcal{O}_F/\varpi)$. For each i, let Y_i denote the blowing-up of Y_{i-1} along the strict transforms in Y_{i-1}

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of the codimension i, $\mathcal{O}_F/\varpi$-rational linear subspaces of Y_0. Then (as observed in [Ito05, Section 6]) the desired variety is Y_{n-1}. In particular, we have the following.

PROPOSITION 3.1. *Let $\overline{s}$ be a geometric point above the closed point of* Spec $\mathcal{O}_F$. *For each prime $l \neq p$, the action of Frobenius on the étale cohomology groups $H^{2i}(Y_{n-1,\overline{s}}, \mathbb{Z}_l)$ is by the scalar q^i. These groups are torsion free. For each odd integer i, $H^i(Y_{n-1,\overline{s}}, \mathbb{Z}_l)$ is zero.*

Proof. This follows from the calculation of the cohomology of the blow-up of a smooth variety along a smooth center; see [Ito05, Section 3]. □

For global applications, we will need to introduce a simple enlargement of $\Omega_{\mathcal{O}_F}$. We write $\mathcal{M}^{\mathrm{split}}$ for the p-adic formal scheme formally locally of finite type over $\mathcal{O}_F$ given by the formula

$$\mathcal{M}^{\mathrm{split}} = \Omega_{\mathcal{O}_F} \times \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \times \mathrm{GL}_n(F)/\mathrm{GL}_n(F)^0,$$

where $\mathrm{GL}_n(F)^0 \subset \mathrm{GL}_n(F)$ is the open subgroup consisting of matrices with determinant a p-adic unit. Here, we identify the sets on the right-hand side with the corresponding constant $\mathcal{O}_F$-formal schemes. We define $\mathcal{M} = \mathcal{M}^{\mathrm{split}}\widehat{\otimes}_{\mathcal{O}_F}\mathcal{O}_{\mathbb{F}}$, where $\mathbb{F}$ denotes the completion of a maximal unramified extension of $\mathcal{O}_F$. The group $\mathbb{Q}_p^\times \times \mathrm{GL}_n(F)$ acts on both of these formal schemes on the left. By definition, $\mathrm{GL}_n(F)$ acts through its usual action on $\Omega_{\mathcal{O}_F}$, trivially on $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times$, and by left multiplication on $\mathrm{GL}_n(F)/\mathrm{GL}_n(F)^0$, and $\mathbb{Q}_p^\times$ acts trivially on $\Omega_{\mathcal{O}_F}$ and $\mathrm{GL}_n(F)/\mathrm{GL}_n(F)^0$ and by left multiplication on $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times$.

The set of irreducible components in the special fiber of $\mathcal{M}$ is in bijection with the set $\mathrm{BT}(0) \times \mathbb{Z} \times \mathbb{Z}$. We define a coloring map $\kappa : \mathrm{BT}(0) \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ by sending (M, a, b) to $\kappa(M, a, b) = \log_q[M : \mathcal{O}_F^n] + b$. We observe that κ is equivariant for the action of the group $\mathbb{Q}_p^\times \times \mathrm{GL}_n(F)$, and its fibers are precisely the orbits of this group.

If we make some more choices, then we can get an even more concrete realization of this set. Let $B \subset U_0 = \mathrm{GL}_n(\mathcal{O}_F)$ denote the standard Iwahori subgroup inside the standard maximal compact subgroup. Let

$$\zeta = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ \varpi & 0 & \dots & 0 & 0 \end{pmatrix}.$$

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For $i = 0, \ldots, n-1$, let $U_i = \zeta^{-i} U_0 \zeta^i$. These maximal compact subgroups stabilize the $n-1$ distinct vertices of the closure of the unique chamber of BT fixed by B, and their intersection is exactly equal to B. Let $x_0, \ldots, x_{n-1}$ denote these vertices. Then we have $\kappa(x_i, 0, 0) = i$, and therefore an isomorphism of $\mathbb{Q}_p^\times \times \mathrm{GL}_n(F)$-sets

$$\mathrm{BT}(0) \times \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Q}_p^\times / \mathbb{Z}_p^\times \times \coprod_{i=0}^{n-1} \mathrm{GL}_n(F)/U_i.$$

For each $i = 0, \ldots, n-1$ there is then a bijection between the set of nonempty $(i+1)$-fold intersections of irreducible components of the special fiber of $\mathcal{M}$ and the set

$$\mathrm{BT}(i) \times \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Q}_p^\times / \mathbb{Z}_p^\times \times \coprod_{E \subset \{0, \ldots, n-1\}} \mathrm{GL}_n(F)/U_E.$$

Here, the disjoint union runs over subsets E of order $i+1$, and by definition we have $U_E = \cap_{i \in E} U_i$. Finally, we have the following.

LEMMA 3.2. 1. *Let $\Gamma \subset \mathrm{GL}_n(F)^0$ denote a discrete cocompact subgroup, and suppose that, for all $x \in \mathrm{BT}(0)$, the stabilizer $Z_\Gamma(x)$ is trivial. Then, for all $x \in \mathrm{BT}(0)$, and for all $\gamma \in \Gamma$, $\gamma \neq 1$, we have $d(x, \gamma \cdot x) \geqslant 2$. The quotient $\Gamma \backslash \Omega_{\mathcal{O}_F}$ exists, and has a canonical algebraization, which is a projective algebraic variety, strictly semistable over $\mathcal{O}_F$. The irreducible components of its special fiber are geometrically irreducible and globally smooth.*

2. *Let $\Gamma \subset \mathbb{Q}_p^\times \times \mathrm{GL}_n(F)$ denote a discrete cocompact subgroup, and suppose that, for all $x \in \mathrm{BT}(0) \times \mathbb{Z} \times \mathbb{Z}$, the stabilizer $Z_\Gamma(x)$ is trivial. Then the quotient $\Gamma \backslash \mathcal{M}^{split}$ exists, and has a canonical algebraization, which is a projective algebraic variety, strictly semistable over $\mathcal{O}_F$. The irreducible components of its special fiber are geometrically irreducible and globally smooth.*

Proof. For the first part, we note that, if $d(x, y) = 1$ then there exists a chamber in BT whose closure contains x, y. Then x, y are represented by $\mathcal{O}_F$-lattices $M_x \subset M_y$. If $\gamma \in \Gamma$ and $\gamma x = y$, then we must therefore have $x = y$, and hence $\gamma = 1$. The formal scheme $\Omega_{\mathcal{O}_F}$ has a covering by Zariski open subsets, formally of finite type over $\mathcal{O}_F$, which are in bijective correspondence with the set BT(0). Two Zariski opens intersect if and only if the corresponding vertices are connected by an edge. Thus Γ acts discontinuously with respect to this covering, and the quotient formal scheme can be obtained by simply gluing these Zariski opens. The ample line bundle which defines the algebraization is the relative dualizing sheaf over Spf $\mathcal{O}_F$; see [Mus78, Theorem 4.1].

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For the second part, let $\Gamma^0 = \Gamma \cap (\mathbb{Z}_p^\times \times \mathrm{GL}_n(F)^0)$. The quotient $\Gamma \backslash \mathcal{M}^{\mathrm{split}}$ is a finite union of quotients of the form $\Gamma^0 \backslash \Omega_{\mathcal{O}_F}$. $\square$

4. A level-raising formalism in banal characteristic

Let $p \neq l$ be distinct prime numbers. Let K be a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$ and residue field k, and let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$ and uniformizer ϖ. We write q for the cardinality of the residue field $\mathcal{O}_F/\varpi$. We fix throughout a choice of square root of q in K. Throughout this section we make the following assumption.

- l is a banal characteristic for $\mathrm{GL}_n(F)$. By definition, this means that l is coprime to the pro-order of $\mathrm{GL}_n(F)$.

In this section, we show how one can prove level-raising results for $\mathrm{GL}_n(F)$-modules under the assumption that suitable cohomology groups are torsion free. Let $G = \mathrm{GL}_n(F)$. Let $T \subset P \subset G$ denote the standard maximal torus and Borel subgroup, and $R \subset \Phi^+ \subset \Phi$ the corresponding subsets of simple roots, positive roots, and roots of GL_n. Let $N \subset P$ denote the unipotent radical; then $P = TN$. Let $T_0 \subset T$ denote the unique maximal compact subgroup, and $B \subset G$ for the Iwahori subgroup. In this section, all admissible representations of G will be considered as being defined over $\overline{\mathbb{Q}}_l$. We fix a choice of isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$.

If $\chi : T \to \overline{\mathbb{Q}}_l^\times$ is a continuous character, we define

$$\mathrm{Ind}_P^G \chi = \{f : G \to \overline{\mathbb{Q}}_l \mid f(bg) = \chi(b) f(g) \forall b \in P\},$$

the unnormalized induction. The normalized induction is defined as

$$\text{n-}\mathrm{Ind}_P^G \chi = \mathrm{Ind}_P^G \delta_P^{1/2} \chi,$$

where $\delta_P : P \to \overline{\mathbb{Q}}_l^\times$ is the modulus character sending tu to $|t_1^{n-1} t_2^{n-3} \cdots t_n^{1-n}|$, and the square root is the one defined by ι. In particular, $\text{n-}\mathrm{Ind}_P^G \delta_P^{-1/2} = \mathrm{Ind}_P^G 1 = C^\infty(G/P)$ may be identified with the space of locally constant functions $G/P \to \overline{\mathbb{Q}}_l$. If π is an admissible representation of G, then we define the normalized restriction

$$r_P^G \pi = \delta_P^{-1/2} \otimes \pi_N,$$

where π_N, the module of N-coinvariants, denotes the usual un-normalized Jacquet module of π. Then $r_G^P \pi$ is an admissible representation of T, and the functor r_P^G is left adjoint to $\text{n-}\mathrm{Ind}_P^G$. If π is an admissible representation of G, and $\alpha \in \overline{\mathbb{Q}}_l^\times$, then we write $\pi(\alpha) = \pi \otimes (\det \circ \lambda_\alpha)$, where λ_α is the unramified character satisfying $\lambda_\alpha(\varpi) = \alpha$.

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We describe the decomposition of $\text{n-Ind}_P^G \delta_P^{-1/2} = C^\infty(G/P, \overline{\mathbb{Q}}_l)$. Let $I \subset R$. We write P_I for the group generated by P and the subgroups $U_{-\alpha}$ for $\alpha \in I$. Thus $P_\emptyset = P$ and $P_R = G$. For each $I \subset J$ there is an injection $C^\infty(G/P_J, \overline{\mathbb{Q}}_l) \hookrightarrow C^\infty(G/P_I, \overline{\mathbb{Q}}_l)$. We define

$$\pi_I = C^\infty(G/P_I)/\sum_{I \subsetneq J} C^\infty(G/P_J).$$

PROPOSITION 4.1. *The π_I are irreducible and pairwise nonisomorphic, and they exhaust the composition factors of* $\text{n-Ind}_P^G \delta_P^{-1/2}$.

Proof. See [BW00, Chapter X]. A convenient reference for this and for some facts below is [Orl05]. □

The π_I may be described in terms of the Zelevinsky classification [Zel80] as follows. The irreducible constituents $\pi(\vec{\Gamma})$ of $\text{n-Ind}_P^G \delta_P^{-1/2}$ are in bijection with the orientations $\vec{\Gamma}$ of the graph Γ with vertices corresponding to the characters $|\cdot|^{(1-n)/2}, \dots, |\cdot|^{(n-1)/2}$, and edges joining two characters whose quotient is $|\cdot|^{\pm 1}$. Given $i = 1, \dots, n-1$, let $\alpha_i : T \to F^\times$ denote the homomorphism which sends an element $t = \operatorname{diag}(t_1, \dots, t_n)$ to t_i/t_{i+1}. Then $R = \{\alpha_1, \dots, \alpha_{n-1}\}$. Given an orientation $\vec{\Gamma}$, we write $I(\vec{\Gamma}) \subset R$ for the subset of roots α_i such that the edge connecting $|\cdot|^{(1-n)/2+i-1}$ and $|\cdot|^{(1-n)/2+i}$ starts at the former and ends at the latter.

PROPOSITION 4.2. *We have $\pi_{I(\vec{\Gamma})} \cong \pi(\vec{\Gamma})$. In particular, $\pi_\emptyset = \mathrm{St}_n$ is the Steinberg representation, and π_R is the trivial representation of G.*

We now introduce part of the theory of the Bernstein center. If π is any admissible representation of G over $\overline{\mathbb{Q}}_l$, then we can endow the Iwahori invariants π^B with an action of the algebra $\overline{\mathbb{Q}}_l[T/T_0] \cong \overline{\mathbb{Q}}_l[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ as follows. Let $X^+ \subset T/T_0$ denote the submonoid consisting of those elements

$$(\varpi^{a_1}, \dots, \varpi^{a_n})T_0 \in T/T_0,$$

where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ are integers. We let an element xT_0 act on π^B by the Hecke operator $[BxB]$. This induces an action of the algebra $\overline{\mathbb{Q}}_l[X^+]$, which extends uniquely to an action of the algebra $\overline{\mathbb{Q}}_l[T/T_0]$. We write $t_i = e_i(X_1, \dots, X_n) \in \overline{\mathbb{Q}}_l[T/T_0]^W$, where e_i is the symmetric polynomial of degree i in n variables. As the notation indicates, these elements are fixed under the natural action of the Weyl group on $\overline{\mathbb{Q}}_l[T/T_0]$.

PROPOSITION 4.3. 1. *For any admissible representation V of G over $\overline{\mathbb{Q}}_l$, there is a functorial isomorphism $V^B \cong (r_P^G V)^{T_0}$ of $\overline{\mathbb{Q}}_l[T/T_0]$-modules.*

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2. *If π is an irreducible admissible representation of G over $\overline{\mathbb{Q}}_l$, and $\pi^B \neq 0$, then π is a subquotient of* $\text{n-Ind}_P^G \chi$ *for some unramified character $\chi = \chi_1 \otimes \cdots \otimes \chi_n$. The operator t_i has the unique eigenvalue $e_i(\chi_1(\varpi), \ldots, \chi_n(\varpi))$ on π^B.*

We introduce reduction modulo l; see [Vig94, Section 1.5]. Let V be an admissible G-module over $\overline{\mathbb{Q}}_l$ of finite length. We say that V admits an integral structure if there exists a G-invariant $\overline{\mathbb{Z}}_l$-lattice $\Lambda \subset V$ such that $\Lambda \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong V$. If V admits an integral structure, then the reduction modulo the maximal ideal of $\overline{\mathbb{Z}}_l$ of Λ is a finite-length admissible representation of G over $\overline{\mathbb{F}}_l$. Its Jordan–Hölder factors are independent of the choice of integral structure.

If π is an irreducible admissible representation of G over $\overline{\mathbb{Q}}_l$, then it admits an integral structure if and only its cuspidal support does. In particular, if π is a subquotient of a principal series representation $\text{n-Ind}_P^G \chi$, then π admits an integral structure if and only if χ takes values in $\overline{\mathbb{Z}}_l^\times \subset \overline{\mathbb{Q}}_l^\times$.

PROPOSITION 4.4. 1. *Each representation π_I admits an integral structure, and its reduction modulo l is irreducible. We write $\pi_{I,\overline{\mathbb{F}}_l}$ for this reduced representation.*

2. *Let $\pi = \text{n-Ind}_Q^G \mathrm{St}_a(\alpha) \otimes \mathrm{St}_b(\beta)$ be an irreducible representation of G over $\overline{\mathbb{Q}}_l$ admitting an integral structure, where $a + b = n$, and Q is the standard parabolic subgroup corresponding to this partition. Then $\alpha, \beta \in \overline{\mathbb{Z}}_l$. Suppose that $\beta \equiv q^a \alpha \mod \mathfrak{m}_{\overline{\mathbb{Z}}_l}$. Then the reduction modulo l of π has exactly two Jordan–Hölder factors, which are both absolutely irreducible. The first is the reduction modulo l of $\pi_\emptyset(\alpha)$. The second is the reduction modulo l of $\pi_I(\alpha)$, where $I \subset R$ is such that $P_{R \setminus I} = Q$.*

Proof. For the first part, the existence of the integral structure is immediate from the remarks above. The irreducibility of the representations $\pi_{I,\overline{\mathbb{F}}_l}$ in banal characteristic seems to have first been noted by Lazarus [Laz00, Theorem 4.7.2]. Here, we refer again to the work of Orlik [Orl05]. (The essential point is that, in banal characteristic, the characters $|\cdot|^{(1-n)/2}, \ldots, |\cdot|^{(n-1)/2}$ remain distinct even after reduction modulo l.) The second part of the proposition follows from the corresponding fact in characteristic zero, see [HT01, Lemma I.3.2], by reduction modulo l and the first part of proposition. □

Suppose that M is a smooth $\mathcal{O}[G]$-module. We define cohomology groups $H^*(M)$ as follows. Let $U_0 = \mathrm{GL}_n(\mathcal{O}_F)$ denote the standard maximal compact subgroup, and let $U_1, \ldots, U_{n-1}$ denote the conjugates of U_0 containing B, as defined in the previous section. Similarly, if $E \subset \{0, \dots, n-1\}$ is a subset, then we write $U_E = \cap_{i \in E} U_i$. We define a complex $C^\bullet(M)$ by the formula

$$C^i(M) = \bigoplus_{E \subset \{0, \dots, n-1\}} M^{U_E},$$

the direct sum being over subsets E of cardinality $i+1$. The differential $d_i : C^i(M) \to C^{i+1}(M)$ is given by the sum of the restriction maps $r_{E,E'} : M^{U_E} \to M^{U_{E'}}$ for $E \subset E'$, each multiplied by the sign $\epsilon(E, E')$, where, if $E' = \{i_1, \dots, i_r\}$, $i_1 < \cdots < i_r$, and $E = E' \setminus \{i_s\}$, then

$$\epsilon(E, E') = (-1)^s. \tag{4.1}$$

We then define $H^*(M)$ to be the cohomology of this complex.

PROPOSITION 4.5. 1. *Suppose that $M = \pi$ is an irreducible admissible representation of G over $\overline{\mathbb{Q}}_l$. Then $H^*(M)$ is nonzero if and only if π is an unramified twist of one of the representations π_I, $I \subset R$.*

2. *If $M = \pi_I(\alpha)$ for some $\alpha \in \overline{\mathbb{Q}}_l$, then $H^i(M)$ is nonzero if and only if $i = \#(R \setminus I)$.*

3. *If $M = \pi_{I,\overline{\mathbb{F}}_l}(\overline{\alpha})$ for some $\alpha \in \overline{\mathbb{F}}_l$, then $H^i(M)$ is nonzero if and only if $i = \#(R \setminus I)$.*

Proof. If $M = \pi$ is an irreducible admissible representation and $H^*(M) \neq 0$, then $\pi^B \neq 0$. In particular, π is a subquotient of an unramified principal series representation, and its central character is unramified. After twisting, we can suppose that the center of G acts trivially on π. Then there is a canonical isomorphism $H^*(M) \cong H^*_{\mathrm{e}}(\mathrm{PGL}_n(F), M)$, these latter groups taken in the sense of [BW00, Ch. X, Section 5]. The first and second parts therefore follow from [BW00, Ch. X, Theorem 4.12]. The third part follows in a similar manner from [Orl05, Theorem 1]. □

We now come to the main result of this section. Suppose that M, N are $\mathcal{O}$-flat admissible $\mathcal{O}[G]$-modules, in the sense that, for each open compact subgroup $U \subset G$, M^U and N^U are finite free $\mathcal{O}$-modules, and these submodules exhaust M and N. Suppose further that $M \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$ and $N \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$ are semisimple, and that all of their irreducible constituents are generic, and that there is a perfect G-equivariant pairing $M \times N \to \mathcal{O}$.

THEOREM 4.6. *With notation as above, suppose that $M^B \neq 0$, and that, if $\pi \subset M \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$ is an irreducible admissible representation of G satisfying $\pi^B \neq 0$, then*

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$\mathrm{rec}_F^T(\pi)$ *has at most two irreducible constituents. Suppose that* $H^{n-2}(N \otimes_{\mathcal{O}} k)$ *and* $H^{n-2}(M \otimes_{\mathcal{O}} k)$ *are both zero. Finally, suppose that there exists* $\overline{\alpha} \in \overline{\mathbb{F}}_l^\times$ *such that, for any maximal ideal* $(t_1 - \alpha_1, \ldots, t_n - \alpha_n) \subset \overline{\mathbb{Q}}_l[T/T_0]^W$ *in the support of* M^B, *we have* $\alpha_i \equiv \overline{\alpha} e_i(q^{(n-1)/2}, \ldots, q^{(1-n)/2})$ *mod* $\mathfrak{m}_{\overline{\mathbb{Z}}_l}$ *for each* $i = 1, \ldots, n$. *(Note that we necessarily have* $\alpha_i \in \overline{\mathbb{Z}}_l$.*) Then there exists* $\alpha \in \overline{\mathbb{Z}}_l^\times$ *lifting* $\overline{\alpha}$ *such that* $\mathrm{St}_n(\alpha) \subset M \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$.

Proof. After twisting by an unramified character, we can assume that $\overline{\alpha} = 1$. Decompose $N \otimes_{\mathcal{O}} k = N_0 \oplus N_1$, where N_0 is generated by $N^B \otimes_{\mathcal{O}} k$ and $N_1^B = 0$. (This is possible since the representations of G with nonzero Iwahori-fixed vectors form a block in the category of admissible representations of G over $\overline{\mathbb{F}}_l$.) Then the irreducible constituents of $N_0 \otimes_k \overline{\mathbb{F}}_l$ are of the form $\pi_\emptyset$ or $\pi_{\{\alpha\}}$ for some $\alpha \in R$, by Proposition 4.4. If π is an irreducible constituent of $N_0 \otimes_k \overline{\mathbb{F}}_l$, then the group $H^i(\pi)$ can be nonzero only if $i = n-2$ or $i = n-1$, by Proposition 4.5. It follows by dévissage that the same statement holds for the groups $H^i(N')$, where N' is any $\overline{\mathbb{F}}_l[G]$-subquotient of $N_0 \otimes_k \overline{\mathbb{F}}_l$.

If there is an embedding $\pi_{\{\alpha\}} \hookrightarrow N \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l$, then $\pi_{\{\alpha\}} \hookrightarrow N_0 \otimes_k \overline{\mathbb{F}}_l$, and we have an exact sequence

$$0 \longrightarrow \pi_{\{\alpha\}} \longrightarrow N_0 \otimes_k \overline{\mathbb{F}}_l \longrightarrow N'' \longrightarrow 0.$$

We have $H^{n-2}(\pi_{\{\alpha\}}) \neq 0$ by Proposition 4.5. It follows from the long exact sequence in cohomology that $H^{n-2}(N_0) \neq 0$; this contradicts our assumption that $H^{n-2}(N \otimes_{\mathcal{O}} k) = 0$. It follows that there must be an embedding $\pi_\emptyset \hookrightarrow N \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l$. By duality, there is a surjection $M \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l \twoheadrightarrow \pi_\emptyset$, and hence a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l \longrightarrow \pi_\emptyset \longrightarrow 0,$$

from which it follows that $H^{n-1}(M \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l) \neq 0$. Now, using the long exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow M \otimes_{\mathcal{O}} k \longrightarrow 0,$$

together with our assumption that $H^{n-2}(M \otimes_{\mathcal{O}} k) = 0$, we deduce that $H^{n-1}(M \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l) \neq 0$. It then follows from Proposition 4.5 that M contains a twist of the Steinberg representation. This completes the proof. □

If π is an irreducible admissible representation of G over $\overline{\mathbb{Q}}_l$ which admits an integral structure, and $\pi^B \neq 0$, then we will say that π *satisfies the level-raising congruence* if there exists $\overline{\alpha} \in \overline{\mathbb{F}}_l^\times$ such that, for each $i = 1, \ldots, n$, the eigenvalue α_i of t_i on π^B satisfies the congruence

$$\alpha_i \equiv \overline{\alpha} e_i(q^{(n-1)/2}, \ldots, q^{(1-n)/2}) \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_l}. \tag{4.2}$$

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5. The weight spectral sequence

Let $\mathcal{O}_F$ be a complete discrete valuation ring, and let $S = \operatorname{Spec} \mathcal{O}_F$. Write s for the closed point of S and η for the generic point. Let $F = \operatorname{Frac} \mathcal{O}_F$, and let $\overline{F}$ denote a fixed algebraic closure. We write $\overline{s}, \overline{\eta}$ for the induced geometric points of S above s and η, respectively. Suppose that $f : X \to S$ is a proper, strictly semistable (in the sense of [Sai03, Section 1.1]) morphism of relative dimension n. Then X_s is a strict normal crossings divisor on X; write $X_1, \dots, X_h$ for its irreducible components. We suppose moreover that each X_i is globally smooth over $\kappa(s)$. For $E \subset \{1, \dots, h\}$, we write X_E for the intersection $\cap_{i \in E} X_E$, and $X^{(m)} = \coprod_{\#E=m+1} X_E$ (disjoint union). Let K be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$, uniformizer λ, and residue field k, where l is coprime to the residue characteristic of $\mathcal{O}_F$. Let $\Lambda = K, \mathcal{O}$, or k, and let V be a local system of flat Λ-modules on X. The weight spectral sequence of Rapoport and Zink is a spectral sequence

$$E_1^{p,q} = \bigoplus_{i \geqslant \max(0,-p)} H^{q-2i}(X_{\overline{s}}^{(p+2i)}, V(-i)) \Rightarrow H^{p+q}(X_{\overline{K}}, V). \tag{5.1}$$

It is equivariant for the natural action of G_F on both sides, and the differentials commute with the action of the group G_F. Note that the groups $E_1^{p,q}$ vanish for $q < 0$ and $q > 2n$. Let us briefly recall the construction of this spectral sequence, following Saito [Sai03]. Consider the following diagram:

$$\begin{tikzcd}
X_{\overline{s}} \arrow[r, "\overline{i}"] \arrow[d] & X_{\mathcal{O}_{\overline{F}}} \arrow[d] & X_{\overline{\eta}} \arrow[l, "\overline{j}"'] \arrow[d] \\
X_s \arrow[r, "i"] & X & X_\eta . \arrow[l, "j"']
\end{tikzcd}$$

The complex $R\Psi V = \overline{i}^* R\overline{j}_* V$ in $D_c^b(X_{\overline{s}}, V)$ of nearby cycles receives an action of the inertia group $I_F \subset G_F = \operatorname{Gal}(\overline{F}/F)$. Let $T \in I_F$ denote an element that maps to a generator of $\mathbb{Z}_l(1)$ under the canonical homomorphism $t_l : I_F \to \mathbb{Z}_l(1)$. Let ν denote the endomorphism of $R\Psi V$ induced by the element $T - 1$. We then have the following (see [Sai03, Section 2]).

PROPOSITION 5.1. 1. *$R\Psi V$ lies in the abelian subcategory* $\operatorname{Perv}(X_{\overline{s}}, \Lambda)[-n]$ *of $-n$-shifted perverse sheaves with Λ-coefficients.*

2. *Let $M_\bullet$ denote the increasing monodromy filtration of the nilpotent endomorphism ν of $R\Psi \Lambda$. For each positive integer $p \geqslant 0$, let $a_p : X_{\overline{s}}^{(p)} \to X_{\overline{s}}$*

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denote the canonical map. Then, for each integer $r \geqslant 0$, *there is a canonical isomorphism*

$$\bigoplus_{p-q=r} a_{p+q,*}V(-p)[-(p+q)] \cong \mathrm{Gr}_r^M R\Psi V,$$

compatible with the action of G_F *on either side.*

The weight spectral sequence is now the spectral sequence associated to the filtered object $R\Psi V$. Note that [Sai03] treats only the case of constant coefficients, but the case of twisted coefficients can be reduced to this one by working étale locally on X.

We compute the first row of the spectral sequence of the pair (X, V), using [Sai03, Proposition 2.10]. By definition, we have

$$E_1^{p,0} = H^0(X_{\bar{s}}^{(p)}, V) \cong \bigoplus_{\#E=p+1} H^0(X_{E,\bar{s}}, V),$$

and the differential

$$d_1^{p,0} : E_1^{p,0} \to E_1^{p+1,0}$$

is the sum of the canonical pullback maps $i_{E,E'}^* : H^0(X_{E,\bar{s}}, V) \to H^0(X_{E',\bar{s}}, V)$, each multiplied by the sign $\epsilon(E, E')$ defined in (4.1). We define a simplicial complex $\mathcal{K}$ as follows: the vertices of $\mathcal{K}$ are in bijection with the X_i, and the set $\{X_{i_1}, \dots, X_{i_r}\}$ corresponds to a simplex σ_E if and only if the intersection X_E is nonempty, $E = \{i_1, \dots, i_r\}$. We define a coefficient system $\mathcal{V}$ on $\mathcal{K}$ by the assignment $\sigma_E \mapsto H^0(X_{E,\bar{s}}, V)$. Let $C^\bullet(\mathcal{K}, \mathcal{V})$ denote the complex calculating the simplicial cohomology of $\mathcal{K}$ with coefficients in $\mathcal{V}$. Thus, by definition, we have

$$C^r(\mathcal{K}, \mathcal{V}) = \bigoplus_{E \subset \{1,\dots,h\}} H^0(X_{E,\bar{s}}, V),$$

the sum being over subsets E of cardinality $r + 1$. The differential $d_r = C^r(\mathcal{K}, \mathcal{V}) \to C^{r+1}(\mathcal{K}, \mathcal{V})$ is given by the direct sum of the restriction maps

$$\mathrm{res}_{E,E'} : H^0(X_{E,\bar{s}}, V) \to H^0(X_{E',\bar{s}}, V),$$

each multiplied by the sign $\epsilon(E, E')$.

PROPOSITION 5.2. *There is a canonical isomorphism of complexes* $E_1^{\bullet,0} \cong C^\bullet(\mathcal{K}, \mathcal{V})$.

Proof. In the case when $V = \Lambda$, this follows immediately from [Sai03, Proposition 2.10]. Again, the case of general V can be reduced to this one by working étale locally. □

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6. Shimura varieties and uniformization

Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and let E be a CM imaginary field with totally real subfield F. We fix a rational prime p, and suppose that p is totally inert in F. We suppose that the unique prime v of F above p is split in E as $v = ww^c$. We let d denote the degree of F over $\mathbb{Q}$. We fix embeddings ϕ_∞, ϕ_p of $\overline{\mathbb{Q}}$ into $\mathbb{C}, \overline{\mathbb{Q}}_p$, respectively. The composite $\phi_\infty \circ \phi_p^{-1}$ induces a bijection of sets

$$\mathrm{Hom}(E, \mathbb{C}) \leftrightarrow \mathrm{Hom}(E, \overline{\mathbb{Q}}_p).$$

Let $n \geqslant 2$ be an integer, and let D be a central division algebra over E of dimension n^2, whose invariants at the places w and w^c are given respectively by $1/n$ and $-1/n$. We suppose that, at every other place of F, D is split. Let $*$ be a positive involution on D. Let $V = D$, viewed as a D-module, and let $\psi : V \times V \to \mathbb{Q}$ be an alternating pairing satisfying the condition $\psi(dv, w) = \psi(v, d^* w)$ for all $d \in D$, $v, w \in V$. Fix a CM-type $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$. Then we can choose an isomorphism $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau \in \Phi} D \otimes_{E,\tau} \mathbb{C} \cong \prod_{\tau \in \Phi} M_n(\mathbb{C})$, such that $*$ corresponds to the operation $X \mapsto {}^t\overline{X}$.

Similarly, we may decompose $V \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\tau \in \Phi} V \otimes_{E,\tau} \mathbb{C}$. We can find for each $\tau \in \Phi$ a complex vector space W_τ and an isomorphism $V \otimes_{E,\tau} \mathbb{C} = \mathbb{C}^n \otimes_{\mathbb{C}} W_\tau$ of $M_n(\mathbb{C})$-modules, with $M_n(\mathbb{C})$ acting on the first factor. The form ψ_τ then admits a decomposition

$$\psi_\tau(z_1 \otimes w_1, z_2 \otimes w_2) = \mathrm{tr}_{\mathbb{C}/\mathbb{R}}({}^t\overline{z}_1 \cdot z_2 h_\tau(w_1, w_2)),$$

where h_τ is a skew-hermitian form on W_τ. We can find a basis $\{e_1, \ldots, e_n\}$ of W_τ such that h_τ is given by the matrix

$$\mathrm{diag}(\underbrace{-i, \ldots, -i}_{r_\tau}, \underbrace{i, \ldots, i}_{r_{\tau^c}}),$$

where $r_\tau + r_{\tau^c} = n$. We define algebraic groups over $\mathbb{Q}$ by their functors of R-points:

$$G(R) = \{g \in \mathrm{GL}_D(V \otimes R) \mid \psi(gv, gw) = c(g)\psi(v, w), c(g) \in R^\times\}.$$
$$G_1(R) = \{g \in \mathrm{GL}_D(V \otimes R) \mid \psi(v, w) = \psi(v, w)\}.$$

The choices above give rise to an embedding $G_{\mathbb{R}} \hookrightarrow \prod_{\tau \in \Phi} GU(r_\tau, r_{\tau^c})$. We write $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$ for the homomorphism which corresponds under this identification to the map

$$h : z \in \mathbb{C}^\times \mapsto (\mathrm{diag}(\underbrace{z, \ldots, z}_{r_\tau}), \mathrm{diag}(\underbrace{\overline{z}, \ldots, \overline{z}}_{r_{\tau^c}}))_{\tau \in \Phi}.$$

Let X denote the $G(\mathbb{R})$-conjugacy class of h inside the set of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$.

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We now suppose that Φ corresponds under the identification $\mathrm{Hom}(E, \mathbb{C}) \leftrightarrow \mathrm{Hom}(E, \overline{\mathbb{Q}}_p)$ to the set of embeddings inducing the p-adic place w of E. Write $\tau_1, \ldots, \tau_d$ for the elements of Φ; we suppose that $r_{\tau_1} = 1$ and $r_{\tau_i} = 0, i = 2, \ldots, d$. We will also assume that the group G is quasi-split at every finite place not dividing p. PEL data $(D, E, *, F, V, \psi)$ satisfying these assumptions exist provided that $[F : \mathbb{Q}]$ is even, which we always assume in the applications below; see [**HT01**, Lemma I.7.1].

PROPOSITION 6.1. *The pair (G, X) is a Shimura datum. For $U \subset G(\mathbb{A}^\infty)$ a neat open compact subgroup, the Shimura varieties $S(G, U)$ with $S(G, U)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}^f) \times X/U$ are smooth projective algebraic varieties over $\mathbb{C}$, and they admit canonical models over the reflex field $\tau_1(F) \subset \mathbb{C}$.*

The varieties $S(G, U)$ admit p-adic uniformizations. Let $\nu = \phi_p \phi_\infty^{-1}$ denote the induced embedding of $\tau_1(F)$ into $\overline{\mathbb{Q}}_p$. According to [**RZ96**, Section 6], there exists an inner form I of G over $\mathbb{Q}$ of the type considered in Section 2.2, and isomorphisms $I(\mathbb{A}^{p,\infty}) \cong G(\mathbb{A}^{p,\infty})$, $I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(F_v)$, all satisfying the following. Let $\mathbb{F}$ denote the completion of the maximal unramified extension of F_v. The group $I(\mathbb{Q})$ acts on $\Omega_{\mathcal{O}_{F_v}} \widehat{\otimes}_{\mathcal{O}_{F_v}} \mathcal{O}_{\mathbb{F}}$ via the scalar extension of its action on $\Omega_{\mathcal{O}_{F_v}}$ through the map $I(\mathbb{Q}) \subset I(\mathbb{Q}_p) \to \mathrm{PGL}_n(F_v)$. It also acts on $G(\mathbb{A}^\infty)/U_p$, where $U_p \subset G(\mathbb{Q}_p)$ is the unique maximal compact subgroup, as follows. There is an isomorphism $G(\mathbb{A}^\infty)/U_p = G(\mathbb{A}^{p,\infty}) \times G(\mathbb{Q}_p)/U_p \cong I(\mathbb{A}^{p,\infty}) \times G(\mathbb{Q}_p)/U_p$. $I(\mathbb{Q})$ acts diagonally under this identification via the natural action on $I(\mathbb{A}^{p,\infty})$, and as follows on $G(\mathbb{Q}_p)/U_p$. The choice of place w of E induces a canonical isomorphism $G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times D_w^\times$. Let $\Pi \in D_w^\times$ denote a uniformizer. Then an element $(c, a) \in I(\mathbb{Q}_p)$ acts by the formula (see [**RZ96**, Lemma 6.45])

$$(c, a) \cdot (c', a') \mod U_p = (cc', \Pi^{\mathrm{val}_{F_v} \det a} a') \mod U_p,$$

where val_{F_v} is normalized so that $\mathrm{val}_{F_v}(F_v^\times) = \mathbb{Z}$. The following theorem now follows from [**RZ96**, Corollary 6.51]. In what follows, we say that an open compact subgroup of $G(\mathbb{A}^{p,\infty}) \cong I(\mathbb{A}^{p,\infty})$ is sufficiently small if there exists a prime $q \neq p$ such that the projection of U to $G(\mathbb{Q}_q)$ contains no nontrivial elements of finite order.

THEOREM 6.2. *With notation as above, for each sufficiently small open compact subgroup $U^p \subset G(\mathbb{A}^{p,\infty})$, there is an integral model of $S(G, U^p U_p) \otimes_{\tau_1(F),\nu} F_v$ over $\mathcal{O}_{F_v}$, and a canonical isomorphism of formal schemes over* $\mathrm{Spf}\, \mathcal{O}_{\mathbb{F}}$

$$I(\mathbb{Q})\backslash \left[\mathcal{M} \times G(\mathbb{A}^{p,\infty})/U^p\right] \cong \left(S(G, U^p U_p) \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{\mathbb{F}}\right)^{\widehat{}}.$$

This isomorphism is equivariant with respect to the action of the prime-to-p Hecke algebra $\mathcal{H}(G(\mathbb{A}^{p,\infty})/\!\!/U^p)$ on either side.

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From now on, we shall write $S(G, U^pU_p)$ to mean this integral model over $\mathcal{O}_{F_v}$. We will only consider open compact subgroups $U = U^pU_p$, with U_p maximal compact, so that this will always be defined. As is well known, the left-hand side in the above equation can be rewritten as a finite union of quotients of $\Omega_{\mathcal{O}_{F_v}} \widehat{\otimes}_{\mathcal{O}_{F_v}} \mathcal{O}_{\mathbb{F}}$. Indeed, the double quotient $I(\mathbb{Q})\backslash G(\mathbb{A}^{p,\infty})/U^p$ is finite. Let $g_1, \ldots, g_s$ be representatives, and let $\Gamma_i = I(\mathbb{Q}) \cap (g_i U^p g_i^{-1} \times \widetilde{U}_p)$, the intersection taken inside $I(\mathbb{A}^\infty)$. Here, $\widetilde{U}_p \subset I(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \mathrm{GL}_n(F_v)$ is the subgroup $\mathbb{Z}_p^\times \times (\mathrm{val}_{F_v} \circ \det)^{-1}(0)$. Each $\Gamma_i \subset \mathbb{Q}_p^\times \times \mathrm{GL}_n(F_v)$ is a discrete cocompact subgroup, and there is an isomorphism (see Lemma 3.2):

$$(S(G, U) \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{\mathbb{F}})\widehat{} \cong \coprod_{i=1}^{s} \Gamma_i \backslash \mathcal{M}.$$

6.1. Automorphic local systems. From now on, we consider only sufficiently small open compact subgroups $U = U^pU_p$ as in Theorem 6.2. We now introduce some local systems on the varieties $S(G, U)$ corresponding to algebraic representations of G. Corresponding to the infinity type Φ, there is an isomorphism

$$G(\mathbb{C}) \cong \mathbb{C}^\times \times \prod_{\tau \in \Phi} \mathrm{GL}_n(\mathbb{C}).$$

We write $T \subset G \otimes_{\mathbb{Q}} \mathbb{C}$ for the product of the diagonal maximal tori:

$$T(\mathbb{C}) \cong \mathbb{C}^\times \times \prod_{\tau \in \Phi} \underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{n}.$$

Then there is a canonical isomorphism $X^*(T) \cong \mathbb{Z} \times (\mathbb{Z}^n)^\Phi$, and we write $X^*(T)_+$ for the subset of dominant weights $\boldsymbol{\mu} = (c, (\mu_\tau)_{\tau \in \Phi})$, namely those satisfying the condition

$$\mu_{\tau,1} \geqslant \mu_{\tau,2} \geqslant \cdots \geqslant \mu_{\tau,n}$$

for each embedding $\tau : E \hookrightarrow \mathbb{C}$ in Φ. If l is a rational prime, we say that $\boldsymbol{\mu}$ is l-small if, for each $\tau \in \Phi$, we have

$$0 \leqslant \mu_{\tau,i} - \mu_{\tau,j} < l \tag{6.1}$$

for all $0 \leqslant i < j \leqslant n$. If l is unramified in E and $\boldsymbol{\mu}$ is l-small, we associate to $\boldsymbol{\mu}$ an l-adic local system on $S(G, U)$ as follows; see [HT01, Section III.2], [Har13, Section 7.1]. Fix a choice of isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, and let K be a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$, maximal ideal λ, and residue field k. Let $U_l \subset G(\mathbb{Q}_l)$ be a hyperspecial maximal compact subgroup. We suppose that the algebraic representation of $G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_l$ of highest weight $\iota^{-1}\boldsymbol{\mu}$ can be defined over K. Let $W_{\boldsymbol{\mu},K}$ denote this representation. There is, up to homothety,

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a unique U_l-invariant $\mathcal{O}$-lattice of $W_{\mu,K}$. Choose one, and write it as $W_{\mu,\mathcal{O}}$. It is unique since, by the l-small hypothesis, the reduced lattice $W_{\mu,k} = W_{\mu,\mathcal{O}} \otimes_{\mathcal{O}} k$ is an irreducible representation of U_l, and up to isomorphism does not depend on the choice of invariant lattice.

Given an integer $m \geqslant 1$, let $U(m) = U^p(m)U_p \subset U$ denote a normal open compact subgroup which acts trivially on $W_{\mu,\mathcal{O}/\lambda^m} = W_{\mu,\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}/\lambda^m$. Then U acts on the constant sheaf $\underline{W_{\mu,\mathcal{O}/\lambda^m}}$ on $S(G, U(m))$ in a way covering its action on $S(G, U)$, and the quotient defines an étale local system on $S(G, U)$, which we write as $V_{\mu,\mathcal{O}/\lambda^m}$. The sections of $V_{\mu,\mathcal{O}/\lambda^m}$ over an étale open $T \to S(G, U)$ can be identified with the set of functions $f : \pi_0(S(G, U(m)) \times_{S(G,U)} T) \to W_\mu$ such that, for all $\sigma \in U$, $C \in \pi_0(S(G, U(m)) \times_{S(G,U)} T)$, we have the relation $f(C\sigma) = \sigma^{-1} f(C)$. We then take $V_{\mu,\mathcal{O}} = \varprojlim_m V_{\mu,\mathcal{O}/\lambda^m}$ and $V_{\mu,K} = V_{\mu,\mathcal{O}} \otimes_{\mathcal{O}} K$. These local systems are isomorphic to the local systems ${}_{\text{ét}}\underline{V}_{[\mu]}$ constructed in [LS12, Section 4.3] using geometric means. A proof of this comparison for local systems with $\overline{\mathbb{Q}}_l$-coefficients is given in [HT01, Section III.2]. The same argument works in the current context when the weight is l-small.

6.2. A split descent. The scheme $S(G, U) \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{\mathbb{F}}$ has another descent $S(G, U)^{\text{split}}$ to $\mathcal{O}_{F_v}$ whose p-adic formal completion is given by

$$S(G, U)^{\text{split}} = I(\mathbb{Q}) \backslash \left[\mathcal{M}^{\text{split}} \times G(\mathbb{A}^{p,\infty})/U^p\right] \cong \coprod_i \Gamma_i \backslash \mathcal{M}^{\text{split}}.$$

This is *not* the descent defined by $S(G, U)$. However, the local systems $V_{\mu,\Lambda}$, where $\Lambda = K$, $\mathcal{O}$ or $\mathcal{O}/\lambda^m$, also admit descents to $S(G, U)^{\text{split}}$, using exactly the same recipe as before. We write $V_{\mu,\Lambda}^{\text{split}}$ for the local systems defined this way. Our reason for introducing $S(G, U)^{\text{split}}$ is that we will be able to use Lemma 6.3 to prove the degeneration at E_2 of the weight spectral sequence for $S(G, U)^{\text{split}}$ with coefficients in $V_{\mu,k}^{\text{split}}$. Since $S(G, U)^{\text{split}}$ and $S(G, U)$ become isomorphic after extension of scalars to the maximal unramified extension, this will allow us to deduce consequences for the cohomology of $S(G, U)$.

LEMMA 6.3. *The pullback of $V_{\mu,k}^{split}$ to any irreducible (hence geometrically irreducible) component Y of the special fiber of $S(G, U)^{split}$ is a constant sheaf. If $Y_1, \dots, Y_s$ are irreducible components of the special fiber of $S(G, U)^{split}$, then the action of the Frobenius element is by the scalar $q_v^{i/2}$ on the group $H^i((Y_1 \cap \cdots \cap Y_s)_{\overline{s}}, V_{\mu,k}^{split})$. (We recall that this group is zero if i is odd.)*

Proof. Let $Y \subset S(G, U(1))^{\text{split}}$ denote an irreducible component of the special fiber of this scheme. Let $\pi : S(G, U(1))^{\text{split}} \to S(G, U)^{\text{split}}$ denote the natural projection. Then the restriction $\pi|_Y$ induces an isomorphism from Y to its image in $S(G,U)^{\mathrm{split}}$. Pulling back $\mathcal{V}_{\mu,k}^{\mathrm{split}}$ by the inverse of this isomorphism now gives the first assertion. The second assertion now follows from the first and Proposition 3.1. □

6.3. Hecke actions and weight spectral sequence. We now compute the complex of abelian groups $C^{\bullet}(\mathcal{K},\mathcal{V})$ of Proposition 5.2 for the local system $V_{\mu,k}$ on the Shimura variety $S(G,U)$ in terms of the p-adic uniformization

$$I(\mathbb{Q})\backslash\left[\mathcal{M}\times G(\mathbb{A}^{p,\infty})/U^p\right]\cong (S(G,U)\otimes_{\mathcal{O}_{F_v}}\mathcal{O}_{\mathbb{F}})\widehat{\ }.$$

Since U is sufficiently small, the irreducible components of the special fiber are in bijection with the set

$$\begin{aligned}&I(\mathbb{Q})\backslash\left[\mathrm{BT}(0)\times\mathbb{Q}_p^{\times}/\mathbb{Z}_p^{\times}\times\mathrm{GL}_n(F_v)/\mathrm{GL}_n(F_v)^0\times I(\mathbb{A}^{p,\infty})/U^p\right]\\ &\cong\coprod_{i=0}^{n-1}I(\mathbb{Q})\backslash I(\mathbb{A}^{\infty})/\mathbb{Z}_p^{\times}U_iU^p,\end{aligned}$$

where the subgroup $U_i\subset\mathrm{GL}_n(F_v)$ is as in Section 3. For each $i=0,\ldots,n-1$, there is now a bijection

$$\pi_0(S(G,U)_{\bar{s}}^{(i)})\cong\coprod_{E\subset\{0,\ldots,n-1\}}I(\mathbb{Q})\backslash I(\mathbb{A}^{\infty})/\mathbb{Z}_p^{\times}U_EU^p,$$

the union running over subsets E of cardinality $i+1$. If $x\in I(\mathbb{Q})\backslash I(\mathbb{A}^{\infty})/\mathbb{Z}_p^{\times}U_EU^p$, then the images of x under the natural maps

$$I(\mathbb{Q})\backslash I(\mathbb{A}^{\infty})/\mathbb{Z}_p^{\times}U_EU^p\to I(\mathbb{Q})\backslash I(\mathbb{A}^{\infty})/\mathbb{Z}_p^{\times}U_{E\backslash\{i\}}U^p,$$

$i\in E$, correspond exactly to those i-fold intersections of irreducible components which contain the $(i+1)$-fold intersection corresponding to x.

In order to write down the weight spectral sequence for $S(G,U)$, we must first choose a partial ordering of the set of irreducible components of the special fiber which restricts to a total ordering on all subsets of irreducible components which have nontrivial intersection. We choose the partial ordering on $\mathbb{Z}/n\mathbb{Z}$ given by $0\leqslant\cdots\leqslant n-1$, and pull this back to the set

$$I(\mathbb{Q})\backslash\left[\mathrm{BT}(0)\times\mathbb{Q}_p^{\times}/\mathbb{Z}_p^{\times}\times\mathrm{GL}_n(F_v)/\mathrm{GL}_n(F_v)^0\times I(\mathbb{A}^{p,\infty})/U^p\right]$$

via the function κ defined in Section 3. Let $E_1^{p,q}\Rightarrow H^{p+q}(S(G,U)_{\bar{\eta}},V_{\mu,k})$ denote the weight spectral sequence of Section 5. It follows from Lemma 6.3 and Proposition 3.1 that the groups $E_1^{p,q}$ are zero if q is odd, and if $q=2k$ is even then the groups $E_1^{p,2k}$ are nonzero only if $-k\leqslant p\leqslant n-1-k$.

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PROPOSITION 6.4. 1. *For each* $i = 0, \dots, n-1$, *there is a canonical isomorphism*

$$E_1^{i,0} \cong \bigoplus_{E \subset \{0,\dots,n-1\}} \mathcal{A}(\mathbb{Z}_p^\times U^p U_E, W_{\mu,k}),$$

the direct sum running over the set of all subsets E *of order* $i+1$.

2. *There is a canonical isomorphism of complexes*

$$E_1^{\bullet,0} \cong C^\bullet(\mathcal{A}(\mathbb{Z}_p^\times U^p, W_{\mu,k})),$$

and hence, for each $i = 0, \dots, n-1$,

$$E_2^{i,0} \cong H^i(\mathcal{A}(\mathbb{Z}_p^\times U^p, W_{\mu,k})).$$

Proof. By definition, we have $E_1^{i,0} = H^0(S(G,U)_{\bar{s}}^{(i)}, V_{\mu,k})$, and this space can be identified with the set of all functions $f : \pi_0(S(G,U(1))_{\bar{s}}^{(i)}) \to W_{\mu,k}$ satisfying the relation $f(C\sigma) = \sigma^{-1} f(C)$ for all $C \in \pi_0(S(G,U(1))_{\bar{s}}^{(i)})$, $\sigma \in U$. We have identified the set $\pi_0(S(G,U(1))_{\bar{s}}^{(i)})$ with $\coprod_E I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/\mathbb{Z}_p^\times U_E U^p$, compatibly as U varies. The isomorphism of the first part of the proposition now follows from the very definition of the spaces $\mathcal{A}(\mathbb{Z}_p^\times U^p U_E, W_{\mu,k})$.

It remains to show that the differentials in the two complexes correspond under the isomorphism of the first part. This follows after noting that the restriction maps of sections correspond under this isomorphism to the natural inclusions $\mathcal{A}(\mathbb{Z}_p^\times U_E U^p, W_{\mu,k}) \to \mathcal{A}(\mathbb{Z}_p^\times U_{E'} U^p, W_{\mu,k})$, and that the signs that must be inserted in either complex agree because of the choices we have made. $\square$

6.4. Degeneration.

PROPOSITION 6.5. *Let* $r = 2s+1$. *With notation as above, the differentials*

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}$$

are all zero as long as $q_v^s \not\equiv 1$ *modulo* l.

Proof. We recall that the differentials in the weight spectral sequence are Galois equivariant. The proposition would therefore follow if the action of Frobenius on $E_1^{p,q}$ was given by the scalar $q_v^{q/2}$. (We recall that these groups are zero if q is odd.) This is not the case. However, this is the case for the weight spectral sequence of the pair $(S(G,U)^{\mathrm{split}}, V_{\mu,k}^{\mathrm{split}})$, by Lemma 6.3. The weight spectral sequence of a pair (X, V), where X is a strictly semistable scheme over $\mathcal{O}_{F_v}$ and V is a local system on X, viewed as a spectral sequence of abelian groups (forgetting the

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Galois action), depends only on $(X \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{\mathbb{F}}, V)$, that is, the pullback of X to the maximal unramified extension of $\mathcal{O}_{F_v}$. Since the pairs $(S(G, U)^{\mathrm{split}}, V^{\mathrm{split}}_{\boldsymbol{\mu},k})$ and $(S(G, U), V_{\boldsymbol{\mu},k})$ become canonically isomorphic over $\mathcal{O}_{\mathbb{F}}$, we are done. $\square$

COROLLARY 6.6. *Suppose that l is a banal characteristic for $\mathrm{GL}_n(F_v)$. Then the weight spectral sequence for the pair $(S(G, U), V_{\boldsymbol{\mu},k})$ degenerates at E_2, and there is for each $i \geqslant 0$ an injection, equivariant for the prime-to-p Hecke algebra $\mathcal{H}(G(\mathbb{A}^{p,\infty})/\!\!/U^p)$:*

$$H^i(\mathcal{A}(\mathbb{Z}_p^\times U^p, W_{\boldsymbol{\mu},k})) \hookrightarrow H^i(S(G, U_p U^p)_{\overline{F_v}}, V_{\boldsymbol{\mu},k}).$$

Proof. When l is a banal characteristic for $\mathrm{GL}_n(F_v)$, none of the numbers $q_v, q_v^2, \ldots, q_v^{n-1}$ are congruent to 1 modulo l. Since the groups $E_r^{p,q}$ can be nonzero only if $0 \leqslant p \leqslant 2(n-1)$, it follows from Proposition 6.5 that all the differentials $d_r^{p,q}$, $r \geqslant 2$, in the weight spectral sequence are 0. Since the weight spectral sequence is the spectral sequence associated to a filtered complex, we have injections $E_\infty^{0,q} \hookrightarrow H^q(S(G, U_p U^p)_{\overline{F_v}}, V_{\boldsymbol{\mu},k})$ for each $q \geqslant 0$. Combining these observations with the isomorphisms of Proposition 6.4, we obtain for each $q \geqslant 0$ an injection

$$H^q(\mathcal{A}(\mathbb{Z}_p^\times U^p, W_{\boldsymbol{\mu},k})) \cong E_2^{0,q} \cong E_\infty^{0,q} \hookrightarrow H^q(S(G, U_p U^p)_{\overline{F_v}}, V_{\boldsymbol{\mu},k}).$$

The Hecke equivariance is an easy consequence of the construction. This completes the proof. $\square$

6.5. Raising the level. We now suppose in addition that $E = F \cdot E_0$, where E_0 is a quadratic imaginary extension of $\mathbb{Q}$, and that E/F is everywhere unramified. We now change notation slightly, and write v_0 for the place of F above the rational prime p, and w_0 for one of the places of E above it. Let $l \neq p$ be another prime, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. We assume that l is unramified in E.

Let $\boldsymbol{\mu}$ be a choice of l-small dominant weight, and let $U = \prod_q U_q \subset I(\mathbb{A}^\infty)$ denote a open compact subgroup. Then there is defined a finite free $\mathcal{O}$-module $W_{\boldsymbol{\mu},\mathcal{O}}$ on which U_l acts, and a space of automorphic forms $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})$. It is a finite free $\mathcal{O}$-module. We recall that this space has the following interpretation. Let $\mathcal{A}$ denote the space of automorphic forms on I, a semisimple admissible representation of $I(\mathbb{A})$. Let $W_{\boldsymbol{\mu},\mathbb{C}}$ denote the representation of $I(\mathbb{R}) \subset I(\mathbb{C}) \cong \mathbb{C}^\times \times \prod_{\tau \in \Phi} \mathrm{GL}_n(\mathbb{C})$ which is the restriction of the algebraic representation of highest weight $\boldsymbol{\mu}$. Then there is an isomorphism

$$\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}}) \otimes_{\mathcal{O},\iota} \mathbb{C} \cong \mathrm{Hom}_{I(\mathbb{R})}(W^\vee_{\boldsymbol{\mu},\mathbb{C}}, \mathcal{A}).$$

If T is a finite set of rational primes containing l, and such that U_q is a hyperspecial maximal compact subgroup for all $q \notin T$, let $\mathbb{T}_T^{\mathrm{univ}} = \mathcal{O}[\{T_1^v, \ldots, T_n^v, (T_n^v)^{-1}\}]$

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denote the polynomial ring in infinitely many indeterminates corresponding to the unramified Hecke operators at places v of F which split in E and are not contained in T. Then $\mathbb{T}_T^{\mathrm{univ}}$ acts on $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})$ by $\mathcal{O}$-algebra endomorphisms, and on the spaces $H^i(S(G,U)_{\overline{E}}, V_{\boldsymbol{\mu},k})$, via the fixed isomorphism $I(\mathbb{A}^{p,\infty}) \cong G(\mathbb{A}^{p,\infty})$. If σ is an automorphic representation of $I(\mathbb{A})$ such that $(\sigma^\infty)^U \neq 0$ and $\sigma_\infty \cong W^\vee_{\boldsymbol{\mu},\mathbb{C}}$, then we can associate to it a maximal ideal $\mathfrak{m}_\sigma \subset \mathbb{T}_T^{\mathrm{univ}}$ by assigning to each Hecke operator the reduction modulo l of its eigenvalue on $\iota^{-1}(\sigma^\infty)^U \subset \mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$. If σ' is another automorphic representation of $I(\mathbb{A})$, we say that σ' contributes to $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$ if $\sigma'_\infty \cong W^\vee_{\boldsymbol{\mu},\mathbb{C}}$, $(\sigma'^\infty)^U \neq 0$, and the intersection of $(\iota^{-1}\sigma'^\infty)^U$ and $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$ inside $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$ is nontrivial.

There is an isomorphism $\iota_{w_0} : I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{w_0})$, and, if σ_p is an irreducible admissible representation of $I(\mathbb{Q}_p)$, then $(\sigma_p \circ \iota_{w_0})|_{\mathrm{GL}_n(E_{w_0})}$ remains irreducible. We assume that $\iota_{w_0}(U_p) = \mathbb{Z}_p^\times \times B$, where $B \subset \mathrm{GL}_n(E_{w_0})$ is the standard Iwahori subgroup. We write $U'_p \subset G(\mathbb{Q}_p)$ for the unique maximal compact subgroup.

THEOREM 6.7. *Suppose that σ is as above, and let $\mathfrak{m}_\sigma \subset \mathbb{T}_T^{univ}$ denote the associated maximal ideal. Suppose that the following hypotheses hold.*

1. *The group U^p is a sufficiently small open compact subgroup of $I(\mathbb{A}^{p,\infty})$.*
2. *If σ' is another automorphic representation which contributes to $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$, then $(\sigma'_p \circ \iota_{w_0})|_{\mathrm{GL}_n(E_{w_0})}$ is a subquotient of a parabolic induction* $\text{n-Ind}_Q^G \mathrm{St}_a(\alpha) \otimes \mathrm{St}_b(\beta)$ *for some $a + b = n$.*
3. *$\iota^{-1}\sigma_{1,w_0}$ satisfies the level-raising congruence (4.2).*
4. *$\boldsymbol{\mu}$ is l-small (6.1) and l is a banal characteristic for* $\mathrm{GL}_n(E_{w_0})$.
5. *The groups $H^{n-2}(S(G, U^pU'_p)_{\overline{F_v}}, V^\vee_{\boldsymbol{\mu},k})$ and $H^{n-2}(S(G, U^pU'_p)_{\overline{F_v}}, V_{\boldsymbol{\mu},k})$ are zero.*

Then we can raise the level: there exists another irreducible constituent σ' contributing to $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$, and such that σ' is an unramified twist of the Steinberg representation.

We remark that [**LS12**, Theorem 8.12] implies that hypothesis 5 above is satisfied provided that U_l is a hyperspecial maximal compact subgroup, $\boldsymbol{\mu}$ is strictly regular, and the following inequalities hold:

$$2n+\sum_{\tau\in\Phi}\sum_{j=1}^{n}(2\lceil\mu_{\tau,1}/2\rceil-\mu_{\tau,n+1-j}) \leqslant l \quad \text{and} \quad 2n+\sum_{\tau\in\Phi}\sum_{j=1}^{n}(\mu_{\tau,j}-2\lfloor\mu_{\tau,n}/2\rfloor) \leqslant l.$$

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By adding some further local hypotheses at a prime $q \neq p$, we could also appeal to the main result of [Shi].

Proof. Combining hypothesis 5 and Corollary 6.6, we see that the groups $H^i(\mathcal{A}(U^p\mathbb{Z}_p^\times, W_{\mu,k}^\vee))$ and $H^i(\mathcal{A}(U^p\mathbb{Z}_p^\times, W_{\mu,k}))$ vanish when $i = n-2$. On the other hand, there is a perfect pairing

$$\mathcal{A}(U^p\mathbb{Z}_p^\times, W_{\mu,\mathcal{O}}^\vee) \times \mathcal{A}(U^p\mathbb{Z}_p^\times, W_{\mu,\mathcal{O}}) \to \mathcal{O}.$$

Indeed, given an open compact subgroup $V \subset B$ and $f_1 \in \mathcal{A}(U^p\mathbb{Z}_p^\times V, W_{\mu,\mathcal{O}}^\vee)$, $f_2 \in \mathcal{A}(U^p\mathbb{Z}_p^\times V, W_{\mu,\mathcal{O}})$, we define $\langle f_1, f_2\rangle$ by the formula

$$\langle f_1, f_2\rangle = \frac{1}{[B:V]} \sum_{x \in I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/U^p\mathbb{Z}_p^\times V} (f_1(x), f_2(x)).$$

This pairing is independent of the choice of V, and for every such V restricts to a perfect pairing $\mathcal{A}(U^p\mathbb{Z}_p^\times V, W_{\mu,\mathcal{O}}^\vee) \times \mathcal{A}(U^p\mathbb{Z}_p^\times V, W_{\mu,\mathcal{O}}) \to \mathcal{O}$. For any $g \in \mathrm{GL}_n(E_{w_0})$, we have the formula $\langle gf_1, gf_2\rangle = \langle f_1, f_2\rangle$. The action of $\mathbb{T}_T^{\mathrm{univ}}$ on $\mathcal{A}(U^p\mathbb{Z}_p^\times, W_{\mu,\mathcal{O}})$ gives a canonical direct sum decomposition of $\mathcal{O}[\mathrm{GL}_n(E_{w_0})]$-modules:

$$\mathcal{A}(U^p\mathbb{Z}_p^\times, W_{\mu,\mathcal{O}}) = \mathcal{A}(U^p\mathbb{Z}_p^\times, W_{\mu,\mathcal{O}})_{\mathfrak{m}_\sigma} \oplus C,$$

for some C. The hypotheses of Theorem 4.6 are now satisfied with $M = \mathcal{A}(U^p\mathbb{Z}_p^\times, W_{\mu,\mathcal{O}})_{\mathfrak{m}_\sigma}$ and N taken to be the annihilator of C under the pairing $\langle\cdot,\cdot\rangle$. The result follows from this. □

7. Consequences for GL_n

In this section, we deduce our main theorem. We suppose that E is an imaginary CM field of the form $E = E_0 \cdot F$, where F is a totally real number field and E_0 is an imaginary quadratic field. We suppose that E/F is everywhere unramified. Suppose that there exists a prime p which is totally inert in F and split in E_0. Let $v_0 = w_0 w_0^c$ denote the unique place of F above p. Let $n \geqslant 3$ be an integer, and $l \neq p$ a prime. We fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$.

Let n_1, n_2 be positive integers with $n = n_1 + n_2$. Suppose that π_1, π_2 are conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\pi = \pi_1 \boxplus \pi_2$ is regular algebraic. We recall that in Theorem 2.1 we have associated to π a continuous semisimple representation $r_\iota(\pi) : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$.

THEOREM 7.1. *With π as above, suppose that $\iota^{-1}\pi_{w_0}$ satisfies the level-raising congruence (4.2). Suppose further that the following hold.*

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1. *If $t_l \in G_{E_{w_0}}$ is a generator of the l-part of the tame inertia group at w_0, then $\overline{r_\iota(\pi)}(t_l)$ is a unipotent matrix with exactly two Jordan blocks.*
2. *l is a banal characteristic for $\mathrm{GL}_n(E_{w_0})$.*
3. *The weight $\boldsymbol{\lambda} = (\lambda_\tau)_{\tau:E\hookrightarrow\mathbb{C}}$ of π satisfies the following.*
   - *For each τ, and for each $0 \leqslant i < j \leqslant n$, we have $0 < \lambda_{\tau,i} - \lambda_{\tau,j} < l$.*
   - *There exists an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ such that the following inequalities hold:*

$$2n + \sum_{\tau:E\hookrightarrow\mathbb{C}} \sum_{j=1}^{n} (\lambda_{\tau,j} - 2\lfloor \lambda_{\tau,n}/2 \rfloor) \leqslant l,$$

$$2n + \sum_{\tau:E\hookrightarrow\mathbb{C}} \sum_{j=1}^{n} (2\lceil \lambda_{\tau,1}/2 \rceil - \lambda_{\tau,n+1-j}) \leqslant l,$$

   the first sum in each case being over embeddings τ such that the place of E_0 induced by $\iota_p^{-1}\tau$ is the same as the restriction of the place w_0 to E_0.

4. *If π is ramified at a place w of E, then w is split over F.*
5. *π is unramified at the primes of E dividing l, and the prime l is unramified in E and split in E_0.*
6. *$\pi = \pi_1 \boxplus \pi_2$ satisfies the sign condition (2.3), $n_1 \neq n_2$, and $n_1 n_2$ is even.*

*Then there exists an RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ of weight $\boldsymbol{\lambda}$ such that $\overline{r_\iota(\pi)} \cong \overline{r_\iota(\Pi)}$ and Π_{w_0} is an unramified twist of the Steinberg representation. If the places of F above l are split in E, and π is ι-ordinary in the sense of [**Ger**, Definition 5.1.2], then we can even assume that Π is also ι-ordinary.*

Proof. Let I_1 denote the definite unitary group associated to the extension E/F in Section 2.2. By Proposition 2.4, there exists an automorphic representation σ_1 of $I_1(\mathbb{A}_F)$ such that π is the base change of σ_1. Let I denote the corresponding unitary similitude group. By Lemma 2.3, σ_1 extends to an automorphic representation σ of $I(\mathbb{A})$. We apply Theorem 6.7 to σ. Let $U^p = \prod_{q\neq p} U_q$ be a sufficiently small open compact subgroup of $I(\mathbb{A}^{p,\infty})$ with $\sigma^U \neq 0$, where $U = U^p U_p$ and $U_p \subset I(\mathbb{Q}_p)$ corresponds under the isomorphism $\iota_{w_0} : I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{w_0})$ to the product $\mathbb{Z}_p^\times \times B$, where $B \subset \mathrm{GL}_n(E_{w_0})$ is the standard Iwahori subgroup. Suppose in addition that U_l is a hyperspecial maximal compact subgroup.

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In the notation of Theorem 6.7, let $\boldsymbol{\mu}$ be the weight such that σ contributes to the space $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})$. If σ' is an automorphic representation which contributes to the space $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$, then let σ'_1 and π' be the automorphic representations of the groups $I_1(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{A}_E)$ associated to σ' by Lemma 2.3 and Proposition 2.4. Then $\overline{r_\iota(\pi')}|_{G_{E_{w_0}}} \cong \overline{r_\iota(\pi)}|_{G_{E_{w_0}}}$, and hence the former representation maps t_l to a unipotent matrix with exactly two Jordan blocks. If σ' is such a representation, then the representation $\sigma'_{w_0} \circ \iota_{w_0}$ of $\mathrm{GL}_n(E_{w_0})$ has an Iwahori-fixed vector, and it is therefore isomorphic to $\mathrm{St}_{n_1}(\alpha_1) \boxplus \cdots \boxplus \mathrm{St}_{n_s}(\alpha_s)$ for some constants $\alpha_1, \ldots, \alpha_s$ and integers with $n_1 + \cdots + n_s = n$. The nilpotent operator N in the associated Weil–Deligne representation then has a Jordan decomposition corresponding to this partition of n. By hypothesis, the conjugacy class of N specializes to the conjugacy class of a nilpotent matrix with exactly two Jordan blocks. This implies that $s \leqslant 2$, and hence the second hypothesis of Theorem 6.7 is satisfied. Let σ' be the representation whose existence is guaranteed by that theorem. Applying Proposition 2.4 and Lemma 2.3 to σ', we obtain a representation Π satisfying the conclusion of the present theorem. It must be cuspidal, since Π_{w_0} is an unramified twist of the Steinberg representation.

To obtain the last sentence of the theorem, we can enlarge the Hecke algebra $\mathbb{T}_T^{\mathrm{univ}}$ appearing in the proof of Theorem 6.7 to contain the analogs of the U_l operators at the places dividing l, and further localize at a maximal ideal not containing them. We omit the details. □

7.1. Proof of Theorem 1.1. We now give the proof of the theorem in the introduction. We first note the following.

PROPOSITION 7.2. *Let E be an imaginary CM field with totally real subfield F, and let π be an RACSDC automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E)$. *Suppose that w_0 is a place of E and that π_{w_0} is an unramified twist of the Steinberg representation. Let $\mathcal{L}$ denote the set of rational primes l such that, for all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, the residual representation $\overline{r_\iota(\pi)}$ is irreducible, and, if t_l denotes a generator of the pro-l part of the tame inertia group at w_0, then $\overline{r_\iota(\pi)}(t_l)$ is a regular unipotent element. Then $\mathcal{L}$ has Dirichlet density* 1.

Proof. We sketch the proof, by exhibiting for every $\delta \in (0, 1)$ a set $\mathcal{L}_\delta \subset \mathcal{L}$ of lower density at least $1 - \delta$. Replacing E by a soluble extension, we can assume without loss of generality that, for any prime w at which π is ramified, w is split over F.

Suppose that $E_1, \ldots, E_s$ are quadratic imaginary fields such that, for each i, E_i is disjoint over $\mathbb{Q}$ from the compositum of the fields E_j, $j \neq i$. Let E_0 denote the compositum of the fields $E, E_1, \ldots, E_s$. Let F_0 denote the totally real

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subfield of E_0. If a prime l splits in any E_i, then the primes of F_0 above l all split in E_0. Let Π denote the base change of π to E_0. By [TY07, Corollary B] and [BLGGT14, Proposition 5.2.2], there exists a set $\mathcal{M}$ of rational primes l of Dirichlet density 1 such that, for all $l \in \mathcal{M}$ and all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, the residual representation $\overline{r_\iota(\Pi)}|_{G_{E_0(\zeta_l)}}$ is irreducible and $l > 2(n+1)$. This implies *a fortiori* that $\overline{r_\iota(\pi)}$ is irreducible. After casting out finitely many elements of $\mathcal{M}$, we can suppose further that, for all $l \in \mathcal{M}$, E_0 and Π are unramified above l, and, if $\boldsymbol{\lambda}$ denotes the weight of Π, then, for all embeddings $\tau : E_0 \hookrightarrow \mathbb{C}$, we have $\lambda_{\tau,1} - \lambda_{\tau,n} \leqslant l - n - 1$ (this means that the Hodge–Tate weights of $r_\iota(\Pi)$ lie in the Fontaine–Lafaille range).

Choose a place x_0 of E_0 above w_0. It follows from [BLGGT14, Theorem 4.4.1] that, if $l \in \mathcal{M}$ is a prime split in one of $E_1, \ldots, E_s$, $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$ is an isomorphism, and $\overline{r_\iota(\pi)}(t_l)$ is not a regular unipotent element, then we can find an RACSDC automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_{E_0})$ satisfying the following.

- $\overline{r_\iota(\Pi)} \cong \overline{r_\iota(\Pi')}$.
- If w is a place of E_0 and $U_w \subset \mathrm{GL}_n(E_{0,w})$ is an open compact subgroup such that $\Pi_w^{U_w} \neq 0$, then $(\Pi'_w)^{U_w} \neq 0$.
- Π' has weight $\boldsymbol{\lambda}$.
- There exists an open compact subgroup U_{x_0} of $\mathrm{GL}_n(E_{0,x_0})$ strictly containing the Iwahori subgroup, such that $(\Pi'_{x_0})^{U_{x_0}} \neq 0$.

We claim that there can be only finitely many such primes. Indeed, if there are infinitely many, then, by the pigeonhole principle, there exists an automorphic representation Π' of $\mathrm{GL}_{\mathbb{A}_{E_0}}$ satisfying the last three points, and infinitely many primes $l_1, l_2, \ldots \in \mathcal{M}$ with isomorphisms $\iota_i : \overline{\mathbb{Q}}_{l_i} \cong \mathbb{C}$ such that $\overline{r_{\iota_i}(\Pi)} \cong \overline{r_{\iota_i}(\Pi')}$. As Π^∞, $(\Pi')^\infty$ are defined over number fields; this implies that we must have $\Pi \cong \Pi'$, which is a contradiction (see [BG06, Lemma 5.1.7]).

Let $\mathcal{L}_s$ denote the set of primes $l \in \mathcal{M}$ which are split in one of $E_1, \ldots, E_s$. This set has Dirichlet density $1 - 2^{-s}$. The above argument shows that, after casting out finitely many elements, we have $\mathcal{L}_s \subset \mathcal{L}$. This concludes the proof. □

Proof of Theorem 1.1. We take up the notation of the introduction. Thus E/F is a CM imaginary extension of a totally real field, and π_1, π_2 are RACSDC automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E)$, $\mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively. Let $\mathcal{L}$ denote the intersection of the sets $\mathcal{L}_1, \mathcal{L}_2$ of primes associated to the representations π_1, π_2 by Proposition 7.2. After removing finitely many elements from $\mathcal{L}$, we can assume that, for all $l \in \mathcal{L}$ and all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, $\pi = \pi_1 \boxplus \pi_2$ is unramified at every prime of E above l, l is unramified in E, the order of q_{w_0} in $\mathbb{F}_l^\times$ is greater than $2n$, and the weight $\boldsymbol{\lambda}$ of π satisfies the inequalities

$$([E:\mathbb{Q}]+2)n + \sum_{\tau : E \hookrightarrow \mathbb{C}} \sum_{j=1}^{n} (\lambda_{\tau,j} - \lambda_{\tau,n}) \leqslant l/2.$$

Fix a prime $l \in \mathcal{L}$ and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$.

There exist $\alpha, \beta \in \overline{\mathbb{Z}}_l^\times$ such that the Frobenius eigenvalues of $r_\iota(\pi_1)$ and $r_\iota(\pi_2)$ are given, respectively, by

$$\alpha, q_{w_0}\alpha, \ldots, q_{w_0}^{n_1-1} \quad \text{and} \quad \beta, q_{w_0}\beta, \ldots, q_{w_0}^{n_2-1}.$$

Let γ denote the image of $\beta/(\alpha q_{w_0}^{n_1})$ in $\overline{\mathbb{F}}_l^\times$, and let $m \geqslant 1$ denote the order of γ in this group. By the Grunwald–Wang theorem, there exists a cyclic extension K of E of degree m such that w_0 is inert in K and w_0^c splits in K, and K is unramified above the primes of E dividing l. Let $\varphi : G_E \to \overline{\mathbb{F}}_l^\times$ be the character factoring through $\mathrm{Gal}(K/E)$ such that $\varphi(\mathrm{Frob}_{w_0}) = \gamma$, and let ψ be the Teichmüller lift of φ/φ^c. Then $\psi\psi^c = 1$, and $\iota^{-1}(\pi_1 \boxplus (\pi_2 \otimes \iota\psi))_{w_0}$ satisfies the level-raising congruence.

Let E_0 be a quadratic imaginary extension of $\mathbb{Q}$ in which p is inert, and which is split at l and every prime $q \neq p$ of $\mathbb{Q}$ below a place of E at which $\pi_1 \boxplus (\pi_2 \otimes \iota\psi)$ or the extension E/F is ramified. Let $E_1 = E \cdot E_0$. The hypotheses of Theorem 7.1 now apply to the base change of $\pi_1 \boxplus (\pi_2 \otimes \iota\psi)$ to E_1. This completes the proof. □

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