POLYNOMIALS WITH SOME PRESCRIBED ZEROS

Q.I. Rahman and Mohd. Ali Khan

(received September 27, 1966)

1. In connection with various problems concerning polynomials

$$p_n(x) = \Sigma_{\nu=o}^{n} a_\nu x^\nu$$

on the unit interval, the Tchebycheff polynomial

$$T_n(x) = \cos(n \cos^{-1} x) = \frac{1}{2}\{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\}$$

$$= \sum_{\nu=o}^{[n/2]} \frac{(-1)^\nu n}{2(n-\nu)} \binom{n-\nu}{\nu} (2x)^{n-2\nu}$$

is known to play a very important role [11, problem 34]. For example, if

$$|p_n(x)| \leq 1 \qquad \text{for } -1 \leq x \leq 1 ,$$

then in the same interval [12]

$$|p_n'(x)| \leq n^2 ,$$

with equality possible if and only if $p_n(x)$ is the n-th Tchebycheff polynomial $T_n(x)$. Another situation illustrating our remark is the following [7, p.62].

Amongst all polynomials $q_n(x) = x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o$ with leading coefficient 1, the one which minimizes the norm

Canad. Math. Bull. vol. 10, no. 2, 1967

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press

$$\|q_n\| = \max_{[-1,1]} |q_n(x)|$$

is the polynomial $2^{-n+1} T_n(x)$.

It is also known [14, Theorem 17] that if $p_n(x) = \sum_{\nu=o}^{n} a_\nu x^\nu$ is a polynomial of degree n such that $|p_n(x)| \leq 1$ on $[0, 1]$, then for $0 \leq \nu \leq n$

$$|a_\nu| \leq |t_{n,\nu}| ,$$

where $t_{n,\nu}$'s are the coefficients of the polynomial

$$T_n(2x - 1) = \sum_{\nu=o}^{n} t_{n,\nu} x^\nu .$$

The following result of I. Schur ([15], Theorem V; see also [4], Theorem 3) provides yet another illustration.

THEOREM A. _If_ $p_n(x)$ _is a polynomial of degree_ n _such that_ $|p_n(x)| \leq 1$ _on_ $[-1, 1]$ _and_ $p_n(0) = 0$, _then in the same interval_

(1) $$|p_n(x)/x| \leq m ,$$

where $m = n$ _or_ $n-1$ _according as_ n _is odd or even, with equality possible only at_ $x = 0$. _The extremal polynomial is_ $T_n(x)$ _or_ $T_{n-1}(x)$ _according as_ n _is odd or even._

The theorem of Schur can be stated in the following alternative form:

_If_ $p_n(x)$ _is a polynomial of degree_ n _such that_

$$|T_1(x)\, p_n(x)| \equiv |\cos\,(\cos^{-1} x)\, p_n(x)| \equiv |x\, p_n(x)| \leq 1 ,$$

for $-1 \leq x \leq 1$, _then in the same interval_

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press

$$|p_n(x)| \leq m + 1 ,$$

where $m = n$ *or* $n-1$ *according as* n *is even or odd.*

Having realized the importance of Tchebycheff polynomials we seek to generalize the preceding result by assuming that $|T_k(x)\, p_n(x)| \leq 1$ for $-1 \leq x \leq 1$, where $T_k(x)$ is the k-th Tchebycheff polynomial. We prove

THEOREM 1. *If* $p_n(x)$ *is a polynomial of degree* n *such that* $|T_k(x)\, p_n(x)| \leq 1$ *for* $-1 \leq x \leq 1$, *then in the same interval*

$$|p_n(x)| \leq (n + k)/k .$$

For even n, this result includes Theorem A.

For the proof of Theorem 1, we need the following:

LEMMA. *If* $F(\theta)$ *is a real trigonometric polynomial of degree* n *and* $|F(\theta)| \leq 1$ *for real* θ, *then*

$$n^2(F(\theta))^2 + (F'(\theta))^2 \leq n^2 , \qquad \theta \text{ real} . \tag{2}$$

Inequality (2) was first explicitly stated by van der Corput and Schaake [6], although it is implicit in an earlier inequality due to Szegö [16].

Proof of Theorem 1. Let the maximum of $|p_n(\cos\theta)|$ for $0 \leq \theta \leq 2\pi$ occur when $\theta = \theta_o$. If θ_o is either 0 or 2π there is nothing to prove, since then $|T_k(\cos\theta_o)| = 1$ and we get

$$|p_n(\cos\theta)| \leq |p_n(\cos\theta_o)\, T_k(\cos\theta_o)| \leq 1 .$$

Now let $0 < \theta_o < 2\pi$, and choose γ such that $e^{i\gamma}\, p_n(\cos\theta_o)$ is real. Consider the real trigonometric polynomial

$$F(\theta) \equiv \mathrm{Re}\,\{e^{i\gamma}\, p_n(\cos\theta)\} .$$

The maximum modulus of $F(\theta)$ occurs at θ_o and it is a local

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press

maximum as well, i.e., the derivative of $F(\theta)$ vanishes at θ_o. Applying the lemma to the trigonometric polynomial

$$T_k(\cos\theta)\,F(\theta) \equiv \cos k\theta\, F(\theta)$$

we get

$$(n+k)^2 (\cos^2 k\theta)\, F^2(\theta) + \{-k(\sin k\theta)\, F(\theta) + (\cos k\theta)\, F'(\theta)\}^2 \le (n+k)^2 ,$$

for $0 \le \theta \le 2\pi$. For $\theta = \theta_o$ we have in particular

$$\{(n+k)^2 (\cos^2 k\theta_o) + k^2(\sin^2 k\theta_o\}\; F^2(\theta_o) \le (n+k)^2$$

or

$$|F(\theta_o)| \le (n+k)/\{k^2 + (n^2+2nk)\,(\cos^2 k\theta_o)\}^{1/2},$$

and the desired result follows.

If $S(\theta)$ is a trigonometric polynomial of degree n and $|S(\theta)| \le 1$ for real θ, then

(3) $$|S'(\theta)| \le n , \qquad \theta \text{ real} .$$

This result was proved by Bernstein [1], except that in (3) he had $2n$ in place of n. Inequality (3) in the present form first appeared in print in a paper of Fekete [8] who attributes the proof to Fejér. Bernstein [2, p.39] attributes the proof to Landau. Using (3), we can deduce from

$$S(\theta) - S(0) = \int_0^\theta S'(t)\, dt ,$$

that <u>if</u> $S(\theta)$ <u>is a trigonometric polynomial of degree</u> n <u>such that</u> $|S(\theta)| \le 1$, <u>and</u> $S(0) = 0$, <u>then for all real</u> θ

(4) $$|S(\theta)/\theta| \le n .$$

The example $\sin n\theta$ shows that the result is best possible. Note the analogy between this result and Theorem A.

It is well known that the class of trigonometric polynomials of degree n coincides [5] with the class of entire functions of

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press

exponential type $\tau (n \leq \tau < n+1)$, periodic on the real axis with period 2π. Inequality (4) is therefore included in the following result.

THEOREM 2. *If* $f(z)$ *is an entire function of exponential type* τ *such that* $|f(x)| \leq 1$ *for all real* x, *and* $f(0) = 0$, *then*

$$|f(x)/x| \leq \tau . \tag{5}$$

The bound is attained for the function $\sin \tau z$.

Inequality (3) has been extended [3, p. 206] by S. N. Bernstein to entire functions of exponential type and so Theorem 2 follows in exactly the same way as inequality (4).

2. In this section, we obtain L^2 analogues of (1), (4) and (5).

Let $p_n(x)$ be a polynomial of degree $n\ (>1)$ such that $p_n(0) = 0$. Then for $0 < a < n$

$$\int_{-1}^{1} |p_n(x)/x|^2\, dx$$

$$= \int_{|x|\leq a/n} |p_n(x)/x|^2\, dx + \int_{(a/n)<|x|\leq 1} |p_n(x)/x|^2\, dx \tag{6}$$

$$< \frac{2a}{n} \max_{|x|\leq a/n} |p_n(x)/x|^2 + \frac{n^2}{a^2} \int_{-1}^{1} |p_n(x)|^2\, dx .$$

Let $R > 1$ be arbitrary and let E_R denote the ellipse with foci at ± 1 and semi-axes

$$\alpha = \frac{1}{2}(R + R^{-1}), \qquad \beta = \frac{1}{2}(R - R^{-1}) .$$

It is easy to verify that if $|x| \leq \alpha^{-1}$ then the shortest distance D of x from E_R is $\beta(1-x^2)^{1/2}$. By Cauchy's integral formula

$$\left\{\frac{p_n(x)}{x}\right\}^2 = \frac{1}{2\pi i}\int_{E_R}\left\{\frac{p_n(z)}{z}\right\}^2 \frac{dz}{z-x} .$$

Hence for $|x| \leq \alpha^{-1}$,

$$|p_n(x)/x|^2 \leq \frac{1}{2\pi\beta(1-x^2)^{1/2}}\int_{E_R} |p_n(z)/z|^2 \, |dz|$$

$$\leq \frac{R^{2n-1}}{\pi\beta(1-x^2)^{1/2}}\int_{-1}^{1} |p_n(x)/x|^2 \, dx$$

by an inequality of Hille, Szegő, and Tamarkin (see inequality (2.3) on p. 732 of [9]). On putting $R^2 = \frac{n}{n-1}$ we obtain

$$|p_n(x)/x|^2 < \frac{2en}{\pi(1-x^2)^{1/2}}\int_{-1}^{1} |p_n(x)/x|^2 \, dx$$

$$\leq \frac{2en^2}{\pi(n^2-a^2)^{1/2}}\int_{-1}^{1} |p_n(x)/x|^2 \, dx$$

if $|x| \leq a/n$.

Using the last estimate in (6), we obtain

$$\int_{-1}^{1} |p_n(x)/x|^2 \, dx$$

$$< \frac{4aen}{\pi(n^2-a^2)^{1/2}}\int_{-1}^{1} |p_n(x)/x|^2 \, dx + \frac{n^2}{a^2}\int_{-1}^{1} |p_n(x)|^2 \, dx$$

or

$$\int_{-1}^{1} |p_n(x)/x|^2 \, dx < \frac{\pi n^2(n^2-a^2)^{1/2}}{a^2\{\pi(n^2-a^2)^{1/2} - 4aen\}}\int_{-1}^{1} |p_n(x)|^2 \, dx .$$

On putting $a = \frac{\pi}{6e}$ this reduces to

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press

$$\int_{-1}^{1} |p_n(x)/x|^2\, dx$$

$$< (6en/\pi)^2 \{1-(\tfrac{2}{3})/\sqrt{1-(6en/\pi)^{-2}}\}^{-1} \int_{-1}^{1} |p_n(x)|^2\, dx ,$$

which is the analogue of (1) we wanted to prove. It is not asserted that the right-hand side of this inequality is best possible. Thus we may state the following:

THEOREM 3. <u>If</u> $p_n(z)$ <u>is a polynomial of degree</u> $n(>1)$ $p_n(0) = 0$, <u>then</u>

$$\text{(7)} \qquad \int_{-1}^{1} |p_n(x)/x|^2\, dx < K\, n^2 \int_{-1}^{1} |p_n(x)|^2\, dx ,$$

<u>where</u> $K < 83$.

If $F(\theta)$ is a trigonometric polynomial of degree n , such that $F(0) = 0$, then $F(2\theta)/\sin\theta$ is a trigonometric polynomial of degree $2n - 1$, and for $0 < a < n\pi/2$,

$$\int_{-\pi/2}^{\pi/2} |F(2\theta)/\sin\theta|^2\, d\theta$$

$$= \int_{|\theta|\le a/n} |F(2\theta)/\sin\theta|^2\, d\theta + \int_{(a/n)<|\theta|\le\pi/2} |F(2\theta)/\sin\theta|^2\, d\theta$$

$$< \frac{2a}{n} \max_{|\theta|\le a/n} |F(2\theta)/\sin\theta|^2 + \frac{\pi^2}{4} \int_{(a/n)<|\theta|\le\pi/2} |F(2\theta)/\theta|^2\, d\theta ,$$

since $|\sin\theta| \ge \frac{2}{\pi}|\theta|$ for $|\theta| \le \frac{\pi}{2}$. Hence

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press

$$\int_{-\pi/2}^{\pi/2} |F(2\theta)/\sin\theta|^2 \, d\theta$$

$$< \frac{2a}{n} \max_{|\theta| \leq a/n} |F(2\theta)/\sin\theta|^2 + \frac{\pi^2 n^2}{4a^2} \int_{-\pi/2}^{\pi/2} |F(2\theta)/\theta|^2 \, d\theta$$

$$\leq \frac{2a}{n}\,\frac{4n-1}{2\pi} \int_{-\pi}^{\pi} |F(2\theta)/\sin\theta|^2 \, d\theta + \frac{\pi^2 n^2}{4a^2} \int_{-\pi/2}^{\pi/2} |F(2\theta)|^2 \, d\theta$$

by a result of Ibragimov [10, p. 178] which states that, <u>if</u> $S(\theta)$ <u>is</u> <u>a trigonometric polynomial of degree</u> n, <u>then for</u> $1 \leq p \leq 2$

$$\max_{-\pi \leq \theta \leq \pi} |S(\theta)| \leq \left(\frac{2n+1}{2\pi}\right)^{1/p} \left(\int_{-\pi}^{\pi} |S(\theta)|^p \, d\theta\right)^{1/p} .$$

Since $\int_{-\pi}^{\pi} |F(2\theta)/\sin\theta|^2 \, d\theta$ is equal to $2\int_{-\pi/2}^{\pi/2} |F(2\theta)/\sin\theta|^2 \, d\theta$,

we obtain

$$\int_{-\pi/2}^{\pi/2} |F(2\theta)/\sin\theta|^2 \, d\theta < \frac{n^3 \pi^3}{4a^2\{n\pi - 2a(4n-1)\}} \int_{-\pi/2}^{\pi/2} |F(2\theta)|^2 \, d\theta .$$

On putting $a = \frac{\pi}{12}$ this reduces to

$$\int_{-\pi/2}^{\pi/2} |F(2\theta)/\sin\theta|^2 \, d\theta < 6 \times 3 \times 12 \times \left(\frac{n^3}{2n+1}\right) \int_{-\pi/2}^{\pi/2} |F(2\theta)|^2 \, d\theta .$$

Hence

$$\int_{-\pi}^{\pi} |F(\theta)/\sin\tfrac{\theta}{2}|^2 \, d\theta < 6 \times 3 \times 12 \times \left(\frac{n^3}{2n+1}\right) \int_{-\pi}^{\pi} |F(\theta)|^2 \, d\theta .$$

Since $|\frac{\theta}{2}| \geq |\sin\frac{\theta}{2}|$ we get

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press

$$\int_{-\pi}^{\pi} |F(\theta)/\theta|^2 \, d\theta < 54 \times \left(\frac{n^3}{2n+1}\right) \int_{-\pi}^{\pi} |F(\theta)|^2 \, d\theta .$$

Hence we have the following theorem.

THEOREM 4. *If* $F(\theta)$ *is a trigonometric polynomial of degree* n *such that* $F(0) = 0$, *then*

(8) $$\int_{-\pi}^{\pi} |F(\theta)/\theta|^2 \, d\theta < 27 n^2 \int_{-\pi}^{\pi} |F(\theta)|^2 \, d\theta .$$

Inequality (8) is to be compared with (4). Here again we do not claim that the inequality is sharp.

Finally, we prove the following analogue of (5).

THEOREM 5. *If* $f(z)$ *is an entire function of exponential type* τ *belonging to* L^2 *on the real axis and* $f(0) = 0$, *then*

(9) $$\int_{-\infty}^{\infty} |f(x)/x|^2 \, dx \leq 27(\tau/\pi)^2 \int_{-\infty}^{\infty} |f(x)|^2 \, dx .$$

Proof of Theorem 3. For every positive a

$$\int_{-\infty}^{\infty} |f(x)/x|^2 \, dx = \int_{|x| > a/\tau} |f(x)/x|^2 \, dx + \int_{|x| \leq a/\tau} |f(x)/x|^2 \, dx$$

$$< (\tau/a)^2 \int_{-\infty}^{\infty} |f(x)|^2 \, dx + 2(a/\tau) \max_{-\infty<x<\infty} |f(x)/x|^2 .$$

It has been proved by Korevaar [13] that if $F(z)$ is an entire function of exponential type τ belonging to L^2 on the real axis then

$$|F(x)|^2 \leq \frac{\tau}{\pi} \int_{-\infty}^{\infty} |F(x)|^2 \, dx , \qquad -\infty<x<\infty .$$

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press

Hence

$$\int_{-\infty}^{\infty} |f(x)/x|^2 \, dx < (\tau/a)^2 \int_{-\infty}^{\infty} |f(x)|^2 \, dx + 2(a/\pi) \int_{-\infty}^{\infty} |f(x)/x|^2 \, dx$$

or

$$\int_{-\infty}^{\infty} |f(x)/x|^2 \, dx < (\tau/a)^2 \frac{\pi}{\pi - 2a} \int_{-\infty}^{\infty} |f(x)|^2 \, dx .$$

Putting $a = \frac{\pi}{3}$ which makes $(\tau/a)^2 \frac{\pi}{\pi - 2a}$ minimum we get the desired result.

REFERENCES

1. S.N. Bernstein, Sur l'ordre de la meilleure approximation de fonctions continues par des polynômes de degré donné. Mémoires de l'Académie Royale de Belgique, (2), vol.4 (1912), pages 1-103.

2. S.N. Bernstein, Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle. Paris (1926).

3. R.P. Boas, Jr., Entire functions. Academic Press, New York, (1954).

4. R.P. Boas, Jr., Inequalities for polynomials with a prescribed zero. Studies in Mathematical Analysis and related topics (Essays in honour of George Pólya), Stanford University Press, Stanford (1962).

5. R.P. Boas, Jr., Periodic entire functions. American Math. Monthly, vol. 71 (1964), page 782.

6. J.G. van der Corput and G. Schaake, Ungleichungen für Polynome und trigonometrische Polynome. Compositio Mathematica, vol. 2 (1935), pages 321-361. Berichtigung zu: Ungleichungen für Polynome und trigonometrische Polynome. Compositio Mathematica, vol.3 (1936), page 128.

7. P.J. Davis, Interpolation and approximation. Blaisdell Publishing Company (1963).

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press

8. M. Fekete, Über einen Satz des Herrn Serge Bernstein. Journal für die reine und angewandte Mathematik, vol. 146 (1916), pages 88-94.

9. E. Hille, G. Szegö and J.D. Tamarkin, On some generalizations of a theorem of A. Markoff. Duke Mathematical Journal, vol. 3 (1937), pages 729-739.

10. I.I. Ibragimov, Extremal properties of entire functions of finite degree, (in Russian), Baku (1962).

11. Nicholas D. Kazarinoff, Analytic inequalities. Holt, Rinehart and Winston, New York (1961).

12. O.D. Kellogg, On bounded polynomials in several variables. Mathematische Zeitschrift, vol. 27 (1928), pages 55-64.

13. J. Korevaar, An inequality for entire functions of exponential type. Nieuw Arch. Wiskunde (2), vol. 23 (1949), pages 55-62.

14. Q.I. Rahman and K.R. Unni, Extremal problems and polynomials of least deviation. Scripta Mathematica, vol. 27 (1965), pages 303-329.

15. I. Schur, Über das Maximum des absoluten Betrages eines Polynoms in einem gegebenen Intervall. Mathematische Zeitschrift, vol. 4 (1919), pages 271-287.

16. G. Szegö, Über einen Satz des Herrn Serge Bernstein. Schriften der Konigsberger Gelehrten Gesellschaft, vol. 5 (1928), pages 59-70.

Department of Mathematics
University of Montreal
and
Department of Chemistry
Regional Engineering College
Srinagar (India).

https://doi.org/10.4153/CMB-1967-016-5 Published online by Cambridge University Press