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The Smallest Pisot Element in the Field of Formal Power Series Over a Finite Field

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Abstract. Dufresnoy and Pisot characterized the smallest Pisot number of degree $n \geq 3$ by giving explicitly its minimal polynomial. In this paper, we translate Dufresnoy and Pisot's result to the Laurent series case.

The aim of this paper is to prove that the minimal polynomial of the smallest Pisot element (SPE) of degree n in the field of formal power series over a finite field is given by $P(Y) = Y^n - \alpha XY^{n-1} - \alpha^n$, where α is the least element of the finite field $\mathbb{F}_q \backslash \{0\}$ (as a finite total ordered set). We prove that the sequence of SPEs of degree n is decreasing and converges to αX. Finally, we show how to obtain explicit continued fraction expansion of the smallest Pisot element over a finite field.

1 Introduction

In the real case, the closedness of the set of all Pisot numbers $\mathcal{P}$ implies that it has a minimal element. In 1944, Siegel [9] proved that it is the positive root of the equation $x^3 - x - 1 = 0$ (Plastic constant) and is isolated in $\mathcal{P}$. He constructed two sequences of Pisot numbers converging to the golden ratio φ and asked whether φ is the smallest limit point of $\mathcal{P}$. This was later proved by Dufresnoy and Pisot [4], who also determined all elements of $\mathcal{P}$ that are less than φ. Not all of them belong to Siegel's two sequences. Vijayaraghavan proved that $\mathcal{P}$ has infinitly many limit points; in fact, the sequence of derived sets $\mathcal{P}, \mathcal{P}', \mathcal{P}'', \dots$ does not terminate.

In this paper we give smallest Pisot elements (SPE) over a field of Laurent series with coefficients in a finite field.

The paper is organized as follows. In this section, we give some preliminary definitions. In Section 2, we define the lexicographic order over a field of Laurent series with coefficients in a finite field and we give the SPE of degree n in Laurent series case. In Section 3, the continued fraction expansion of the SPE over a field of Laurent series is studied.

Let $\mathbb{F}_q$ be a field with q elements of characteristic p, $\mathbb{F}_q[X]$ the set of polynomials of coefficients in $\mathbb{F}_q$, and $\mathbb{F}_q(X)$ its field of fractions. The set $\mathbb{F}_q((X^{-1}))$ of Laurent series over $\mathbb{F}_q$ is defined

$$\mathbb{F}_q((X^{-1})) = \left\{ \sum_{j=s}^{+\infty} a_j X^{-j} \ : \ a_j \in \mathbb{F}_q, \ a_s \neq 0 \text{ with } s \in \mathbb{Z} \right\}.$$

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Let $\omega = \sum_{j=s}^{+\infty} a_j X^{-j} \in \mathbb{F}_q((X^{-1}))$. We denote its polynomial part by $[\omega]$ and its fractional part by $\{\omega\}$. We remark that $\omega = [\omega] + \{\omega\}$. As in Sprindžuk [11], a non archimedean absolute value on $\mathbb{F}_q((X^{-1}))$ is defined by $|\omega| = e^{-s}$. It is clear that for all $P \in \mathbb{F}_q[X]$, $|P| = e^{\deg P}$ and, for all $Q \in \mathbb{F}_q[X]$ such that $Q \neq 0$, $|\frac{P}{Q}| = e^{\deg P - \deg Q}$.

We know that $\mathbb{F}_q((X^{-1}))$ is complete and locally compact with respect to the metric defined by this absolute value.

We denote by $\overline{\mathbb{F}}_q((X^{-1}))$ an algebraic closure of $\mathbb{F}_q((X^{-1}))$. We note that the absolute value has a unique extension to $\overline{\mathbb{F}}_q((X^{-1}))$. Abusing the notation a little, we will use the same symbol $|\cdot|$ for the two absolute values.

A Pisot element $w \in \mathbb{F}_q((X^{-1}))$ is an algebraic integer over $\mathbb{F}_q[X]$ such that $|w| > 1$, whose remaining conjugates in $\overline{\mathbb{F}}_q((X^{-1}))$ have an absolute value strictly smaller than 1. In 1962 Bateman and Duquette [1] introduced and characterized Pisot elements in a field of Laurent series . They obtained the following results.

Theorem 1.1 *An element w in $\mathbb{F}_q((X^{-1}))$ is a Pisot element if and only if its minimal polynomial can be written as $P(Y) = Y^s + A_{s-1}Y^{s-1} + \cdots + A_0$, $A_i \in \mathbb{F}_q[X]$ for $i = 0, \ldots, s-1$, with $|A_{s-1}| = |w| > 1$ and $|A_i| < |w|$ for $i = 0, \ldots, s-2$.*

Theorem 1.2 *An element $w \in \mathbb{F}_q((X^{-1}))$ satisfying $|w| > 1$ is a Pisot element if and only if there exists a $\lambda \in \mathbb{F}_q((X^{-1})) \setminus \{0\}$ such that $\lim_{n\to+\infty}\{\lambda w^n\} = 0$; moreover, λ can be chosen to belong to $\mathbb{F}_q(X)(w)$.*

The study of the set $\mathcal{P}$ of Pisot elements was resumed in 1967 by Grandet-Hugot [5, 6]. In particular she showed that $\mathcal{P}$ is dense in $\mathbb{F}_q((X^{-1})) \setminus \{w : |w| < 1\}$. For more information about Pisot elements, see [1, 2, 5–8, 10].

Recall that $\mathbb{F}_q((X^{-1}))$ contains Pisot elements of any degree over $\mathbb{F}_q(X)$. Indeed, consider the polynomial $Y^d - aXY^{d-1} - b$, where $a, b \in \mathbb{F}_q \setminus \{0\}$. It can easily be seen, considering its Newton polygon, that the polynomial, which is irreducible over $\mathbb{F}_q(X)$, has a root $\omega \in \mathbb{F}_q((X^{-1}))$ such that $|\omega| > 1$ and all of its conjugates in $\overline{\mathbb{F}}_q((X^{-1}))$ have an absolute value strictly smaller than 1. Moreover, we have the following proposition ([3]).

Proposition 1.3 *There are infinitely many Pisot elements of a fixed degree. Furthermore, if $\mathcal{P}(n, Q)$ denotes the set of Pisot elements of degree $n \geq 2$, with integer part Q, then*

$$\mathrm{Card}(\mathcal{P}(n, Q)) = (q-1)q^{n-2}.$$

2 Smallest Pisot Elements

In the real case, Dufresnoy and Pisot [4] characterized the smallest Pisot number of degree $n \geq 3$. They obtained the following result.

Theorem 2.1 *Let α_n be the smallest Pisot number of degree $n \geq 3$. Then the following assertions hold:*

(i) *$P_n(z) = z^n - z^{n-1} - z^{n-2} + z^2 - 1$ is the minimal polynomial of α_n;*

(ii) *the sequence $(\alpha_n)_{n\geq 1}$ is increasing and converges to $\frac{1+\sqrt{5}}{2}$.*

Before giving the analogues of this result in Laurent series case, we introduce a lexicographic order on $\mathbb{F}_q((X^{-1}))$.

Let $\mathbb{F}_q$ be a finite field equipped with a total order $\prec$. Then the lexicographic order on $\mathbb{F}_q((X^{-1}))$ is defined as follows: If $w = \sum_{i=m}^{+\infty} w_i X^{-i}$ and $v = \sum_{i=m}^{+\infty} v_i X^{-i}$, then $w \prec v$ if $w_m \prec v_m$ or $w_m = v_m, \dots, w_{m+k} = v_{m+k}, w_{m+k+1} \prec v_{m+k+1}$ for some $k = 1, 2, 3, \dots$.

Now, we are prepared to give the main theorem.

Theorem 2.2 *If $\mathcal{P}(n) = \{$Pisot elements of degree $n\}$, $n \geq 2$, and α is the least element of $\mathbb{F}_q \backslash \{0\}$, then $w_n = \inf \mathcal{P}(n)$ is a Pisot element of minimal polynomial*

$$P_n(Y) = Y^n - \alpha X Y^{n-1} - \alpha^n.$$

Moreover, the sequence $(w_n)_{n\geq 1}$ is decreasing and converges to αX.

To prove this theorem we need the following lemmas.

Lemma 2.3 ([8, Lemma 2.2]) *Let $P(Y) = A_n Y^n + \cdots + A_0$ with $A_i \in \mathbb{F}_q[X]$, $A_d \neq 0$ and $|A_{n-1}| > |A_i|$, for all $i \neq n-1$. Then P has only one root $w \in \mathbb{F}_q((X^{-1}))$ satisfying $|w| > 1$. Moreover, $[w] = -\left[\frac{A_{n-1}}{A_n}\right]$.*

Lemma 2.4 *Let w be an algebraic Laurent series in $\mathbb{F}_q((X^{-1}))$ of degree n such that $[w] \neq 0$ and $A_n w^n + \cdots + A_1 w + A_0 = 0$, with $\deg A_{n-1} > \deg A_i$ for all $i \neq n-1$. Then $h = \frac{1}{\{w\}}$ is an algebraic Laurent series satisfying $B_n h^n + \cdots + B_1 h + B_0 = 0$, with $B_i \in \mathbb{F}_q[X]$, $\deg B_{n-1} > \deg B_i$, for all $i \neq n-1$.*

Proof Let $h = \frac{1}{\{w\}}$, then $w = [w] + \frac{1}{h}$ and

$$\begin{aligned} 0 = \sum_{j=0}^{n} A_j \Big([w] + \frac{1}{h} \Big)^j &= \sum_{j=0}^{n} A_j \sum_{k=0}^{j} \binom{j}{k} [w]^{j-k} h^{-k} \\ &= \sum_{k=0}^{n} \Big(\sum_{j=n-k}^{n} A_j \binom{j}{n-k} [w]^{j+k-n} \Big) h^k. \end{aligned}$$

Let

$$B_k = \sum_{j=n-k}^{n} A_j \binom{j}{n-k} [w]^{j+k-n}. \tag{2.1}$$

Then $\sum_{k=0}^{n} B_k h^k = 0$. Let us prove that $\deg B_{n-1} > \deg B_k$ for all $k \neq n-1$. Since $[w] = -\left[\frac{A_{n-1}}{A_n}\right]$, we get $A_{n-1} = -[f] A_n + R$ with $\deg R < \deg A_n$. From (2.1) we deduce

$$B_n = \sum_{j=0}^{n} A_j [w]^j = \Big(\sum_{j=0}^{n-2} A_j [w]^j \Big) + R[w]^{n-1} \tag{2.2}$$

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and

$$B_{n-1} = \sum_{j=1}^{n} jA_j[w]^{j-1} \tag{2.3}$$
$$= \Big(\sum_{j=1}^{n-2} jA_j[w]^{j-1}\Big) + A_n[w]^{n-1} + (n-1)R[w]^{n-2}.$$

Consequently, it turns out that $\deg B_{n-1} = (n-2)\deg[w] + \deg A_{n-1}$ and $\deg B_n < (n-2)\deg[w] + \deg A_{n-1}$. Since $\deg B_k \le \deg A_{n-1} + k\deg[w]$ for $k \in \{0, \dots, n-2\}$, $\deg B_k < \deg B_{n-1}$. ∎

Lemma 2.5 *Let $H(Y) = Y^d - AY^{d-1} - B$, $A, B \in \mathbb{F}_q[X] \setminus \{0\}$, $\deg A > \deg B$. Then H is irreducible over $\mathbb{F}_q[X]$.*

Proof From Lemma 2.3, H has a unique root w such that $|w| > 1$ and $[w] = A$. Let w_i, $2 \le i \le d$, be the other roots of H and set $w_1 = w$.

Since H is a monic polynomial, $\sum_{k=1}^{d} w_i^k \in \mathbb{F}_q[X]$, for each $k \in \mathbb{N}$, hence

$$\lim_{m\to+\infty} \{w^m\} = 0.$$

Let $P(Y) = Y^n + A_{n-1}Y^{n-1} + \cdots + A_0$ be the minimal polynomial of w. It is clear that $A_{n-1} = -A$, since $[w] = A$. From Theorem 1.1, the polynomial P satisfies $\deg A_{n-1} \ge \max_{i\ne n-1} \deg A_i$.

Let now $H(Y) = P(Y)Q(Y)$, with $Q(Y) = Y^m + B_{m-1}Y^{m-1} + \cdots + B_0$. Suppose that $m \ge 1$, then

$$B_{m-1} + A_{n-1} = -A \quad \text{and} \quad A_0B_0 = -B \tag{2.4}$$

$$\sum_{\substack{i+j=s\\0\le i\le n\\0\le j\le m}} A_iB_j = 0; \quad s \in \{1, 2, \dots, d-2\}. \tag{2.5}$$

Since $A_{n-1} = -A$, then, from (2.4), $B_{m-1} = 0$. Let $i_0 \in \{0, 1, \cdots, m\}$ such that $\deg B_{i_0} = \max_{0\le i\le m} \deg B_i$.

If $B_{i_0} \ne 0$, then $\deg(A_{n-1}B_{i_0}) > \deg(A_iB_j)$, $(i, j) \ne (n-1, i_0)$. Consequently,

$$\deg\Big(\sum_{\substack{i+j=n+i_0-1\\0\le i\le n\\0\le j\le m}} A_iB_j\Big) = \deg(A_{n-1}B_{i_0}).$$

This leads to a contradiction with (2.5).

Finally, we obtain $H(Y) = Y^mP(Y)$. If $m \ge 1$, then this contradicts the fact that $B \ne 0$, and consequently $H(Y) = P(Y)$. ∎

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Proof of Theorem 2.2 Let $P_n(Y) = Y^n - \alpha XY^{n-1} - \alpha^n$, where α is the least element of $\mathbb{F}_q\backslash\{0\}$. It follows from Lemmas 2.3 and 2.5 that P is irreducible and it has only one root w_n satisfying $|w_n| > 1$ and $[w_n] = \alpha X$. Since P is a monic polynomial, w_n is a Pisot element of degree n. Furthermore, if $w_n = \alpha X + \frac{1}{h}$, then h is an algebraic Laurent series satisfying

$$\alpha^n h^n - (\alpha X)^{n-1} h^{n-1} - \sum_{k=0}^{n-2} \binom{n-1}{k} (\alpha X)^k h^k = 0,$$

and using Lemma 2.3, we have

$$[h] = \frac{X^{n-1}}{\alpha}. \tag{2.6}$$

Now we consider another Pisot element $v_n \neq w_n$ of degree n such that $[v_n] = \alpha X$, then from Theorem 1.1 the minimal polynomial of v_n can be written as $F(Y) = Y^n - \alpha XY^{n-1} - \sum_{i=0}^{n-2} A_i Y^i$ with $A_i \in \mathbb{F}_q$. If $v_n = \alpha X + \frac{1}{g}$, then from Lemma 2.4, g satisfies an algebraic equation

$$\sum_{k=0}^{n} B_k g^k = 0, \text{ with } \deg B_{n-1} > \max_{i \neq n-1} \deg B_i.$$

In particular, from (2.2), (2.3), and Lemma 2.3

$$[g] = \left[\frac{(\alpha X)^{n-1} - \sum_{i=1}^{n-2} iA_i(\alpha X)^{i-1}}{\sum_{i=0}^{n-2} A_i(\alpha X)^i} \right],$$

- if $(A_1, \dots, A_{n-2}) \neq (0, \dots, 0)$, we have $\deg[g] < n-1 = \deg[h]$, then $\frac{1}{h} \preceq \frac{1}{g}$,
- if $(A_1, \dots, A_{n-2}) = (0, \dots, 0)$, we have $[g] = \frac{\alpha^{n-1}}{A_0} X^{n-1}$ and from (2.6), we obtain $\frac{1}{h} = \alpha X^{-(n-1)} + \cdots \preceq \frac{A_0}{\alpha^{n-1}} X^{-(n-1)} + \cdots = \frac{1}{g}$ ($A_0 \neq \alpha^n$ if not $v_n = w_n$).

Hence, we get in the two cases $\frac{1}{h} \preceq \frac{1}{g}$, which implies that $w_n \preceq f_n$, and consequently w_n is the SPE of degree n.

Since $w_n = \alpha X + \frac{1}{h}$, then from (2.6) $|w_n - \alpha X| = |\frac{1}{h}| = |\frac{\alpha}{X^{n-1}}| = e^{-(n-1)}$, consequently $\lim_{n \to +\infty} w_n = \alpha X$. ∎

3 Continued Fraction Expansion of the SPE

Let $\mathcal{J} = \{ f \in \mathbb{F}_q((X^{-1}))/|f| < 1\}$ and let the map $T\colon \mathcal{J} \to \mathcal{J}$ be given by

$$T(\omega) := \frac{1}{\omega} - \left[\frac{1}{\omega}\right], \quad \omega \neq 0, \quad T(0) = 0.$$

Recall that the map T generates the continued fraction expansion of ω of the form

$$\omega = A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \cdots}}}, \tag{3.1}$$

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where $A_n = \left[\frac{1}{T^{n-1}(w)}\right]$. As a shorthand for (3.1) we write

$$\omega = [A_0; A_1, A_2, \dots].$$

Theorem 3.1 *Let ω be the SPE of degree $q^n + 1$, with $n \in \mathbb{N}$. Then*

$$\omega = [A_0, A_1, \dots, A_s, \dots],$$

where

$$A_s = (-1)^{s+1}(-\alpha^2)^{\frac{q^{(s+(-1)^s)n}+(-1)^s}{q^n+1}}(-\alpha X)^{q^{sn}}.$$

Proof Let $P(Y) = Y^{q^n+1} - \alpha XY^{q^n} - \alpha^2$ be the minimal polynomial of the SPE $w = w_{q^n+1}$. Let $z_0 = \omega$, $A_0 = [z_0] = \alpha X$, $U_0 = 1$, $V_0 = -\alpha X$, $R_0 = 0$, $T_0 = -\alpha^2$, and $z_{s+1} = \frac{1}{z_s - [z_s]}$, then from Lemmas 2.3 and 2.4 we know that z_s satisfies the equation $U_s z_s^{q^n+1} + V_s z_s^{q^n} + R_s z_s + T_s = 0$ with $\deg V_s > \max(\deg U_s, \deg R_s, \deg T_s)$, and

$$\begin{aligned}
U_{s+1} &= U_s A_s^{q^n+1} + V_s A_s^{q^n} + R_s A_s + T_s,\\
V_{s+1} &= U_s A_s^{q^n},\\
R_{s+1} &= V_s + A_s U_s,\\
T_{s+1} &= U_s,\\
A_{s+1} &= -\left[\frac{V_{s+1}}{U_{s+1}}\right].
\end{aligned}$$

Now one shows, using a simple induction on s, that

$$U_{2s} = 1, \quad U_{2s+1} = -\alpha^2, \quad V_{2s} = -(-\alpha^2)^{\frac{q^{(2s+1)n}+1}{q^n+1}}(-\alpha X)^{q^{2sn}},$$

$$V_{2s+1} = (-\alpha^2)^{\frac{q^{2sn}+q^n}{q^n+1}}(-\alpha X)^{q^{(2s+1)n}}, \quad R_s = 0, \quad T_{2s} = -\alpha^2,$$

$$T_{2s+1} = 1, \quad \text{and} \quad A_s = (-1)^{s+1}(-\alpha^2)^{\frac{q^{(s+(-1)^s)n}+(-1)^s}{q^n+1}}(-\alpha X)^{q^{sn}}. \qquad \blacksquare$$

Example 3.2 (i) The minimal polynomial of the SPE w of degree 2 over $\mathbb{F}_2((X^{-1}))$ is $P(Y) = Y^2 - XY - 1$, so $w = \sum_{i=-1}^{\infty} w_i X^{-i}$ is defined by $w_{-1} = w_1 = 1$, $w_0 = 0$, and

$$w_{2n} = 0, \quad w_{2n+1} = w_n \quad \text{for all } n \geq 0.$$

Its continued fraction expansion is $w = [\,\overline{X}\,]$.

(ii) Let $w = \sum_{i=-1}^{\infty} w_i X^{-i}$ the SPE of degree 3 over $\mathbb{F}_2((X^{-1}))$ and let $P(Y) = Y^3 - XY^2 - 1 \in \mathbb{F}_2[X][Y]$ be its minimal polynomial. It is easy to see that w is a root of the polynomial $Q(Y) = Y^4 - X^2Y^2 - Y - X$. The sequence (w_i) is then defined by $w_{-1} = w_2 = 1$ and

$$w_{4n} = w_n + w_{2n+1}, \quad w_{4n+1} = w_{4n+3} = 0, \quad w_{4n+2} = w_{2n+2}, \quad \text{for all } n \geq 0.$$

Its continued fraction is $w = [X, X^2, X^4, \dots, X^{2^s}, \dots]$.

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References

[1] P. T. Bateman and A. L. Duquette, *The analogue of Pisot-Vijayaraghavan numbers in fields of power series.* Illinois J. Math. **6**(1962) 594–606.

[2] M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, and J.-P. Schreiber, *Pisot and Salem numbers*. Birkhauser Verlag, Basel, 1992.

[3] A. Chandoul, M. Jellali, and M. Mkaouar, *Irreducible polynomials and number of Pisot elements.* Communications in Algebra, to appear.

[4] J. Dufresnoy and C. Pisot, *Etude de certaines fonctions méromorphes bornées sur le cercle unité. Application à un ensemble fermé d'entiers algébriques*. Ann. Sci. Ecole Norm. Sup. (3) **72**(1955), 69–92.

[5] M. Grandet-Hugot, *Sur une propriété des "nombres de Pisot" dans un corps de série formelles.* C. R. Acad. Sér. A-B **266,**(1967), A39–A41; errata, ibid. **265**(1967), A551.

[6] ———, *Nombres de Pisot dans un corps de séries formelles.* Séminaire Delange-Pisot-Poitou, Théorie des nombres **8**(1966–1967), Exp. No. 4, 1–12.

[7] ———, *Eléments algébriques remarquables dans un corps de série formelles*, Acta Arith. *14*(1967/1968), 177–184.

[8] M. Hbaid and M. Mkaouar, *Sur le bêta développemnt de 1 dans le corps des séries formelles.* Int. J. Number Theory **2**(2006), no. 3, 365–378. http://dx.doi.org/10.1142/S1793042106000619

[9] C. L Siegel, *Algebraic number whose conjugates lie in the unit circle.* Duke Math. J **11**(1944), 597–602. http://dx.doi.org/10.1215/S0012-7094-44-01152-X

[10] K. Scheicher, *β-expansion in algebraic function fields over finite fields.* Finite Fields Appl. **13**(2007), no. 2, 394–410. http://dx.doi.org/10.1016/j.ffa.2005.08.008

[11] V. G Sprindžuk, *Mahler's problem in metric number theory.* Translations of Mathematical Monographs, 25, American Mathematical Society, Providence, RI, 1969.

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