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ARTICLE

Tight bound for the Erdős–Pósa property of tree minors

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Abstract

Let T be a tree on t vertices. We prove that for every positive integer k and every graph G, either G contains k pairwise vertex-disjoint subgraphs each having a T minor, or there exists a set X of at most $t(k-1)$ vertices of G such that $G-X$ has no T minor. The bound on the size of X is best possible and improves on an earlier $f(t)k$ bound proved by Fiorini, Joret, and Wood (2013) with some fast-growing function $f(t)$. Moreover, our proof is short and simple.

Keywords: Graph minors; pathwidth; Erdős–Pósa property

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1. Introduction

In 1965, Erdős and Pósa [6] showed that every graph G either contains k vertex-disjoint cycles or contains a set X of $\mathcal{O}(k\log k)$ vertices such that $G-X$ has no cycles. The $\mathcal{O}(k\log k)$ bound on the size of X is best possible up to a constant factor. Using their Grid Minor Theorem, Robertson and Seymour [9] proved the following generalisation: for every planar graph H, there exists a function $f_H(k)$ such that every graph G contains either k vertex-disjoint subgraphs each having an H minor, or a set X of at most $f_H(k)$ vertices such that $G-X$ has no H minor. For $H=K_3$, this corresponds to the setting of the Erdős–Pósa theorem.

The theorem of Robertson and Seymour is best possible in the sense that no such result holds when H is not planar. The original upper bound of $f_H(k)$ on the size of X depends on bounds from the Grid Minor Theorem and is large as a result (though it is polynomial in k if we use the polynomial version of the Grid Minor Theorem, see [4]). Chekuri and Chuzhoy [3] subsequently showed an improved upper bound of $\mathcal{O}_H(k\log^c k)$ for a fixed planar graph H, where c is some large but absolute constant. This was in turn improved to $\mathcal{O}_H(k\log k)$ by Cames van Batenburg, Huynh, Joret, and Raymond [2], thus matching the original bound of Erdős and Pósa for cycles.

An $\mathcal{O}_H(k\log k)$ bound is best possible when H contains a cycle. However, when H is a forest, it turns out that one can obtain a linear in k bound on the size of X, as proved by Fiorini, Joret, and Wood [7]. Their proof gives an $\mathcal{O}_H(k)$ bound with a non-explicit constant factor that grows very fast as a function of $|V(H)|$. This is due to the use of MSO-based tools in the proof, among others. In this short note, we give a simple proof of their result with an optimal dependence on t and k when H is a tree.

Theorem 1. *Let T be a tree on t vertices. For every positive integer k and every graph G, either G contains k pairwise vertex-disjoint subgraphs each having a T minor, or there exists a set X of at most $t(k-1)$ vertices of G such that $G-X$ has no T minor.*

Observe that the bound on the size of X in Theorem 1 is tight: if G is a complete graph on $tk-1$ vertices, then G does not contain k pairwise vertex-disjoint subgraphs each having a T minor, and every set X of vertices such that $G-X$ has no T minor has size at least $|V(G)|-(t-1)=t(k-1)$.

Theorem 1 follows immediately from the following more general result for forests.

Theorem 2. *Let F be a forest on t vertices and let t' be the maximum number of vertices in a component of F. For every positive integer k and every graph G, either G contains k pairwise vertex-disjoint subgraphs each having an F minor, or there exists a set X of at most $tk-t'$ vertices of G such that $G-X$ has no F minor.*

Let us also point out the following corollary of Theorem 1 (proved in the next section).

Corollary 3. *For all positive integers p and k, and for every graph G, either G contains k vertex-disjoint subgraphs each of pathwidth at least p, or G contains a set X of at most $2\cdot 3^{p+1}k$ vertices such that $G-X$ has pathwidth strictly less than p.*

2. Proof

For a positive integer k, we use the notation $[k] := \{1, \dots, k\}$, and when $k=0$ let $[k] := \varnothing$.

Let G be a graph. We denote by $V(G)$ and $E(G)$, the vertex set and edge set of G, respectively. Let $X \subseteq V(G)$. Then $G[X]$ denotes the subgraph of G induced by the vertices in X and $G-X = G[V(G)-X]$. We define the *boundary* of X in G to be $\partial_G X := \{v \in X \mid vw \in E(G),\ w \in V(G-X)\}$. We omit the subscript G when the graph G is clear from the context.

A *path decomposition* of G is a sequence $(B_1, B_2, \dots, B_q)$ of vertex subsets of G called *bags* satisfying the following properties: (1) every vertex of G appears in a non-empty set of consecutive bags, and (2) for every edge uv of G, there is a bag containing both u and v. The *width* of the path decomposition is the maximum size of a bag minus 1. The *pathwidth* $pw(G)$ of G is the minimum width of a path decomposition of G.

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. Robertson and Seymour [8] proved that there exists a function $f:\mathbb{N}\to\mathbb{N}$ such that for every graph G and every forest F on t vertices, if $pw(G)\geqslant f(t)$ then G contains F as a minor. Bienstock, Robertson, Seymour, and Thomas [1] later showed that one can take $f(t)=t-1$, which is best possible. Diestel [5] subsequently gave a short proof of this result. Our proof of Theorem 2 builds on the following slightly stronger result, which appears implicitly in Diestel's proof [5].

Lemma 4 ([5]). *Let G be a graph, let t be a positive integer, and let F be a forest on t vertices. If $pw(G)\geqslant t-1$, then there exists $Y\subseteq V(G)$ such that*

1. *$G[Y]$ has a path decomposition $(B_1, \dots, B_q)$ of width at most $t-1$ such that $\partial Y \subseteq B_q$, and*
2. *$G[Y]$ contains F as a minor.*

We now turn to the proof of Theorem 2.

Proof of Theorem 2. We prove the following strengthening of Theorem 2: Let G be a graph, let c be a positive integer, let $t_1 \leqslant \cdots \leqslant t_c$ be non-negative integers, let $T_1, \dots, T_c$ be trees with $|V(T_i)| = t_i$ for every $i\in[c]$, let $x_1, \dots, x_c$ be non-negative integers, at least one of which is non-zero, and let $I := \{i \in [c] \mid x_i \geqslant 1\}$. Then either

1. G contains pairwise vertex-disjoint subgraphs $\{M_{i,j} \mid i\in[c],\ j\in[x_i]\}$ such that, for each $i\in[c]$ and $j\in[x_i]$, $M_{i,j}$ contains a T_i minor, or

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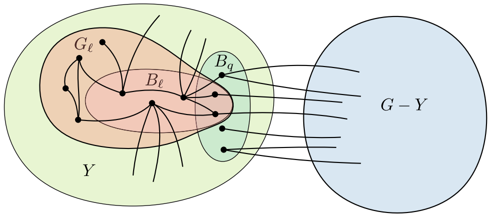

Figure 1. The set Y and the graph G_ℓ whose boundary in G is contained in B_ℓ.

2. there exists $X \subseteq V(G)$ with $|X| \leqslant \sum_{i\in I} x_i t_i - t_{\max(I)}$ and $G - X$ does not contain T_i as a minor for some $i \in I$.

We call the tuple $(G, c, T_1, \dots, T_c, x_1, \dots, x_c)$ an *instance*. Theorem 2 follows by letting $T_1, \dots, T_c$ be the components of the forest F and letting $x_1 = x_2 = \dots = x_c = k$.

Roughly, the proof describes an inductive procedure that attempts to find a pairwise disjoint collection of models, where the number of models of each tree T_i is x_i. Induction is on the number $\sum_{i\in[c]} x_i$ of models still missing from the collection. Failing to find one of the missing models at some step will establish (2).

Let $(G, c, T_1, \dots, T_c, x_1 \dots, x_c)$ be an instance, and let $m := \min(I)$. Then T_m is a smallest tree among $T_1, \dots, T_c$ such that $x_m \geqslant 1$, that is, such that we are still missing a model of T_m. In the base case, $\sum_{i\in[c]} x_i = x_m = 1$, and either G has a T_m minor and the first outcome of the statement holds, or G has no such minor and the second outcome holds with $X := \emptyset$, since $\sum_{i\in I} x_i t_i - t_{\max(I)} = t_m - t_m = 0$.

For the inductive case, assume that $\sum_{i\in[c]} x_i \geqslant 2$ and that the statement holds for instances with smaller values of the sum. If, for every $i \in I$, G has no T_i minor, then the second outcome of the statement holds with $X := \emptyset$ again. Thus, we may assume that G has a T_i minor for some $i \in I$.

If G has pathwidth at least $t_m - 1$, apply Lemma 4 with $t = t_m$ and $F = T_m$, and let Y be the resulting subset of vertices of G. If G has pathwidth less than $t_m - 1$, simply let $Y := V(G)$. In either case, $G[Y]$ has pathwidth at most $t_m - 1$ and has a path decomposition $(B_1, B_2, \dots, B_q)$ with $|B_\ell| \leqslant t_m$ for all $\ell \in [q]$, and such that $\partial_G Y \subseteq B_q$. See Figure 1. Furthermore, observe that in both cases $G[Y]$ has a T_i minor for some $i \in I$, by our assumption on G.

Let $\ell \in [q]$ be the smallest index such that $G_\ell := G[B_1 \cup \dots \cup B_\ell]$ contains a T_i minor for some $i \in I$, and let i' be an index in I such that

$$G_\ell \text{ contains a } T_{i'} \text{ minor.} \qquad (\star)$$

Observe that

$$G_\ell - B_\ell \text{ has no } T_i \text{ minor for every } i \in I. \qquad (\star\star)$$

We claim that

$$\text{there is no edge in } G \text{ between vertices of } G_\ell - B_\ell \text{ and vertices of } G - V(G_\ell). \qquad (\star\star\star)$$

To see this, suppose for a contradiction that uv is such an edge, with $u \in V(G_\ell) - B_\ell$ and $v \in V(G) - V(G_\ell)$. First, note that $u \in B_1 \cup \dots \cup B_{\ell-1}$. If $v \in Y$, then u and v appear together in some bag B_j of the path decomposition $(B_1, B_2, \dots, B_q)$ of $G[Y]$, and $j > \ell$ since $v \notin B_1 \cup \dots \cup B_\ell$. However, since $u \in B_1 \cup \dots \cup B_{\ell-1}$ and $u \in B_j$, we conclude that u belongs also to B_ℓ, a contradiction. If $v \notin Y$, then $u \in \partial Y$, and thus $u \in B_q$. Again, we deduce similarly that $u \in B_\ell$, a contradiction. This completes the proof of ($\star\star\star$).

Let $G' := G - V(G_\ell)$. Let $x'_i := x_i$ for each $i \in [c] - \{i'\}$ and let $x'_{i'} := x_{i'} - 1$. Let $I' = \{i \in [c] \mid x'_i \geqslant 1\}$. Apply induction to the instance $(G', c, T_1, \dots, T_c, x'_1, \dots, x'_c)$. If it results in a set of vertex-disjoint subgraphs $\{M'_{i,j} \mid i \in [c], j \in [x'_i]\}$, with $M'_{i,j}$ containing a T_i minor for each $i \in [c]$ and

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$j \in [x'_i]$, then we let $M_{i,j} := M'_{i,j}$ for each $i \in [c]$ and $j \in [x'_i]$, and $M_{i',x_{i'}} := G_\ell$, which using ($\star$) results in the desired collection of vertex-disjoint subgraphs. Otherwise, we obtain a set X' of at most $\sum_{i\in I'} x'_i t_i - t_{\max(I')}$ vertices such that $G' - X'$ does not contain T_a as a minor for some $a \in I'$.

Let $X := X' \cup B_\ell$. Observe that

$$|X| = |X'| + |B_\ell| \leqslant \sum_{i\in I'} x'_i t_i - t_{\max(I')} + t_m \leqslant \sum_{i\in I} x_i t_i - (t_{\max(I')} + t_{i'} - t_m)$$
$$\leqslant \sum_{i\in I} x_i t_i - t_{\max(I)}.$$

To see why the last inequality holds, there are two cases to consider: (i) if $\max(I') = \max(I)$, then the inequality follows immediately since $t_{i'} \geqslant t_m$. (ii) If $\max(I') < \max(I)$, then $i' = \max(I)$ and $\max(I') \geqslant \min(I') = m$, so $t_{\max(I')} + t_{i'} - t_m \geqslant t_{i'} = t_{\max(I)}$.

Now, let us show that $G - X$ does not contain T_i as a minor, for some $i \in I$. Let $a \in I'$ be such that $G' - X'$ does not contain T_a as minor. We will show that we can take $i = a$. To do so, it is enough to show that X meets every inclusion-wise minimal subgraph of G containing a T_a minor. Let M be such a subgraph of G. Note that M is connected, since T_a is connected. Now, observe that by ($\star\star\star$), either M is contained in G', or M is contained in $G_\ell - B_\ell$, or M contains a vertex of B_ℓ. In the first case, M contains a vertex of $X' \subseteq X$, by the choice of a. The second case is ruled out by ($\star\star$). In the third case, M contains a vertex of $B_\ell \subseteq X$. Thus, we conclude that M contains a vertex of X. This concludes the proof. □

We may now turn to the proof of Corollary 3. We will use the following lemma, which is a special case of a more general result of Robertson and Seymour [Statement (8.7) in [9]].

Lemma 5. *For every graph G, for every path decomposition $(B_1, B_2, \ldots, B_q)$ of G, for every family $\mathcal{F}$ of connected subgraphs of G, for every positive integer d, either:*

1. *there are d pairwise vertex-disjoint subgraphs in $\mathcal{F}$, or*
2. *there is a set X that is the union of at most $d-1$ bags of $(B_1, B_2, \ldots, B_q)$ such that $V(F) \cap X \neq \varnothing$ for every $F \in \mathcal{F}$.*

Proof of Corollary 3. It is known (and an easy exercise to show) that, for every positive integer p, the complete ternary tree T_p of height p has pathwidth p. First, apply Theorem 1 on G with the tree T_p. If G contains k vertex-disjoint subgraphs each containing a T_p minor, we are done. So we may assume that the theorem produces a set X_1 of at most $|V(T_p)|(k-1) \leqslant 3^{p+1}(k-1)$ vertices such that $G - X_1$ has no T_p minor.

By Lemma 4, $G - X_1$ has a path decomposition $(B_1, B_2, \ldots, B_q)$ of width strictly less than 3^{p+1}. It is easily checked that every inclusion-wise minimal subgraph of $G - X_1$ with pathwidth at least p is connected. Apply Lemma 5 on $G - X_1$ with the path decomposition $(B_1, B_2, \ldots, B_q)$, with $d = k$, and with the family $\mathcal{F}$ of connected subgraphs of $G - X_1$ with pathwidth at least p. If $\mathcal{F}$ contains k pairwise vertex-disjoint members, we are done. So we may assume that the lemma produces a set X_2 of at most $3^{p+1}(k-1)$ vertices such that X_2 hits every member of $\mathcal{F}$. It follows that $G - X_1 - X_2$ has pathwidth strictly less than p. Let $X := X_1 \cup X_2$. Since $|X| \leqslant 3^{p+1}(k-1) + 3^{p+1}(k-1) \leqslant 2 \cdot 3^{p+1} k$, the set X has the desired properties. □

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