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Topologies generated by relations

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Let R be a relation on a set X, and if $A \subset X$ set $RA = \{x \mid (x,a) \in R$ for some $a \in A\}$ and $AR = \{x \mid (a,x) \in R$ for some $a \in A\}$. Also A is called an antiset in case no two distinct elements of A are related. If $\mathcal{A}$ is a collection of antisets, then we generate a topology $T(\mathcal{A})$ by taking sets of the form RA or AR (or X or $\emptyset$) as subbasic open sets. Then conditions are given under which this topology satisfies separation axioms, or is compact or connected. For example, Theorem: Let $\mathcal{A}$ contain the singletons. If for each $x \in X$ and $y \in X \setminus x$, there is a $z \in X$ such that $(x,z) \in R$ $((z,x) \in R)$ and $(y,z) \notin R$ $((z,y) \notin R)$, then $T(\mathcal{A})$ is a T_1-topology. The conditions used to obtain compactness or connectedness are analogous to the conditions used to get the same properties for the order topology on a totally ordered set. Finally, by modifying the definition of $T(\mathcal{A})$ slightly, we obtain conditions so that if X is a tree and R the cutpoint order, then $T(\mathcal{A})$ is the original topology.

1. Introduction

Let X be a set and let R be a relation on X (i.e. $R \subset X \times X$). A subset $A \subset X$ is an *antiset* (with respect to R) iff no two distinct elements of A are R related. Let $A \subset X$; we shall use the following notation:

$$RA = \{x \mid (x,a) \in R \text{ for some } a \in A\} ,$$
$$AR = \{x \mid (a,x) \in R \text{ for some } a \in A\} .$$

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Also $x\,R\,y$ iff $(x,y) \in R$. A set A such that $RA \subset A$ is called *decreasing* and a set A such that $AR \subset A$ is called *increasing*. Let $\mathcal{A}$ be a collection of subsets of X . Let $D(\mathcal{A}) = \{RA \mid A \in \mathcal{A}\}$ and $I(\mathcal{A}) = \{AR \mid A \in \mathcal{A}\}$. For $A \in \mathcal{A}$, set $I(A) = AR$ and $D(A) = RA$. If $\mathcal{A}$ is a collection of antisets of X , set $S = D(\mathcal{A}) \cup I(\mathcal{A}) \cup \{X\} \cup \{\emptyset\}$. Then $T(R,\mathcal{A})$ (or $T(\mathcal{A})$ when the relation R is fixed) is the topology with S as a subbase for the *closed* sets. If X is the real numbers and R the usual reflexive order, the topology $T(\mathcal{A})$, where $\mathcal{A}$ is the set of singletons, is the usual topology.

2. Separation axioms

In the following we shall assume that R is a relation on a set X and that $\mathcal{A}$ is a collection of R-antisets.

PROPOSITION 1. *Suppose that for each pair* $x \neq y \in X$ *there exists a* $z \in X$ *such that either*

(i) $(x,z) \in R$ *and* $(y,z) \notin R$ $((z,x) \in R$ *and* $(z,y) \notin R)$ *or*

(ii) $(y,z) \in R$ *and* $(x,z) \notin R$ $((z,y) \in R$ *and* $(z,x) \notin R)$.

Then if $\mathcal{A}$ *contains the singletons,* $T(\mathcal{A})$ *is a* T_0*-topology.*

Proof. Let $x \neq y$. If z is an element such that $(x,z) \in R$ and $(y,z) \notin R$, then $R\{z\}$ is a closed set containing x but not y , hence $y \notin \overline{\{x\}}$. If *(ii)* holds, $R\{z\}$ is a closed set containing y and not x .

PROPOSITION 2. *Let* $\mathcal{A}$ *contain the singletons. If for each* $x \in X$ *and* $y \in X \setminus x$ *, there is a* $z \in X$ *such that* $(x,z) \in R$ $((z,x) \in R)$ *and* $(y,z) \notin R$ $((z,y) \notin R)$ *, then* $T(\mathcal{A})$ *is a* T_1*-topology.*

Proof. As above $y \notin \overline{\{x\}}$ for all $y \in X \setminus x$; hence $\overline{\{x\}} = \{x\}$ and so $T(\mathcal{A})$ is a T_1-topology.

COROLLARY. *If* R *is reflexive and antisymmetric, and if* $\mathcal{A}$ *contains the singletons, then* $T(\mathcal{A})$ *is a* T_1*-topology.*

Proof. If $y \neq x$, then $(x,y) \notin R$ or $(y,x) \notin R$. In either case let $z = x$ and Proposition 2 implies that $T(\mathcal{A})$ is a T_1-topology.

DEFINITION. A collection $\mathcal{A}$ of antisets is called *separating* (or separates X) if and only if for $x \in X$ and $y \in X \setminus x$ there is an $A \in \mathcal{A}$ such that $x \in I(A)$ and $y \notin I(A)$ or $x \in D(A)$ and $y \notin D(A)$.

PROPOSITION 3. *If* $\mathcal{A}$ *is separating, then* $T(\mathcal{A})$ *is a* T_1*-topology.*

DEFINITION. A collection $\mathcal{A}$ of antisets *separates points of* X iff for $x \neq y$ there exist $A_1, A_2 \in \mathcal{A}$ such that $x \in A_1 \setminus A_2$ and $y \in A_2 \setminus A_1$.

COROLLARY. *If* R *is reflexive, transitive, and antisymmetric, (i.e., a partial order), and if* $\mathcal{A}$ *separates points of* X, *then* $T(\mathcal{A})$ *is a* T_1*-topology.*

Proof. Suppose $y \in I(A)$ and $y \in D(A)$ then there is a $z_1 \in A$ and a $z_2 \in A$ such that $z_1 R y$ and $y R z_2$, then $z_1 R z_2$ (by transitivity) and hence, $z_1 = z_2$, but then by antisymmetry $z_1 = y \in A$ which is a contradiction. Since R is reflexive, we get $\mathcal{A}$ separating, and hence $T(\mathcal{A})$ is a T_1-topology.

If R is a relation on X and $x \neq y$, then there is a maximal antiset A such that $x \in A$ and $y \notin A$. Thus, it will be possible to construct collections of antisets which satisfy the following definition. In order to simplify the notation we let B or B_i denote either $I(A)$ or $D(A)$ where A is some antiset in some collection $\mathcal{A}$.

DEFINITION. A collection $\mathcal{A}$ of antisets *completely separates* points of X iff for $x \neq y$ there exist $B_1, \ldots, B_k$ such that $X = \bigcup_{i=1}^{k} B_i$ and $x \in B_i$ implies $y \notin B_i$.

THEOREM 4. *If* $\mathcal{A}$ *completely separates points of* X, *then* $(X, T(\mathcal{A}))$ *is* T_2.

Proof. Let $X_1 = \bigcup\{B_i \mid y \notin B_i\}$, and $X_2 = \bigcup\{B_i \mid y \in B_i\}$. Then $X = X_1 \cup X_2$, X_1, X_2 are closed, $x \in X_1$, $y \in X_2$, $y \notin X_1$ and $x \notin X_2$. Hence $(X, T(\mathcal{A}))$ is T_2.

DEFINITION. A relation is called *full* iff whenever x is not related to y there are elements $z_1 \neq x$, $z_2 \neq y$ such that z_1 and z_2 are not related and either $x \in D(z_1)$, $y \in I(z_2)$ or $x \in I(z_1)$, $y \in D(z_2)$.

THEOREM 5. *If* R *is a full partial order, and if* $\mathcal{A}$ *contains the maximal antisets, then* $\mathcal{A}$ *completely separates points of* X. *Hence,*

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$(X, T(\mathcal{A}))$ is T_2 .

Proof. Let $x \neq y$ and suppose that $x R y$. We have two subcases. First suppose there exists a z such that $x \neq z \neq y$, $x R z$, and $z R y$. Let A be a maximal antiset containing z . Let $B_1 = D(A)$ and $B_2 = I(A)$. Then $x \in B_1$, $y \in B_2$, and $X = B_1 \cup B_2$. Also by applying the transitivity and antisymmetry we find that $y \notin B_1$ and $x \notin B_2$, and we are done. Now suppose that no such z exists. Let A be a maximal antiset containing x . Let $A_1 = (A \setminus x) \cup \{y\}$ and let A_2 be a maximal antiset in A_1 which contains y . Then let A_3 be a maximal antiset in X containing A_2 . If $D(A_1) \cup I(A_3) = X$, we choose $B_1 = D(A_1)$ and $B_2 = I(A_3)$ and we are done. So let $z \in X \setminus (D(A_1) \cup I(A_3))$. Let C be the maximal antiset in $X \setminus (D(x) \cup I(y))$ containing z . If x is not related to any element of C , then $C \cup \{x\} = C_1$ is a maximal antiset and $y \notin D(C_1)$ follows from the transitivity and the choice of C . Similarly if y is not related to any element of C , then $C_1 = C \cup \{y\}$ is maximal and $x \in I(C_2)$. Then take $B_1 = D(C_1)$ and $B_2 = I(C_2)$ and B_1 , B_2 are the desired sets. If $y \in I(C)$, we take $B_1 = D(C_1)$ and $B_2 = I(A_3)$. If $x \in D(C)$ and $y \in I(C) \cup D(C)$ we take $B_1 = D(A_1)$ and $B_2 = I(C_2)$. Finally, if $x \in D(C)$ and $y \in I(C)$, C is already maximal and $B_1 = D(C)$, $B_2 = I(C)$ will work. Note that $x \in I(C)$ and $y \in D(C)$ are impossible by the choice of C . We can verify in all cases that B_1 and B_2 are the desired sets. This completes the case with $x R y$.

Now suppose that x is not related to y . Let z_1 , z_2 be such that $x \in D(z_1)$, $y \in I(z_2)$ (similar arguments will work when $x \in I(z_1)$, $y \in D(z_2)$) . Since z_1 and z_2 are not related let A be a maximal antiset containing z_1 and z_2 . Then set $B_1 = D(A)$ and $B_2 = I(A)$, and we are done.

By assuming a richer collection $\mathcal{A}$ of antisets we can obtain the same result without requiring that the relation be full. Therefore an antiset A is *nearly maximal* iff the addition of a finite number of points to A will produce a maximal antiset. We shall use the convention that the empty set is finite and hence, that each maximal antiset is nearly maximal.

THEOREM 6. *Suppose that R is a partial order and that $\mathcal{A}$ contains the nearly maximal antisets. Then $\mathcal{A}$ completely separates points of X . Hence, $(X, T(\mathcal{A}))$ is T_2 .*

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Proof. First suppose that $x \neq y$ and $x\,R\,y$. Then proceed as in the first part of Theorem 5. Next suppose that x and y are not related, and let A be a maximal antiset containing x and y . Let $A_1 = A \setminus y$ and $A_2 = A \setminus x$. Then let $B_1 = D(A_1)$, $B_2 = I(A_1)$, $B_3 = D(A_2)$ and $B_4 = I(A_2)$. Now $x \in B_1 \cap B_2$, $y \in B_3 \cap B_4$ but $y \notin B_1 \cup B_2$ and $x \notin B_3 \cup B_4$. Finally $X = B_1 \cup B_2 \cup B_3 \cup B_4$ and we are done.

Suppose that R is a partial order and that A contains the maximal antisets. If $A \in \mathrm{A}$ is maximal, then we can show that $D(A) \setminus A$ and $I(A) \setminus A$ are open sets. We can then get results similar to Theorems 5 and 6. In fact in the case of a linear order we obtain the usual result that $(X, T(\mathrm{A}))$ is T_2 , when A is the set of singletons.

3. Compact spaces

In this section we shall develop conditions under which the space $(X, T(\mathrm{A}))$ is compact. In the following we shall assume that we have a fixed relation R on a set X . For this note that if A_0 is the set of singletons, if A_1 is the finite antisets and if $\mathrm{A}_0 \subset \mathrm{A} \subset \mathrm{A}_1$, then $T(\mathrm{A}_0) = T(\mathrm{A}) = T(\mathrm{A}_1)$. Thus whenever we are using a set of finite antisets which includes the singletons to generate the topology, we may assume that it is the singletons. Generally we shall use the terminology of bounded ordered sets for a general relation R . For example, a set is R-bounded above iff there is a point x such that $a\,R\,x$ for all $a \in A$.

THEOREM 7. *Let R be transitive, and let X be bounded and complete. If A is the singletons, then $(X, T(\mathrm{A}))$ is compact.*

Proof. Let F be a collection of closed subbasic sets with the finite intersection property. Since A is the singletons we may write $F = F_1 \cup F_2$ where $F_1 = \{I(x_\alpha) \mid \alpha \in \Gamma_1\}$ and $F_2 = \{D(x_\alpha) \mid \alpha \in \Gamma_2\}$. Let x_0 be the supremum of $\{x_\alpha \mid \alpha \in \Gamma_1\}$. Then $x_0 \in I(x_\alpha)$ for all $\alpha \in \Gamma_1$. Let $\gamma \in \Gamma_2$; then by f.i.p. $I(x_\alpha) \cap D(x_\gamma) \neq \emptyset$ for all $\alpha \in \Gamma_1$. Thus, for $\alpha \in \Gamma_2$, there is a z such that $x_\alpha\,R\,z$ and $z\,R\,x_\gamma$. Hence, $x_\alpha\,R\,x_\gamma$ by the transitivity of R . Therefore x_γ is an upper bound of $\{x_\alpha \mid \alpha \in A_1\}$ and thus, $x_0\,R\,x_\gamma$. Consequently, $x_0 \in D(x_\alpha)$ for all

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$\alpha \in \Gamma_2$, and so $x_0 \in \bigcap F$. Therefore, $(X, T(A))$ is compact by the Alexander subbase Lemma.

EXAMPLES. It is relatively easy to find examples of noncompact spaces $(X, T(A))$ where A is not restricted to finite sets but where R is reflexive and transitive, and where X is complete and bounded. We now sketch an example which shows that we cannot drop the assumption of transitivity in Theorem 7.

In the above let $X = \{(n,i) \mid n \in \omega , i = 0,1,2\} \cup \{(0,-1)\} \cup \{(1,3)\}$, where ω is the positive integers. Define R as follows:

(i) R is reflexive with $(1,3)$ as largest element and $(0,-1)$ as smallest element.

(ii) $(n,0) \; R \; (m,1)$ for all $m \geqq n$.

(iii) $(m,1) \; R \; (n,2)$ for all $m \geqq n$.

(iv) $(n,1) \; R \; (m,1)$ for all $m \geqq n$.

Then X is complete and bounded, and R is reflexive but not transitive. Further the following collection has the f.i.p: $\mathcal{B} = \{I(0,0)\} \cup \{D(n,2) \mid n \in \omega\}$. But $\bigcap\mathcal{B} = \emptyset$.

REMARK. If A_1 is any collection of finite antisets and if A is the singletons, then $T(A_1) \subset T(A)$, hence the identity map $i : (X, T(A)) \to (X, T(A_1))$ is continuous and $(X, T(A_1))$ is compact whenever $(X, T(A))$ is compact.

4. Connected Spaces

In this section we shall derive conditions similar to the well known conditions on totally ordered spaces under which the space is connected. Simple examples show that it is necessary to assume that the relation R must contain points other than the diagonal in order to get connectedness. A subset $C \subset X$ is an *R-chain* (or chain for short) iff for each $x , y \in C , x \neq y$, either $x \; R \; y$ or $y \; R \; x$. By a chain between two points x_1 , x_2 , with $x_1 \; R \; x_2$, we mean a chain C with x_1 as smallest element and x_2 as largest element. The set X is *strongly complete* if and only if X is complete, and if $C \subset X$ is a maximal chain, and if $A \subset C$ is bounded, then $\sup A$ and $\inf A$ are members of C . The set X

is *R-dense* iff, whenever $x \neq y$ and $x\,R\,y$, there is a $z \notin \{x,y\}$ such that $x\,R\,z$ and $z\,R\,y$. Further, if $y\,R\,x$ is false, then $z\,R\,x$ and $y\,R\,z$ are false. A relation R is *nondiscrete* iff whenever $X = X_1 \cup X_2$, with X_1, X_2 nonempty, there exist points $x_1 \in X_1$ and $x_2 \in X_2$ such that either $x_1\,R\,x_2$ or $x_2\,R\,x_1$.

We are now ready to state the main theorem of this section.

THEOREM 8. *Let R be a reflexive, transitive, nondiscrete relation on the set X. Let* A *be the singletons. If X is strongly complete and R-dense, then $(X, T(A))$ is connected.*

Proof. Suppose that X is not connected, then $X = X_1 \cup X_2$ with X_1, X_2 nonempty, closed sets. Since R is nondiscrete, there exist points $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1\,R\,x_2$ (in case $x_2\,R\,x_1$ we change the subscripts). Let C be the maximal chain between x_1 and x_2. (The chain C is contained in a maximal chain and hence is complete.) Let $A = \{x \in C \mid z\,R\,x \text{ and } z \in C \text{ implies } z \in X_1\}$. Note that A contains x_1 and hence is nonempty. Let x_0 be the supremum of A. We now show that $x_0 \in \overline{A}$, and hence $x_0 \in X_1$. If $x_0 \notin \overline{A}$, then there must be a finite number of subbasic closed sets, say $B_1,\ldots,B_k$, such that $A \subset \bigcup_{i=1}^{k} B_i$, but $x_0 \notin B_i$ for all $i = 1,\ldots,k$. Since $x_0 = \sup A$, and since R is transitive, each B_i must be of the form, $B_i = D(b_i)$. Now let $A_i = \{x \in A \mid x \in B_i\}$ and let a_i be the supremum of A_i (necessarily in C). If $i \neq j$, then $a_i\,R\,a_j$ or $a_j\,R\,a_i$. If $a_i\,R\,a_j$, then $a_i\,R\,b_j$, and $x\,R\,b_j$ for all $x \in A_i \cup A_j$. Consequently, there is a j such that $A \subset B_j = D(b_j)$. But then b_j is an upper bound for A. But by the hypothesis that X is strongly complete, $x_0\,R\,b_j$, which is a contradiction. Thus it follows that $x_0 \in \overline{A} \subset X_1$, and hence $x_0 \in A$. Let $B = \{x \in C \mid x \in X_2\}$ and let $z_0 = \inf B$. By an argument similar to the above $z_0 \in \overline{B} \subset X_2$. Hence $x_0 \neq z_0$. Then there exists a y such that $x_0\,R\,y$, $y\,R\,z_0$ with $x_0 \neq y \neq z_0$. If $z_0\,R\,x_0$, then any closed set which contains x_0, contains z_0 and conversely. So $(z_0,x_0) \notin R$ and we may choose y so that $(y_1,x_0) \notin R$ and $(z_0,y) \notin R$. But then $y \notin X_1$

and $y \notin X_2$, a contradiction. (That $y \notin X_2$ follows from the definition of z_0, and $y \notin X_1$ since $y \in X_1$ implies there exists z such that $z \, R \, y$ and $z \notin X_1$, since $y \notin A$. But then $z_0 \, R \, z$ and $z_0 \, R \, y$, another contradiction.) Consequently we conclude that X is connected.

We can obtain a corollary analogous to the corollary in the preceding section.

REMARK. Simple examples show that the condition that X be strongly complete, or a similar condition, is needed. Also, other examples show that the particular way in which the concept of R-density was defined is also justified. Further we note that in the case of a totally ordered space, R-dense becomes order dense. Finally, it is conjectured that the transitivity is necessary in order to assure the validity of Theorem 8. However, no example is available to show this.

5. Relations in topological spaces

Suppose (X,T) is a topological space and R is a relation on X. In this section we investigate the relationship between T and $T(A)$ for certain classes of antisets A. In particular, if X is a tree, R the cutpoint order (see Ward [1]), and A suitably chosen, then $T = T(A)$.

LEMMA 9. *Suppose that (X,T) is a T_2 space, and that R is a reflexive, compact relation on X. Let A be any closed subset of X. Then $D(A)$ and $I(A)$ are closed sets.*

Proof. Let $x \in \text{cl}(D(A))$. If $x \in A$, then, since R is reflexive, $x \in D(A)$. Hence, we may assume that $x \notin A$. Let V be any open set containing x. We may assume that $V \subset X \setminus A$. Since $x \in \text{cl}(D(A))$, there is a $y \in V \cap D(A)$. Thus we can find a net (y_α, a_α) where $y_\alpha \to x$, $a_\alpha \in A$, and $y_\alpha \, R \, a_\alpha$. Since R is compact, there is a limit point (y_0, a_0) of this set in R, but since (X,T) is T_2, $y_0 = x$ and since A is closed, $a_0 \in A$. Thus $x \in D(A)$ and we are done.

Ward [1] has shown that the cutpoint order is closed, hence compact. Thus for any collection A of closed antisets of a tree (X,T), we have $T(A) \subset T$.

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REMARK. Since we have a topology on X, we can broaden the class of sets which are used to construct the topology $T(A)$. For example, we might use the closures of antisets and in fact we do this in the next result.

THEOREM 10. *Let (X,T) be a tree and let R be the cutpoint order with minimal element e. Let A consist of the finite antisets, and the closures of sets of maximal elements. Then $T(A) = T$.*

Proof. We have $T(A) \subset T$ and so we must show that $T \subset T(A)$. Let $x \in X$ and let U be an open set containing x. Let V be an open set such that $x \in V \subset U$, and such that the boundary of V is finite. Let $M = \{y \mid y$ is maximal and $x\ R\ y$ is false$\}$. Since $I(x) \setminus x$ is open (see Ward [2]), $\overline{M} \cap I(x) = \phi$. Let $b(V) = \{x_1, \ldots, x_k\}$ be the boundary of V. Consider the following $T(A)$-closed set:

$$B = D(\overline{M}) \cup (\cup\{D(x_k) \mid x \notin D(x_k)\}) \cup (\cup\{I(x_k) \mid x \notin I(x_k)\}) .$$

We claim $x \in X \setminus B \subset V$; that $x \in X \setminus B$ is clear. Let $y \in X \setminus B$. If $y \in I(x) \setminus V$, then the arc from x to y meets $b(V)$, and hence some point in $b(V)$ is smaller than y. So $y \notin X \setminus B$ contrary to assumption.

If $y \in D(x) \setminus V$, then y must be smaller than some element of $b(V)$. Finally if $y \notin D(x)$ or $I(x)$, y must be in $D(\overline{M})$, and hence $y \notin X \setminus B$. Thus $y \in X \setminus B$ implies that $y \in V$ and we are done.

That we must assume that A contains more than the singletons is shown by the following example. Let $X_0 = [0,1]$. At $1/2^n$ erect an interval of length $1/2^n$. Let $e = 0$. Then if A is the finite antisets, each neighborhood of 1 (with respect to $T(A)$) contains points arbitrarily close to 0. Thus, $T(A)$ is not the same as the usual topology for this space.

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