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Eisenstein Cohomology and the Construction of p-Adic Analytic L-Functions

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Abstract. Let π be a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$, unramified at p and of cohomological type at infinity. We construct p-adic L-functions, which interpolate the critical values of $L(\pi, s)$ and which satisfy a logarithmic growth condition. We obtain these functions as p-adic Mellin transforms of certain distributions μ_π on $\mathbb{Z}_p^*$ having values in some fixed number field and which are of moderate growth. In the p-ordinary case we obtain the bound $|\mu_\pi(U)|_p \leqslant |\mu_{\mathrm{Haar}}(U)|_p$ for open subsets $U \leqslant \mathbb{Z}_p^*$, where μ_{Haar} denotes the invariant distribution on $\mathbb{Z}_p^*$.

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Introduction

Let $\mathbb{A}$ denote the ring of adèles of the field of rational numbers and let π be a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A})$. We want to study the p-adic analytic properties of the critical values of the automorphic L-function $L(\pi, s)$ attached to π. This depends on a p-adic variation of $L(\pi, s)$ as follows. Let $l \in \mathbb{N}$ be any critical integer for π. The corresponding critical integer on the left-hand side of the functional equation is $1 - l$ and we shall concentrate on the critical integers on this side. We fix a prime number $p > 2$ as well as a character $\eta : \mathbb{Q}^* \backslash \mathbb{A}^* \to \mathbb{C}^*$ of finite order and we define X_p to be the group of all continuous, $\mathbb{C}_p$-valued characters on $\mathbb{Z}_p^*$. X_p has the structure of a p-adic Lie group and it contains all characters $\chi\colon \mathbb{Q}^* \backslash \mathbb{A}^* \to \mathbb{C}^*$ of conductor $f_\chi = p^e$ a p-power, which are of finite order. We then want to study the relations among the twisted values $L(\pi \otimes \eta\chi, 1 - l)$ as χ varies over all characters with conductor a p-power and infinity component $\chi_\infty = \mathrm{id}$. In particular we may ask whether the function $\chi \mapsto L(\pi \otimes \eta\chi, 1 - l)$, after dividing by some period $\Omega(\pi) \in \mathbb{C}^*$, can be continued to a *p-adic analytic* function $L_p\colon X_p \to \mathbb{C}_p$ and which are the properties of this analytic extension.

The existence of (bounded) p-adic L-functions has been proved for automorphic representations of GL_1 and GL_2 (cf., for example, [M-SwD]). In the higher rank case examples of p-adic L-functions have been obtained only under very special assumptions on the representation π: they are known to exist in the case of the symmetric square L-function on GL_2, which using the Jacquet–Gelbart lift provides examples of p-adic L-functions for GL_3 (cf. [Sch 1]) and for automorphic representations on GL_{2n} having an H-model (cf. [A-G]).

Our aim is to examine the group GL_3, the only assumptions that we will make on the representation π being that π appears in the cohomology of the symmetric space of GL_3 with trivial coefficients and that the p-component of π is unramified. In particular, unlike in the case of the symmetric square L-function on GL_2, it is not possible to apply the q-expansion principle. Under these assumptions we prove the existence of p-adic analytic functions, which interpolate the values $L(\pi \otimes \eta\chi, 1-l)$ and which satisfy a logarithmic growth condition.

The construction of these p-adic L-functions is based on a representation of the twisted values as an integral of χ against a certain distribution. For the moment we shall assume that π has nonvanishing cohomology with coefficients in some finite dimensional representation of $GL_3(\mathbb{C})$ and we let $l \in \mathbb{N}$ run through the critical integers of $\pi \otimes \eta$ (cf. Remark 1.6). Only using the fact that π_p is spherical we construct a family of distributions $\mu_{\pi,l} = \mu^{\eta}_{\pi,l}$ on $\mathbb{Z}_p^*$ such that for all idèle class characters χ with conductor a p-power and infinity component $\chi_\infty = \mathrm{id}$

$$\int_{\mathbb{Z}_p^*} \chi_p \eta_p^2 \, d\mu^{\eta}_{\pi,l} = \text{some explicit factors} \times L(\pi \otimes \eta\chi, 1-l),$$

whereas this integral vanishes if $\chi_\infty = \mathrm{sgn}$ (cf. Corollary 1 in Section 2.2 for a precise statement). We then prove that the occurrence of π in cohomology implies:

- There is an $\Omega(\pi) \in \mathbb{C}^*$ such that $\mu_{\pi,l}(U)/\Omega(\pi)$ has values in a finite extension $E/\mathbb{Q}$ for all critical integers l and all open subsets $U \subset \mathbb{Z}_p^*$.

Let us now assume that π embeds into cohomology with trivial coefficients. In this case the only critical integers are $s = 1, 0$ and we further obtain for $\mu_\pi = \mu_{\pi,1}$ (cf. the Notations for the definition of $|\cdot|_p$):

- There is a number $h \in \mathbb{N}$ such that $|\mu_\pi(U)/\Omega(\pi)|_p \leqslant |\mu_{\mathrm{Haar}}(U)|_p^h$ for all open subsets $U \leqslant \mathbb{Z}_p^*$, where μ_{Haar} denotes the invariant distribution on $\mathbb{Z}_p^*$.
- For all prime numbers $\ell \neq p$ the absolute values $|\mu_\pi(U)/\Omega(\pi)|_\lambda$ are bounded for any λ extending the ℓ-adic valuation on E.

(cf. Theorems 2 and 4 in Sections 3.2 and 5.1). If, in addition, we assume π to be p-ordinary with respect to i_p we obtain the bound

- $|\mu_\pi(U)/\Omega(\pi)|_p \leqslant |\mu_{\mathrm{Haar}}(U)|_p$ for all open subsets $U \leqslant \mathbb{Z}_p^*$.

(cf. Remark 5.4). In particular, we can not deduce boundedness of μ_π.

Using the integration theory developed in [V], we then obtain the p-adic L-functions as the Mellin transform of μ_π (cf. Corollary 3 in Section 5.2).

Our construction is based on a formula which gives us control over the behaviour of the values $L(\pi \otimes \eta\chi, 1-l)$ as χ varies. This formula is proved in Section 1

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and it expresses the twisted values as certain linear combinations of period integrals, which involve a fixed cusp form belonging to π but also an Eisenstein series belonging to an induced representation $\mathrm{Ind}(1,\chi)$. In Section 2 we construct the distribution $\mu_{\pi,l}$ and using the formula from Section 1 calculate its integral against characters. In Section 3 we use the integrality structure on the cohomology to deduce the algebraicity of the values of $\mu_{\pi,l}/\Omega(\pi)$. In Section 5 we finally prove in the case of constant coefficients the bounds on the growth of μ_π and explain how to apply the integration theory of [V] to construct p-adic L-functions. The main difficulty here is to control the behaviour of the Eisenstein cohomology classes constructed from the representations $\mathrm{Ind}(1,\chi)$. In particular Section 4 will be entirely devoted to a calculation of the denominators of these classes in the non-torsion part of the (integral) cohomology using the theory developed in [Ha 2]. The restriction to constant coefficients is mainly made to simplify the calculations in Section 4 (but cf. also Remark 3.5).

We want to remark that the cohomological interpretation of the values of the L-function obtained in this work seems to indicate that the boundedness of the distribution μ_π is equivalent to a certain relation between the restriction of cuspidal cohomology classes from GL_3 to GL_2 and Eisenstein cohomology classes on GL_2 (cf. Remark 5.4).

We finally want to mention that the (purely analytic) construction of the distributions $\mu_{\pi,l}$ described in this work generalizes to the groups GL_n over any number field.

NOTATIONS

We denote by $K_{n,\infty}$ the compact subgroup $\mathrm{SO}_n(\mathbb{R}) < \mathrm{GL}_n(\mathbb{R})$ and by $Z_n^0(\mathbb{R})$ the connected component of 1 of the center $Z_n(\mathbb{R})$ of $\mathrm{GL}_n(\mathbb{R})$. $\mathfrak{gl}_n$ resp. $\mathfrak{so}_n$ is the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ resp. $\mathrm{SO}_n(\mathbb{R})$. We denote by id the trivial character of $\mathbb{R}^*$, sgn is the signum homomorphism $\mathrm{sgn}(x) := x/|x|$, $x \in \mathbb{R}^*$ and we set $\alpha(x) := |x|$ for $x \in \mathbb{A}^*$. We shall use the following level groups: $K(n, p^e) \leqslant \mathrm{GL}_n(\mathbb{Z}_p)$ is the subgroup of matrices which are congruent to 1 mod p^e and $K_0(n, p^e)$ resp. $K_1(n, p^e)$ denotes the subgroup of matrices $(k_{ij}) \in \mathrm{GL}_n(\mathbb{Z}_p)$ satisfying $k_{nj} \equiv 0\,(p^e)$, $j = 1, \dots n-1$ resp. $k_{nj} \equiv 0\,(p^e)$, $j = 1, \dots n-1$ and $k_{nn} \equiv 1\,(p^e)$.

We also fix an additive character $\tau = \otimes_\ell \tau_\ell \colon \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^*$ of conductor $\mathbb{Z}$, i.e.$\mathbb{Z}_\ell$ is the largest ideal contained in the kernel of τ_ℓ for $\ell \neq \infty$.

We denote by $\delta_{n,p}$ resp. $\delta_{n,\mathbb{A}}$ the modulus of $B_n(\mathbb{Q}_p)$ resp. $B_n(\mathbb{A})$, where B_n is the group of upper triangular matrices. Unless stated otherwise, any induction will be unitary.

Finally $\mathbb{C}_\ell$ is the completion of an algebraic closure of $\mathbb{Q}_\ell$ and $|\cdot|_\ell$ is the absolute value on $\mathbb{C}_\ell$ normalized by $|\ell|_\ell = \ell^{-1}$.

We let $i_p \colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ be an embedding and we also denote by $|\cdot|_p$ the absolute value on $\bar{\mathbb{Q}}$ induced by i_p.

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1. A Calculation of the Twisted Values of L-Functions

Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A})$. We fix a rational prime $p > 2$ and assume that the p-component π_p is unramified and that there is an $l_0 \in 2\mathbb{N}$ such that the component at infinity π_∞ is isomorphic to the induced representation $\mathrm{Ind}(D_{l_0}, \mathrm{id})$. Here, by D_{l_0} we understand the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest weight $l_0 + 1$ and the representation is induced from the parabolic subgroup $P \leqslant \mathrm{GL}_3(\mathbb{R})$ of type $(2, 1)$. Because π_p is unramified we have $\pi_p \cong \mathrm{Ind}(\mu_1, \mu_2, \mu_3)$ with unramified characters $\mu_i\colon \mathbb{Q}_p^* \to \mathbb{C}^*$, $i = 1, 2, 3$.

We let $\chi\colon \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ be an idèle class character with conductor $f = p^e$ a power of p and infinity component $\chi_\infty = \mathrm{id}$. We choose a (auxiliary) prime q different from p such that π_q is unramified and we make a choice of a pair of idèle class characters $\eta, \eta'\colon \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$, which are of finite order and satisfy the conditions

- $f_{\eta'} = p,\ f_\eta = pq \quad$ (i.e. $\eta_q|_{\mathbb{Z}_q^*} \neq 1$),
- $\eta_\infty = \eta'_\infty$,
- $\eta_p|_{\mathbb{Z}_p^*} = \eta'_p|_{\mathbb{Z}_p^*}$.

This is equivalent to a choice of primitive Dirichlet characters $\tilde{\eta}'\colon (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{C}^*$ and $\tilde{\eta}\colon (\mathbb{Z}/pq\mathbb{Z})^* \to \mathbb{C}^*$ such that $\tilde{\eta}(-1) = \tilde{\eta}'(-1)$ and $\tilde{\eta}|_{(\mathbb{Z}/p\mathbb{Z})^*} = \tilde{\eta}'|_{(\mathbb{Z}/p\mathbb{Z})^*}$. We set $\flat = 0$ if $\eta_\infty = \mathrm{id}$ and $\flat = 1$ if $\eta_\infty = \mathrm{sgn}$ and we put $\eta_0 := \eta\eta'^{-1}$. We let $l \in \mathbb{N}$ be any integer satisfying $0 < l \leqslant l_0/2$ and $l \equiv \flat$ modulo 2 and for any such l we define the induced representation

$$\Pi(\chi) := \mathrm{Ind}_{B_2(\mathbb{A})\uparrow \mathrm{GL}_2(\mathbb{A})}(\eta'\alpha^{l-1/2}, \eta\chi\alpha^{-(l-1/2)}).$$

For $l > 2$ the intertwining operator

$$\begin{aligned}\mathrm{Eis}\colon \Pi(\chi) &\to \mathcal{A}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))\\ \psi &\mapsto \textstyle\sum_{\gamma\in B_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{Q})} \psi(\gamma g)\end{aligned}$$

is then defined by an absolutely convergent series. In the case $l = 2$ we note that the representation $\Pi(\chi)$ always ramifies at q and Eis is defined using an appropriate analytic continuation. We denote by $\mathcal{A}_\chi$ the image of $\Pi(\chi)$ under Eis. Similarly we define $V(\pi)$ to be the subspace of $L_0^2(\mathrm{GL}_3(\mathbb{Q})\backslash\mathrm{GL}_3(\mathbb{A}))$, on which the representation π can be realized.

We want to choose a pair of automorphic forms (ϕ, E_χ) on $\mathrm{GL}_3 \times \mathrm{GL}_2$ belonging to the representation $\pi \times \Pi(\chi)$ and such that its Mellin transform computes the values $L(\pi \otimes \chi\eta, 1 - l)$. A Mellin transform for forms on $\mathrm{GL}_3 \times \mathrm{GL}_2$ is given by the zeta integral of the *Rankin–Selberg–Convolution*, which for any two forms $\phi \in V(\pi)$, $E \in \mathcal{A}_\chi$ reads

$$I(\phi, E, s) = \int_{\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})} \phi\left[\begin{pmatrix} g & \\ & 1\end{pmatrix}\right] E(g)\,|\det g|^{s-1/2}\, dg$$

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(cf. [J-S 1], ch. 3). The integral converges because ϕ is rapidly decreasing and E is slowly increasing (cf. [Pi-Sh], Theorem 3).

We first deal with E_χ and choose a section

$$\psi_\chi^0 = \otimes_\ell \psi_{\chi,\ell}^0 \in \mathrm{Ind}(\alpha^{l-1/2}, \eta_0\chi\alpha^{-(l-1/2)}) = \eta'^{-1} \otimes \Pi(\chi)$$

as follows.

For any finite place $\ell \neq p, q$ the representation $\mathrm{Ind}(\alpha_\ell^{l-1/2}, \eta_{0,\ell}\chi_\ell\alpha_\ell^{-(l-1/2)})$ is unramified and we let $\psi_{\chi,\ell}^0$ be the spherical function normalized by $\psi_{\chi,\ell}^0(1) = 1$.

At the place q we let $\psi_{\chi,q}^0$ be the essential vector in $\mathrm{Ind}(\alpha_q^{l-1/2}, \eta_{0,q}\chi_q\alpha_q^{-(l-1/2)})$. Since the restrictions of η_q and η'_q to $\mathbb{Z}_q^*$ are different this representation has conductor q and using [Ca], p. 306 we find that $\psi_{\chi,q}^0$ is supported on $B_2(\mathbb{Q}_q)K_0(2, q)$. We therefore may normalize $\psi_{\chi,q}^0$ by setting $\psi_{\chi,q}^0(1) = 1$.

At the place p we distinguish: If $\chi \neq 1$ we let $\psi_{\chi,p}^0$ be the essential vector in $\mathrm{Ind}(\alpha_p^{l-1/2}, \eta_{0,p}\chi_p\alpha_p^{-(l-1/2)})$. This representation has conductor $p^e, e \geqslant 1$ and using [Ca], p. 306 we find that $\psi_{\chi,p}^0$ is supported on $B_2(\mathbb{Q}_p)K_0(2, p^e)$. Hence, we may normalize $\psi_{\chi,p}^0(1) = 1$ and obtain

$$\psi_{\chi,p}^0(bk) = \chi_p\eta_{0,p}(b_2)|b_1/b_2|_p^l\chi_p(d)$$

for

$$b = \begin{pmatrix} b_1 & u \\ & b_2 \end{pmatrix} \in B_2(\mathbb{Q}_p), \qquad k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(2, p^e).$$

If $\chi = 1$ we let $\psi_{\chi,p}^0$ be the unique section in $\mathrm{Ind}(\alpha_p^{l-1/2}, \eta_{0,p}\alpha_p^{-(l-1/2)})$, which is supported on $B_2(\mathbb{Q}_p)K_0(2, p)$ and which is given by $\psi_{\chi,p}^0(g) = \eta_{0,p}(b_2)|b_1/b_2|_p^l$ for elements $g = bk$ in the support.

At infinity we know that there is a proper, irreducible submodule $\Pi_\infty^0 < \mathrm{Ind}(\alpha_\infty^{l-1/2}, \alpha_\infty^{-(l-1/2)})$, which is isomorphic to D_{2l-1}. We let $\psi_{\chi,\infty}^0 = \psi_\infty^0$ be an arbitrary but fixed section in Π_∞^0 i.e. $\psi_{\chi,\infty}^0$ does not depend on χ.

We set $\psi_\chi := \eta' \otimes \psi_\chi^0$ and $E_\chi := \mathrm{Eis}(\psi_\chi)$.

Next, to define ϕ, we choose a Whittaker function $w = \otimes_\ell w_\ell \in W(\pi, \tau)$ as follows.

For any finite place $\ell \neq p, q$ we choose w_ℓ to be the essential vector in $W(\pi_\ell, \tau_\ell)$.

For $\ell = q$ we let $w_q \in W(\pi_q, \tau_q)$ be a Whittaker function such that the restriction of w_q to $\mathrm{GL}_2(\mathbb{Q}_q)$ via the embedding $g \mapsto \left(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\right)$ is supported on $N_2(\mathbb{Q}_q)\left(\begin{smallmatrix} 1 & \\ 1 & 1 \end{smallmatrix}\right)K_0(2, q)$ and for $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(2, q)$ we have

$$w_q\left[\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}k\right] = \eta_q(d)^{-1}.$$

It is an immediate consequence of Theorem F in [Gel-Kaj] that such a Whittaker function exists.

For $\ell = p$ we proceed as follows. We denote by $\mathcal{I} \leqslant \mathrm{GL}_3(\mathbb{Z}_p)$ the Iwahori subgroup consisting of elements $k \in \mathrm{GL}_3(\mathbb{Z}_p)$, which modulo p are congruent to upper triangular matrices. We let $\psi_p^1 \in \mathrm{Ind}(\mu_1, \mu_2, \mu_3)$ be the $\mathcal{I}$-invariant vector defined by

$$\psi_p^1(g) := \begin{cases} \mu(b)\delta_{3,p}^{1/2}(b), & \text{for} \quad g = bw_0\beta,\ b \in B_3(\mathbb{Q}_p),\ \beta \in \mathcal{I} \\ 0, & \text{else.} \end{cases}$$

Here, by w_0 we understand the Weyl group element

$$w_0 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

and μ is given by

$$\mu(b) := \prod_i \mu_i(b_i) \quad \text{for } b = \begin{pmatrix} b_1 & & * \\ & b_2 & \\ & & b_3 \end{pmatrix} \in B_3(\mathbb{Q}_p).$$

We then define $w_p \in W(\pi_p, \tau_p)$ as the image of ψ_p^1 under the isomorphism

$$\begin{array}{ccc} \mathrm{Ind}(\mu_1, \mu_2, \mu_3) & \to & W(\pi_p, \tau_p) \\ \psi & \mapsto & w(g) := \int_{N_3(\mathbb{Q}_p)} \psi(w_0 n g)\bar{\tau}_p(n)\, dn \end{array}$$

(cf. [J-S 2], (3.2) Proposition).

At infinity, again, let w_∞ be an arbitrary but fixed Whittaker function in $W(\pi_\infty, \tau_\infty)$.

We set $w = \otimes_\ell w_\ell$ and let $\phi \in V(\pi)$ be the cusp form belonging to the global Whittaker function w.

We will use the following convention: we identify $\mathrm{GL}_3(\mathbb{Q}_p)$ with its image in $\mathrm{GL}_3(\mathbb{A})$ under the map given by embedding into the p-component. Thus by $u \in \mathrm{GL}_3(\mathbb{Q}_p)$ we also understand the adelic matrix $(1, \ldots, 1, u, 1, \ldots, 1)$, where all entries away from p are equal to 1. Also, we write for short $X_{\mathbb{A}} := \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$, $G(\chi_p) := \sum_{i \in (\mathbb{Z}/f\mathbb{Z})^*} \chi_p(i)\tau_p(i/f)$ denotes the local Gauss sum, $\tilde{\chi}$ the primitive Dirichlet character attached to the idèle class character χ and we write $L_S(\pi, s)$ to indicate that we omit the local factors at all places $\ell \in S$ from the Euler product.

THEOREM 1. *Let* $\chi\colon \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ *be a character with conductor* $f = p^e$ *where* $e \geqslant 2$ *and infinity component* $\chi_\infty = \mathrm{id}$. *Then, for all* $l \in \mathbb{N}$ *such that* $0 < l \leqslant l_0/2$

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and $l \equiv \flat \pmod 2$ *we have*

$$P(1/2) \cdot L(\pi \otimes \chi\eta, 1 - l)$$
$$= \mathbf{A} \cdot \sum_{\substack{i \in (\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p)^*,\, j \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^* \\ y \in \mathbb{Z}_p/p^e\mathbb{Z}_p}} \chi_p\eta_p^2(j)\eta_p(i) \int_{X_{\mathbb{A}}} \phi\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & ip/f & y/f \\ & 1 & j/f \\ & & 1 \end{pmatrix}\right] E_\chi(g)\, dg.$$

In the above formula $P(T) = P_{w_\infty, \psi_\infty}(T) \in \mathbb{C}[T]$ *is a polynomial, which only depends on the choice of the infinity components* ψ_∞ *and* w_∞ *and*

$$\mathbf{A} = \mathbf{A}'_l \boldsymbol{\pi}^{3l-\flat}\zeta^{-e} f^{l+1}\, G(\chi_p)^{-1} G(\chi_p\eta_p^2)^{-1} G(\tilde{\chi}^{-1}\tilde{\eta_0}^{-1}) L(\tilde{\chi}\tilde{\eta_0}, 1 - 2l) L_{\{q\}}(\pi \otimes \eta', l)^{-1},$$

where $\zeta = \mu_3\eta'_p\eta_{0,p}^{-1}(p)$ *and the algebraic number* $\mathbf{A}'_l \in \bar{\mathbb{Q}}^*$ *is independent of* χ. *Moreover, the coset* $\mathbf{A}'_l \cdot \mathbb{Q}^*$ *is even independent of* l.

The *Proof* of Theorem 1 will occupy the rest of this section. First we define the Whittaker function

$$v_\chi(g) := L(\tilde{\chi}^{-1}\tilde{\eta_0}^{-1}, 2l) \int_{N_2(\mathbb{A})} E_\chi(ng)\tau(n)\, dn.$$

Here, $L(\tilde{\chi}\tilde{\eta_0}, s)$ denotes the Dirichlet L-Function. Of course, $L(\tilde{\chi}^{-1}\tilde{\eta_0}^{-1}, 2l)^{-1}\, v_\chi$ is the Whittaker function of E_χ. v_χ decomposes into an infinite product

$$v_\chi = \otimes_{\ell \neq \infty} v_{\chi,\ell} \otimes v_\infty,$$

and using [Ge-Sha], p. 80/81 we know that $v_{\chi,\ell}(1) = 1$ for $\ell \neq p, q$. For any two Whittaker functions $w_\ell \in W(\pi_\ell, \tau_\ell)$ and $v_\ell \in W(\Pi_\ell(\chi), \bar{\tau}_\ell)$ we denote by

$$I(w_\ell, v_\ell, s) := \int_{N_2(\mathbb{Q}_\ell)\backslash \mathrm{GL}_2(\mathbb{Q}_\ell)} w_\ell\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right] v_\ell(g) |\det g|_\ell^{s-1/2}\, dg$$

the local zeta integral of the Rankin–Selberg convolution. The starting point of the proof of Theorem 1 is the decomposition of the global zeta integral into a product of local integrals

$$I(\phi, E, s) = \prod_\ell I(w_\ell, v_\ell, s),$$

where $w = \otimes_\ell w_\ell$ resp. $v = \otimes_\ell v_\ell$ is the Whittaker function of ϕ resp. E. The proof of this equality is the same as the one of (3.3) Proposition in [J-S 1], even if E is not cuspidal. We write for short

$$u = u(i, j, y; f) := \begin{pmatrix} 1 & ip/f & y/f \\ & 1 & j/f \\ & & 1 \end{pmatrix} \in N_3(\mathbb{Q}_p)$$

and denote by $h \cdot \phi(g) := \phi(gh)$ the right translate of ϕ by $h \in \mathrm{GL}_3(\mathbb{A}_f)$. Then in our case this decomposition takes the form

$$\sum_{i,j,y} \eta_p(i)\chi_p\eta_p^2(j)\, I(u\cdot\phi, E_\chi, s)$$
$$= L(\tilde{\chi}^{-1}\tilde{\eta}_0^{-1}, 2l)^{-1} \prod_{\ell\neq p} I(w_\ell, v_{\chi,\ell}, s)\cdot \sum_{i,j,y}\eta_p(i)\chi_p\eta_p^2(j)\, I(u\cdot w_p, v_{\chi,p}, s).$$

To prove the Theorem we therefore have to examine the local integrals appearing on the right-hand side.

At all finite places $\ell \neq p, q$ the Whittaker function w_ℓ is the essential vector und $v_{\chi,\ell}$ is the unique $\mathrm{GL}_2(\mathbb{Z}_\ell)$-invariant vector satisfying $v_{\chi,\ell}(1) = 1$. As an immediate consequence of (4.1) Théorème in [J-P-S 2] we get

$$I(w_\ell, v_{\chi,\ell}, s) = L(\pi_\ell \otimes \eta'_\ell, s + l - 1/2)L(\pi_\ell \otimes \chi_\ell\eta_\ell, s - (l - 1/2)).$$

Next we look to infinity. Using the notation of [Kn], ch. 3, the representations of the Weil group $W_\mathbb{R}$ of $\mathbb{R}$ attached to π_∞ and $\pi_\infty \otimes \mathrm{sgn}$ read

$$\pi_\infty^w = (l_0, 0) \oplus (+, 0), \qquad (\mathrm{sgn}\otimes\pi_\infty)^w = (l_0, 0)\oplus(-, 0).$$

In particular, we deduce from [Kn], (3.6)

$$L(\pi_\infty, s) = 2(2\boldsymbol{\pi})^{-(s+l_0/2)}\boldsymbol{\pi}^{-s/2}\Gamma(s + l_0/2)\Gamma(s/2)$$

and

$$L(\pi_\infty\otimes\mathrm{sgn}, s) = 2(2\boldsymbol{\pi})^{-(s+l_0/2)}\boldsymbol{\pi}^{-(s+1)/2}\Gamma(s + l_0/2)\Gamma((s+1)/2).$$

This implies that $L(\pi_\infty \otimes \eta_\infty, s)$ does not have a pole at $s = l$ and at $s = 1 - l$ for all l satisfying $0 < l \leqslant l_0/2$ and which are congruent to $\flat$ modulo 2. Using [J-P-S 3] we deduce that there is a polynomial $P(T) = P_{w_\infty,\psi_\infty}(T) \in \mathbb{C}[T]$ such that

$$I(w_\infty, v_\infty, s) = P(s)\, L(\pi_\infty \times \eta_\infty \otimes \Pi_\infty^0, s).$$

We want to determine the central value of the quotient of local factors

$$L_\infty(s) = \frac{L(\pi_\infty\times\eta_\infty\otimes\Pi^0_\infty, s)}{L(\pi_\infty\otimes\eta_\infty, s + l - 1/2)L(\pi_\infty\otimes\eta_\infty, s-(l-1/2))}.$$

LEMMA 1.1. *We have*

$$L_\infty(1/2) = \begin{cases} 2^{1-2l}\boldsymbol{\pi}^{-l}\,\Gamma(l)^2\,\Gamma(l/2)^{-2}, & \text{if } \eta_\infty = \mathrm{id},\\ (-1)^{(l-1)/2}2^{2-2l}\boldsymbol{\pi}^{1-l}\,\Gamma(l)^2\,\Gamma((l+1)/2)^{-2}, & \text{if } \eta_\infty = \mathrm{sgn}.\end{cases}$$

Proof. Using the formula for the decomposition of the tensor product of two representations of the Weil group

$$(l, t)\otimes(m, r) = (l + m, t + r)\oplus(l - m, t + r) \qquad (l > m)$$

and [Kn], ch. 3.6, we see that

$$L(D_{l_0} \otimes D_{2l-1}, s) = L(D_{l_0}, s + l - 1/2)L(D_{l_0}, s - (l - 1/2)).$$

Since the local factors are muliplicative in direct sums and $\pi_\infty \otimes \eta_\infty \cong \mathrm{Ind}(D_{l_0}, \eta_\infty)$ as well as $\Pi^0_\infty \otimes \eta_\infty \cong D_{2l-1}$ we deduce that

$$L_\infty(s) = \frac{L(D_{2l-1}, s)}{L(\eta_\infty, s + l - 1/2)L(\eta_\infty, s - (l - 1/2))}.$$

We first assume $\eta_\infty = \mathrm{id}$, i.e. $l \equiv 0 \pmod 2$. Specializing $s \mapsto 1/2$ and using the formulas in [Kn], ch. 3.6, we obtain

$$L_\infty(1/2) = 2^{1-l}\boldsymbol{\pi}^{1/2-l}\Gamma(l)\Gamma(l/2)^{-1}\Gamma(1/2 - l/2)^{-1}.$$

Applying the duplication formula of the Γ-function to $\Gamma(-l/2 + 1/2)$ this becomes

$$L_\infty(1/2) = 2^{-2l}\boldsymbol{\pi}^{-l}\frac{\Gamma(l)\Gamma(-l/2)}{\Gamma(l/2)\Gamma(-l)}.$$

Using the rule

$$\Gamma(-z) = -\Gamma(z)^{-1}\frac{\boldsymbol{\pi}}{z\sin(\boldsymbol{\pi}z)}$$

as well as $\sin(\boldsymbol{\pi}l)/\sin(\boldsymbol{\pi}l/2) = 1$, we see that $L_\infty(1/2)$ equals the expression in the Lemma. The case $\eta_\infty = \mathrm{sgn}$ being analogous this proves the Lemma.

Using the lemma we finally obtain at the Archimedean place

$$I(w_\infty, v_\infty, s) = AP(s)\boldsymbol{\pi}^{\flat-l}\,L(\pi_\infty \otimes \eta_\infty, s + l - 1/2)L(\pi_\infty \otimes \eta_\infty, s - (l - 1/2)),$$

where $A \in \mathbb{Q}^*$ does not depend on χ.

At the place q we know since π_q is unramified and $\eta_q\chi_q$ is ramified that the local factor does not depend on χ

$$L(\pi_q \otimes \eta_q\chi_q, s) = 1.$$

On the other hand the behaviour of the restriction of w_q to $\mathrm{GL}_2(\mathbb{Q}_q)$ implies that $\mathrm{res}_{\mathrm{GL}_2(\mathbb{Q}_q)}w_q(g)v_{\chi,q}(g)$ does not depend on g and Lemma 1.4 b.) (see below) then proves

$$I(w_q, v_{\chi,q}, s) = \mathrm{vol}(K_0(2, q))q^{-1}.$$

At the place $\ell = p$ again we know since η'_p and $\chi_p\eta_p$ are ramified that

$$L(\pi_p \otimes \eta'_p, s) = L(\pi_p \otimes \eta_p\chi_p, s) = 1.$$

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Collecting our results so far we arrive at the equation

$$\begin{aligned}&\sum_{i,j,y} \eta_p(i)\chi_p\eta_p^2(j)\, I(u\cdot\phi, E_\chi, s)\\ &\quad = L_{\{q\}}(\pi\otimes\eta', s+l-1/2)L(\pi\otimes\chi\eta, s-(l-1/2))\times\\ &\qquad \times L(\tilde{\chi}^{-1}\tilde{\eta_0}^{-1}, 2l)^{-1}q^{-1}\mathrm{vol}(K_0(2,q))\, A\boldsymbol{\pi}^{\flat-l}P(s)\times\\ &\qquad \times \sum_{i,j,y} \eta_p(i)\chi_p\eta_p^2(j)\, I(u\cdot w_p, v_{\chi,p}, s)\end{aligned}$$

and it remains to calculate the local integral at p. We denote by $w_{\chi_p}|_{\mathrm{GL}_2}$ the restriction of w_{χ_p} to $\mathrm{GL}_2(\mathbb{Q}_p)$ via the embedding $g\mapsto \left(\begin{smallmatrix} g & \\ & 1\end{smallmatrix}\right)$.

PROPOSITION 1.2. *Let $w\in W(\pi_p,\tau_p)$ be invariant on the right under the Iwahori subgroup $\mathcal{I}\leqslant \mathrm{GL}_3(\mathbb{Z}_p)$. For every character χ_p of $\mathbb{Q}_p^*$ of conductor $f=p^e$ we define the Whittaker function*

$$w_{\chi_p}(g) := \sum_{\substack{i\in(\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p)^*,\ j\in(\mathbb{Z}_p/p^e\mathbb{Z}_p)^*\\ y\in\mathbb{Z}_p/p^e\mathbb{Z}_p}} \eta_p(i)\chi_p\eta_p^2(j)\, w\left[g\begin{pmatrix}1 & ip/f & y/f\\ & 1 & j/f\\ & & 1\end{pmatrix}\right].$$

For $e\geqslant 2$ the restricted function $w_{\chi_p}|_{\mathrm{GL}_2}$ satisfies the properties:

- *The support of $w_{\chi_p}|_{\mathrm{GL}_2}$ is contained in $N_2(\mathbb{Q}_p)\begin{pmatrix}p^{e-2} & \\ & 1\end{pmatrix}K_0(2,f)$.*
- *For $k=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in K_0(2,f)$ we have*

$$\begin{aligned}&w_{\chi_p}|_{\mathrm{GL}_2}\left[\begin{pmatrix}p^{e-2} & \\ & 1\end{pmatrix}k\right]\\ &\quad = fp^{e-2}G(\chi_p\eta_p^2)G(\eta_p)\, w\left[\begin{pmatrix}p^{e-2} & & \\ & 1 & \\ & & 1\end{pmatrix}\right]\eta_p^{-1}(ad)\chi_p^{-1}(d).\end{aligned}$$

In particular $w_{\chi_p}|_{\mathrm{GL}_2}$ is completely determined by these properties.

Proof. For any $k=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in K_0(2,f)$ we have the Iwasawa decomposition

$$\begin{pmatrix}a & b & \\ c & d & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & ip/f & y/f\\ & 1 & j/f\\ & & 1\end{pmatrix} = \begin{pmatrix}1 & \dfrac{api}{cpi+fd} & \dfrac{ay}{f}\Big(1-\dfrac{cpi}{cpi+fd}\Big)+\dfrac{bj}{f}\\ & 1 & dj/f\\ & & 1\end{pmatrix}\cdot\tilde{k},$$

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where

$$\tilde{k} = \begin{pmatrix} \dfrac{ad}{cpi/f + d} & b & 0 \\ c & cip/f + d & cy/f \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{I}.$$

Together with the right $\mathcal{I}$-invariance of w_{χ_p} this yields

$$\begin{aligned} & w_{\chi_p}\left[g \begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} \right] \\ & = \sum_{i,j,y} \eta_p(i)\chi_p\eta_p^2(j)\, w\left[g \begin{pmatrix} 1 & \dfrac{pi}{f}\dfrac{ad^{-1}}{1+cpid^{-1}/f} & \dfrac{ay}{f}\left(1 - \dfrac{cpi}{fd+cpi}\right) + \dfrac{bj}{f} \\ & 1 & dj/f \\ & & 1 \end{pmatrix} \right]. \end{aligned}$$

Changing the summation variables according to the bijections

$$i \mapsto i\big(ad^{-1} - i(cp/fd)\big)^{-1}, \qquad y \mapsto y(a(1-(cpi/fd+cpi)))^{-1} \quad \text{and} \quad j \mapsto d^{-1}j$$

we obtain

$$\begin{aligned} & w_{\chi_p}\left[g \begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} \right] \\ & = \sum_{i,j,y} \eta_p\left(i\left(ad^{-1} - i\frac{cp}{fd}\right)^{-1}\right)\chi_p\eta_p^2(d^{-1}j)\, w\left[g \begin{pmatrix} 1 & ip/f & y/f + bj/f \\ & 1 & j/f \\ & & 1 \end{pmatrix} \right] \\ & = \eta_p(a^{-1}d)\chi_p\eta_p^2(d^{-1}) \sum_{i,j,y} \eta_p(i)\chi_p(j)\, w\left[g \begin{pmatrix} 1 & ip/f & y/f \\ & 1 & j/f \\ & & 1 \end{pmatrix} \right]. \end{aligned}$$

The last equality holds because $ad^{-1} - i(cp/fd) \equiv ad^{-1}\ (p)$ and $f_{\eta_p} = p$. Thus, comparing with the definition of w_{χ_p} we obtain

$$w_{\chi_p}\left[g \begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} \right] = \eta_p(a^{-1}d^{-1})\chi_p(d^{-1})\, w_{\chi_p}(g)$$

for any $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(2, f)$, which is already a partial prove of the second claim.

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To prove the Proposition we now start to calculate the values $w_{\chi_p}|_{\mathrm{GL}_2}(g)$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$ as follows. A simple matrix calculation yields

$$w_{\chi_p}|_{\mathrm{GL}_2}(g) = \sum_{i,j,y} \eta_p(i)\chi_p\eta_p^2(j)\, w\left[\begin{pmatrix} 1 & & ay/f+bj/f \\ & 1 & cy/f+dj/f \\ & & 1 \end{pmatrix}\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & ip/f & \\ & 1 & \\ & & 1 \end{pmatrix}\right].$$

Using the behaviour of the Whittaker functions on the left under $N_3(\mathbb{Q}_p)$ we obtain

$$w_{\chi_p}|_{\mathrm{GL}_2}(g) = \sum_{j\in(\mathbb{Z}_p/p^e\mathbb{Z}_p)^*} \chi_p\eta_p^2(j)\tau_p(dj/f) \sum_{y\in\mathbb{Z}_p/p^e\mathbb{Z}_p} \tau_p(cy/f)\times$$
$$\times \sum_{i\in(\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p)^*} \eta_p(i)\, w\left[\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & ip/f & \\ & 1 & \\ & & 1 \end{pmatrix}\right].$$

The first sum evaluates to

$$\sum_{j\in(\mathbb{Z}_p/p^e\mathbb{Z}_p)^*} \chi_p\eta_p^2(j)\tau_p(dj/f) = \begin{cases} \chi_p^{-1}\eta_p^{-2}(d)\, G(\chi_p\eta_p^2), & \text{for} \quad d \in \mathbb{Z}_p^* \\ 0, & \text{else.} \end{cases}$$

Together with the character relations

$$\sum_{y\in\mathbb{Z}_p/p^e\mathbb{Z}_p} \tau_p(cy/f) = \begin{cases} f & \text{for} \quad c \in f\mathbb{Z}_p \\ 0 & \text{else} \end{cases}$$

we see that $w_{\chi_p}|_{\mathrm{GL}_2}(g) \neq 0$ implies $d \in \mathbb{Z}_p^*$ and $c \in f\mathbb{Z}_p$. The decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b/d \\ & 1 \end{pmatrix}\begin{pmatrix} \det g & \\ & 1 \end{pmatrix}\begin{pmatrix} d^{-1} & \\ c & d \end{pmatrix}$$

then proves that the non-vanishing of $w_{\chi_p}|_{\mathrm{GL}_2}(g)$ implies

$$g \in N_2(\mathbb{Q}_p)\begin{pmatrix} \mathbb{Q}_p^* & \\ & 1 \end{pmatrix} K_0(2, f).$$

Taking into account the behaviour of w_{χ_p} on the right under $K_0(2, f)$ we only have to calculate the values of $w_{\chi_p}|_{\mathrm{GL}_2}$ at $\mathrm{diag}(a, 1)$, $a \in \mathbb{Q}_p^*$ to completely determine $w_{\chi_p}|_{\mathrm{GL}_2}$. From the above expression for $w_{\chi_p}|_{\mathrm{GL}_2}(g)$ we deduce

$$w_{\chi_p}|_{\mathrm{GL}_2}\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right] = fG(\chi_p\eta_p^2) \sum_{i\in(\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p)^*} \eta_p(i)\, w\left[\begin{pmatrix} a & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & ip/f & \\ & 1 & \\ & & 1 \end{pmatrix}\right]$$
$$= fG(\chi_p\eta_p^2) \sum_{i\in(\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p)^*} \eta_p(i)\tau_p(ai/p^{e-1})\, w\left[\begin{pmatrix} a & & \\ & 1 & \\ & & 1 \end{pmatrix}\right].$$

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Again, using that

$$\sum_{i\in(\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p)^*} \eta_p(i)\tau_p(ai/p^{e-1}) = 0$$

for $\mathrm{v}_p(a) \neq e-2$ we find

$$\mathrm{supp}\,(w_{\chi_p}|_{\mathrm{GL}_2}) \subset N_2(\mathbb{Q}_p)\begin{pmatrix} p^{e-2} & \\ & 1\end{pmatrix} K_0(2,f),$$

which proves the first claim of the proposition. To prove the second claim only the value of $w_{\chi_p}|_{\mathrm{GL}_2}$ at $\mathrm{diag}(p^{e-2},1)$ has to be determined, but this is easily done: since $\sum_{i\in(\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p)^*} \eta_p(i)\tau_p(p^{e-2}i/p^{e-1}) = p^{e-2}G(\eta_p)$ the above expression immediately yields

$$w_{\chi_p}|_{\mathrm{GL}_2}\left[\begin{pmatrix} p^{e-2} & \\ & 1\end{pmatrix}\right] = fG(\chi_p\eta_p^2)G(\eta_p)p^{e-2}\, w\left[\begin{pmatrix} p^{e-2} & & \\ & 1 & \\ & & 1\end{pmatrix}\right].$$

Thus the Proposition is proven.

Applying Proposition 1.2 and taking into account the behaviour of $v_{\chi,p}$ under elements of $K_0(2,p^e)$ we find for our local integral at the place p

$$\begin{aligned}&\sum_{i,j,y}\eta_p(i)\chi_p\eta_p^2(j)\, I(u\cdot w_p, v_{\chi,p}, s)\\ &\quad = fp^{e-2}G(\chi_p\eta_p^2)G(\eta_p)w_p\left[\begin{pmatrix} p^{e-2} & & \\ & 1 & \\ & & 1\end{pmatrix}\right]v_{\chi,p}\left[\begin{pmatrix} p^{e-2} & \\ & 1\end{pmatrix}\right]|p^{e-2}|_p^{s-1/2}\int_{K_0(2,f)} dk.\end{aligned}$$

To prove Theorem 1 we therefore have to determine the values of the Whittaker functions.

LEMMA 1.3. *For $\alpha \in \mathbb{Z}_p$ we have*

$$w_p\left[\begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1\end{pmatrix}\right] = |\alpha|_p\mu_3(\alpha).$$

Proof. The support of ψ_p^1 is contained in $B_3(\mathbb{Q}_p)w_0\mathcal{I}$ and in view of the definition of w_p we have to determine the set of elements $n\in N_3(\mathbb{Q}_p)$, which satisfy

$$w_0 n\begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1\end{pmatrix} \in B_3(\mathbb{Q}_p)w_0\mathcal{I}.$$

By a quite elementary calculation this set is found to consist of all elements

$$n = \begin{pmatrix} 1 & u & v \\ & 1 & w \\ & & 1 \end{pmatrix} \in N_3(\mathbb{Q}_p)$$

satisfying the conditions $u \in p^k\mathbb{Z}_p$, $v \in p^k\mathbb{Z}_p$, $c \in \mathbb{Z}_p$, where $k := \mathrm{v}_p(\alpha)$ and in this case we have

$$w_0 n \begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \alpha \end{pmatrix} w_0 \begin{pmatrix} 1 & u/\alpha & v/\alpha \\ & 1 & w \\ & & 1 \end{pmatrix}.$$

Thus, using the definition of ψ_p^1, we get

$$w_p\left[\begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix}\right] = \int_{\substack{u\in p^k\mathbb{Z}_p,\, v\in p^k\mathbb{Z}_p \\ c\in\mathbb{Z}_p}} |\alpha^{-1}|_p \mu_3(\alpha)\, \bar{\tau}_p(u+w)\, du\, dv\, dw.$$

Since τ_p is trivial on $\mathbb{Z}_p$ the last expression is equal to $p^{-k}\mu_3(\alpha)$ and the Lemma is proven.

LEMMA 1.4. (*a*) *The restriction of $v_{\chi,p}$ to $\mathbb{Q}_p^*$ is given by*

$$v_{\chi,p}\left[\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\right] = \begin{cases} \eta_p'(\alpha)|\alpha|_p^l f^{-2l}\eta_{0,p}^{-1}(f)G(\chi_p) & \text{for} \quad \mathrm{v}_p(\alpha) \geqslant 0 \\ 0 & \text{else.} \end{cases}$$

(*b*) *For $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(2, q)$ we have*

$$v_{\chi,q}\left[\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} k\right] = q^{-1}\eta_q(d).$$

Proof. (a) Using [Ge-Sha], p. 80/81 and taking into account that $\psi_{\chi,p} = \eta_p' \otimes \psi_{\chi,p}^0$ we obtain from the definition of $v_{\chi,p}$

$$v_{\chi,p}\left[\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\right] = \eta_{0,p}(-1)\eta_p(\alpha)\chi_p(-\alpha)|\alpha|_p^{-l} \sum_{n\in\mathbb{Z}} \int_{\mathrm{v}_p(u)=n} \psi_{\chi,p}^0\left[\begin{pmatrix} & 1 \\ 1 & u/\alpha \end{pmatrix}\right]\tau_p(u)\, du.$$

To evaluate the integral we have to determine the values $\psi_{\chi,p}^0\left[\begin{pmatrix} & 1 \\ 1 & u/\alpha \end{pmatrix}\right]$. We know that the essential vector $\psi_{\chi,p}^0$ is supported on the subset $B_2(\mathbb{Q}_p)K_0(2,f) \subset \mathrm{GL}_2(\mathbb{Q}_p)$. On the other hand $\mathrm{GL}_2(\mathbb{Q}_p)$ is the disjoint union

$$\mathrm{GL}_2(\mathbb{Q}_p) = \dot{\bigcup}_{i=0,\ldots,e} B_2(\mathbb{Q}_p)\begin{pmatrix} 1 & \\ p^i & 1 \end{pmatrix} K_0(2,f).$$

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and we have

$$\begin{pmatrix} & 1 \\ 1 & u/\alpha \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & u/\alpha - 1 \\ & 1 \end{pmatrix} \in B_2(\mathbb{Q}_p)\begin{pmatrix} 1 & \\ p^0 & 1 \end{pmatrix}K_0(2, f),$$

if $\mathrm{v}_p(u/\alpha) \geqslant 0$ and

$$\begin{aligned}\begin{pmatrix} & 1 \\ 1 & u/\alpha \end{pmatrix} &= \begin{pmatrix} -\alpha/u & \frac{\alpha}{u}p^{\mathrm{v}_p(u/\alpha)} \\ & p^{\mathrm{v}_p(u/\alpha)} \end{pmatrix}\begin{pmatrix} 1 & \\ p^{-\mathrm{v}_p(u/\alpha)} & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & \frac{u}{\alpha}p^{-\mathrm{v}_p(u/\alpha)} \end{pmatrix} \\ &\in B_2(\mathbb{Q}_p)\begin{pmatrix} 1 & \\ p^{-\mathrm{v}_p(u/\alpha)} & 1 \end{pmatrix}K_0(2, f).\end{aligned}$$

if $\mathrm{v}_p(u/\alpha) < 0$. This implies that $\psi^0_{\chi,p}\left[\begin{pmatrix} & 1 \\ 1 & u/\alpha \end{pmatrix}\right] \neq 0$ only if $\mathrm{v}_p(u) \leqslant -e + \mathrm{v}_p(\alpha)$ and using the above decomposition together with the definition of $\psi^0_{\chi,p}$ we obtain

$$v_{\chi,p}\left[\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\right] = \eta'_p(\alpha)|\alpha|^l_p \sum_{n \leqslant -e+\mathrm{v}_p(\alpha)} \eta_{0,p}(p^n)p^{2ln}\int_{\mathrm{v}_p(u)=n} \chi_p(u)\tau_p(u)\,du.$$

The integral occurring in the above line is given by

$$\int_{\mathrm{v}_p(u)=n} \chi_p(u)\tau_p(u)\,du = \begin{cases} G(\chi_p) & \text{if} \quad n = -e, \\ 0 & \text{else,} \end{cases}$$

which immediately implies $v_{\chi,p} = 0$ for $\mathrm{v}_p(\alpha) < 0$. On the other hand, for $\mathrm{v}_p(\alpha) \geqslant 0$ we obtain

$$v_{\chi,p}\left[\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\right] = \eta'_p(\alpha)|\alpha|^l_p\eta_{0,p}(p^{-e})f^{-2l}G(\chi_p)$$

and part (a) of the Lemma is proven.

(b) Taking into account the behaviour of $v_{\chi,q}$ under $K_0(2, q)$ we see that it is enough to calculate the value

$$v_{\chi,q}\left[\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\right] = \eta'_q\left[\det\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\right]\int_{\mathbb{Q}_q} \psi^0_{\chi,q}\left[\begin{pmatrix} 1 & 1 \\ 1+u & u \end{pmatrix}\right]\tau_q(u)\,du.$$

We know that $\psi^0_{\chi,q}$ is supported on $B_2(\mathbb{Q}_q)K_0(2, q)$ and comparing entries we see that

$$\begin{pmatrix} 1 & 1 \\ 1+u & u \end{pmatrix} \in \operatorname{supp}\psi^0_{\chi,q}$$

is equivalent to $u \in -1 + q\mathbb{Z}_q$. Moreover, for $u \in -1 + q\mathbb{Z}_q$ we have

$$\begin{pmatrix} 1 & 1 \\ 1+u & u \end{pmatrix} = \begin{pmatrix} -u^{-1} & 1 \\ & u \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1+u^{-1} & 1 \end{pmatrix}$$

and since $\psi^0_{\chi,q}$ is the essential vector we deduce that

$$\psi_{\chi,q}\left[\begin{pmatrix} 1 & 1 \\ 1+u & u \end{pmatrix}\right] = |u|^{-2l}\chi_q\eta_{0,q}(u) = 1$$

for $u \in -1 + q\mathbb{Z}_q$. We thus obtain

$$v_{\chi,q}\left[\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\right] = \int_{-1+q\mathbb{Z}_q} du,$$

which proves the Lemma.

Using Lemma 1.3 and 1.4 and $\mathrm{vol}(K_0(2, p^e)) = (1+p^{-1})^{-1}p^{-2e}$, the local integral $\sum_{i,j,y}\eta_p(i)\chi_p\eta_p^2(j)\,I(u\cdot w_p, v_{\chi,p}, s)$ is now easily calculated, which completes the computation of the global integral. Together with the functional equation

$$L(\tilde{\chi}^{-1}\tilde{\eta}_0^{-1}, 2l) = \frac{(-1)^l}{2\Gamma(2l)}\left(\frac{2\pi}{fq}\right)^{2l} G(\tilde{\chi}^{-1}\tilde{\eta}_0^{-1})\,L(\tilde{\chi}\tilde{\eta}_0, 1-2l)$$

we obtain the claim of Theorem 1. This finishes the proof.

Remark 1.5. The special values $L_{\{q\}}(\pi\otimes\eta', l)$, $l \geqslant 1$ appearing in the definition of $\mathbf{A}$ do not vanish. This follows from the non-vanishing of $L(\pi\otimes\eta', l)$, which is due to the convergence of the Euler product in the case $l > 1$ and due to [J-S 3] in the case $l = 1$ and the regularity of the local factor $L(\pi_q\otimes\eta'_q, s)$ for $\mathrm{Re}(s) \geqslant 1$ (cf. [J-S 1], Proposition 1.5. (iii)).

Remark 1.6. In the proof of Theorem 1 we have seen, that the critical integers for $\eta\otimes\pi$ are all integers

$$l \in \{-l_0/2 + 1, -l_0 + 2, \ldots, l_0/2\}$$

such that

$$l \equiv \begin{cases} \flat & (\mathrm{mod}\ 2) \quad \text{if} \quad l > 0 \\ 1+\flat & (\mathrm{mod}\ 2) \quad \text{if} \quad l \leqslant 0. \end{cases}$$

Hence, the integers $1-l$, where $0 < l \leqslant l_0/2$ and $l \equiv \flat \pmod 2$ are precisely the critical integers on the left-hand side of the functional equation.

Remark 1.7. The crucial part in the proof of Theorem 1 was the construction of a Whittaker function w_{χ_p} such that the local zeta integral at the place p does not vanish. This was due to the properties of w_{χ_p} as described in Proposition 1.2 and the existence of such a Whittaker function follows from Theorem F in [Gel-Kaj]. We were using their idea to construct w_{χ_p}.

2. Construction of the Distribution

In this section we shall construct a family of distributions μ_l on $\mathbb{Z}_p^*$, whose p-adic Mellin transform interpolates the critical values $L(\pi\otimes\eta\chi, 1-l)$. The construction is based on the properties of certain period integrals, which we are going to define first.

2.1. PERIOD INTEGRALS ATTACHED TO π

We let $\nu:\ \mathbb{Q}^*\backslash\mathbb{A}^*\to\mathbb{C}^*$ be an idèle class character with conductor $f_\nu=p^{e_\nu}$ a p-power and infinity component $\nu_\infty=\mathrm{id}$, i.e. ν corresponds to an even Dirichlet character $\tilde{\nu}:\ (\mathbb{Z}/f_\nu\mathbb{Z})^*\to\mathbb{C}^*$. In particular ν satisfies the same properties as χ and in Section 1 we have defined the vectors $\psi_\nu\in\Pi(\nu)$ (with ν replaced by χ). For any character $\nu\neq 1$ and $e\geqslant e_\nu$ we define the section ψ_{ν,p^e} by

$$\psi_{\nu,p^e}(g):=p^{-(e-e_\nu)l}\,\psi_\nu\left[g\begin{pmatrix}p^{-1}&\\&1\end{pmatrix}^{e-e_\nu}\right].$$

Here and in the following we keep the convention from Section 1 and identify $\begin{pmatrix}p^{-1}&\\&1\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Q}_p)$ with the adelic matrix whose components outside p are equal to 1. In particular we have $\psi_{\nu,p^{e_\nu}}=\psi_\nu$ for $\nu\neq 1$. We also define sections ψ_{1,p^e} for $e\geqslant 1$ by

$$\psi_{1,p^e}(g):=p^{-(e-1)l}\,\psi_1\left[g\begin{pmatrix}p^{-1}&\\&1\end{pmatrix}^{e-1}\right].$$

In particular we have $\psi_{1,p}=\psi_1$.

For any $\epsilon\in\mathbb{Z}_p^*$ and $e\geqslant 1$ we now define the vectors ψ_{ϵ,p^e} as the Fourier transform

$$\psi_{\epsilon,p^e}(g):=\frac{2}{\phi(p^e)}\sum_\nu \nu_p^{-1}(\epsilon)\,\psi_{\nu,p^e}(g),$$

where the sum runs over all characters $\nu:\ \mathbb{Q}^*\backslash\mathbb{A}^*\to\mathbb{C}^*$ with conductor $f_\nu|p^e$ and infinity component $\nu_\infty=\mathrm{id}$. We also define sections ψ^0_{ν,p^e} and ψ^0_{ϵ,p^e} by replacing ψ_ν in the above definitions by ψ^0_ν, in other words, since $\eta'_p(p)=1$ we have $\psi_{\epsilon,p^e}=\eta'\otimes\psi^0_{\epsilon,p^e}$. The sections ψ_{ϵ,p^e} still factorize

$$\psi_{\epsilon,p^e}=\psi_{\epsilon,p^e,f}\otimes\psi_\infty,$$

where the finite part $\psi_{\epsilon,p^e,f}$ is defined by replacing ψ_ν in the above definitions by $\psi_{\nu,f}$ and ψ_∞ is the infinity component of ψ_ν as defined in Section 1. We also note the following properties:

- $\psi_{\epsilon,p^e}=\psi_{\epsilon',p^e}$ for $\epsilon\equiv\pm\epsilon'\ (\mathrm{mod}\ p^e)$, (2.1)
- $\psi_{\epsilon,p^e}\left[g\begin{pmatrix}p^k&\\&p^k\end{pmatrix}\right]=\eta_p(p^k)\,\psi_{\epsilon,p^e}(g)$ for $k\in\mathbb{Z}$, (2.2)

- $\psi_{\epsilon,p^e}(gk) = \eta_p(ad)\,\psi_{\epsilon d^{-1},p^e}(g)$ for $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(2, p^e)$. (2.3)

The first two properties are immediate (note that $v_p(p) = 1$) the third one follows from an easy calculation, taking into account the behaviour of ψ_{v,p^e} under $K_0(2, p^e)$ and $\eta_p|_{\mathbb{Z}_p^*} = \eta'_p|_{\mathbb{Z}_p^*}$.

The family of sections $\{\psi_{\epsilon,p^e}\}_{\epsilon,p^e}$ as well as the cusp form ϕ satisfy a distribution relation. We set

$$\gamma := \mu_2(p)\mu_3(p^2)p^2. \tag{2.4}$$

LEMMA 2.1. (*a*) *For every $\epsilon \in \mathbb{Z}_p^*$ and $e \geqslant 2$ the following holds:*

$$\psi_{\epsilon,p^e}\left[g\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}\right] = p^l \sum_{\substack{\epsilon' \in (\mathbb{Z}_p/p^{e+1}\mathbb{Z}_p)^* \\ \epsilon' \equiv \epsilon\,(p^e)}} \psi_{\epsilon',p^{e+1}}(g).$$

(*b*) *We have*

$$\gamma\,\phi(g) = \sum_{\substack{u,w=0,\dots,p-1 \\ v=0,\dots,p^2-1}} \phi\left[g\begin{pmatrix} p^2 & & \\ & p & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & u/p & v/p^2 \\ & 1 & w/p \\ & & 1 \end{pmatrix}\right].$$

Proof. (a) The claim follows from a straightforward calculation using

$$\psi_{v,p^{e+1}}(g) = p^{-l}\,\psi_{v,p^e}\left[g\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}\right]$$

and the character relations

$$\sum_{\substack{\epsilon' \in (\mathbb{Z}_p/p^{e+1}\mathbb{Z}_p)^* \\ \epsilon' \equiv 1\,(p^e)}} v_p^{-1}(\epsilon') = \begin{cases} 0 & \text{for } f_v = p^{e+1} \\ p & \text{for } f_v | p^e. \end{cases}$$

(b) By an easy calculation one verifies that the vector ψ_p^1 satisfies the relation

$$\mu\delta_3^{1/2}\left[\begin{pmatrix} 1 & & \\ & p & \\ & & p^2 \end{pmatrix}\right]\psi_p^1(g) = \sum_{\substack{u,w=0,\dots,p-1 \\ v=0,\dots,p^2-1}} \psi_p^1\left[g\begin{pmatrix} p^2 & pu & v \\ & p & w \\ & & 1 \end{pmatrix}\right].$$

which immediately implies the claim, because the above relation is preserved under intertwining operators.

We define the Eisenstein series E_{ϵ,p^e} as the images of the vectors ψ_{ϵ,p^e} under Eis. It is immediate that the Eisenstein series E_{ϵ,p^e} satisfy the same properties (2.1)–(2.3) as well as the distribution relation

$$E_{\epsilon,p^e}\left[g\begin{pmatrix}p^{-1} & \\ & 1\end{pmatrix}\right] = p^l \sum_{\substack{\epsilon'\in(\mathbb{Z}_p/p^{e+1}\mathbb{Z}_p)^* \\ \epsilon'\equiv\epsilon\,(p^e)}} E_{\epsilon',p^{e+1}}(g)$$

for $\epsilon \in \mathbb{Z}_p^*$ and $e \geqslant 2$.

We may now introduce the period integrals on which our analysis of the values of the automorphic L-function will be based: for any elements $i, j \in \mathbb{Z}_p^*$ and $y \in \mathbb{Z}_p$ we define

$$P(i,j,y;p^e) := \int_{\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})} \phi\left[\begin{pmatrix}g & \\ & 1\end{pmatrix}\begin{pmatrix}1 & ip/f & y/f \\ & 1 & j/f \\ & & 1\end{pmatrix}\right] E_{1,p^e}(g)\,dg.$$

We note that $P(i,j,y;p^e)$ still depends on the choice of the components at infinity w_∞ and ψ_∞ as well as on the integer l and the character η. We write P_l^η if we want to indicate the dependence on l and η.

LEMMA 2.2. *Let $e \geqslant 2$. For any $j' \in \mathbb{Z}_p^*$ satisfying $j' \equiv 1$ mod p^e we have*

$$P(i,jj',y;p^e) = \eta_p^{-1}(i)\,P(1,j,0;p^e).$$

Proof. Since $j' \equiv 1 \bmod p^e$ and $f_\eta = p$ we obtain using (2.1) and (2.3)

$$E_{1,p^e}\left[g\begin{pmatrix}1 & \\ & j'^{-1}\end{pmatrix}\right] = E_{1,p^e}(g).$$

Therefore, changing the integration variable like $g \mapsto g\begin{pmatrix}1 & \\ & j'^{-1}\end{pmatrix}$ and using right invariance of ϕ under $\mathcal{I}$ we find

$$P(i,jj',y;p^e) = \int_{X_{\mathbb{A}}} \phi\left[\begin{pmatrix}g & \\ & 1\end{pmatrix}\begin{pmatrix}1 & ij'/p^{e-1} & y/f \\ & 1 & j/f \\ & & 1\end{pmatrix}\right] E_{1,p^e}(g)\,dg.$$

Changing the integration variable like $g \mapsto g\begin{pmatrix}1 & -j^{-1}y \\ & 1\end{pmatrix}$ we obtain

$$P(i,jj',y;p^e) = \int_{X_{\mathbb{A}}} \phi\left[\begin{pmatrix}g & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \dfrac{ij' - p^{e-1}j^{-1}y}{p^{e-1}} & 0 \\ & 1 & j/f \\ & & 1\end{pmatrix}\right] E_{1,p^e}(g)\,dg.$$

Finally, changing the integration variable like $g \mapsto g\begin{pmatrix}(ij'-p^{e-1}j^{-1}y)^{-1} & \\ & 1\end{pmatrix}$ and taking

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into account that ϕ is invariant on the right under $\mathcal{I}$, we obtain

$$P(i,jj',y;p^e) = \int_{X_{\mathbb{A}}} \phi\left[\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & \frac{1}{p^{e-1}} & 0 \\ & 1 & j/f \\ & & 1\end{pmatrix}\right] E_{1,p^e}\left[g\begin{pmatrix}(ij'-p^{e-1}j^{-1}y)^{-1} & \\ & 1\end{pmatrix}\right] dg.$$

Since $ij'-p^{e-1}j^{-1}y \equiv i \pmod p$, property (2.3) implies

$$E_{1,p^e}\left[g\begin{pmatrix}(ij'-p^{e-1}j^{-1}y)^{-1} & \\ & 1\end{pmatrix}\right] = \eta_p^{-1}(i)\, E_{1,p^e}(g),$$

which proves the lemma.

From now on we shall assume that $e \geqslant 2$. It is then obvious by Lemma 2.2 that the periods $P(i,j,y;p^e)$ only depend on $j \pmod{p^e}$ and $i \pmod p$. For any $j \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^*$ we therefore may define $P(j;p^e) := P(1,j,0;p^e)$.

PROPOSITION 2.3. *For all $e \geqslant 2$, the period integrals $P(j;p^e)$ satisfy the distribution relation*

$$P(j;p^e) = \eta_p(p^{-1})\gamma^{-1}p^{4+l}\sum_{w\in\mathbb{Z}_p/p\mathbb{Z}_p} P(j+wf;p^{e+1}).$$

Proof. Replacing $\phi\left[\begin{pmatrix} g & \\ & 1\end{pmatrix}u\right]$ by the expression in Lemma 2.1 we obtain

$$\gamma\, P(j;p^e) = \sum_{u,v,w}\int_{X_{\mathbb{A}}} \phi\left[\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} p^2 & & \\ & p & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & \frac{1+p^{e-1}u}{f} & \frac{w+p^{e-1}v}{pf} \\ & 1 & \frac{j+fw}{pf} \\ & & 1\end{pmatrix}\right] E_{1,p^e}(g)\, dg$$
$$= \eta_p(p^{-1})\sum_{u,v,w}\int \phi\left[\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & \frac{1+p^{e-1}u}{f} & \frac{w+p^{e-1}v}{pf} \\ & 1 & \frac{j+fw}{pf} \\ & & 1\end{pmatrix}\right] E_{1,p^e}\left[g\begin{pmatrix} p^{-1} & \\ & 1\end{pmatrix}\right] dg.$$

The last equality follows by the change of integration variable $g \mapsto g\begin{pmatrix} p^{-2} & \\ & p^{-1}\end{pmatrix}$ and using Equation (2.2). Replacing $E_{1,p^e}\left[g\begin{pmatrix} p^{-1} & \\ & 1\end{pmatrix}\right]$ by the expression in Lemma 2.1 and using (2.3), we further obtain for the right hand side of the above equation (note

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that $\eta_p(\epsilon) = 1$ since $e \geqslant 2$)

$$\eta_p(p^{-1})p^l \sum_{\substack{u,v,w \\ \epsilon \equiv 1\,(p^e)}} \int \phi\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1+p^{e-1}u}{f} & \frac{w+p^{e-1}v}{pf} \\ & 1 & \frac{j+fw}{pf} \\ & & 1 \end{pmatrix}\right] E_{1,p^{e+1}}\left[g\begin{pmatrix} 1 & \\ & \epsilon^{-1} \end{pmatrix}\right] dg.$$

Changing variables $g \mapsto g\begin{pmatrix} 1 & \\ & \epsilon \end{pmatrix}$ this equals

$$\eta_p(p^{-1})p^l \sum_{\substack{u,v,w \\ \epsilon \equiv 1\,(p^e)}} \int_{X_{\mathbb{A}}} \phi\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{\epsilon^{-1}+\epsilon^{-1}p^{e-1}u}{f} & \frac{w+p^{e-1}v}{pf} \\ & 1 & \frac{\epsilon j+\epsilon fw}{pf} \\ & & 1 \end{pmatrix}\right] E_{1,p^{e+1}}(g)\, dg.$$

Using the periods this may be written

$$\gamma\, P(j; p^e) = \eta_p(p^{-1})p^l \sum_{\substack{u,v,w \\ \epsilon \equiv 1\,(p^e)}} P(\epsilon^{-1}+\epsilon^{-1}p^{e-1}u, \epsilon j+\epsilon fw, w+p^{e-1}v; fp).$$

Since $\epsilon^{-1}+\epsilon^{-1}p^{e-1}u \equiv 1 \bmod p$, Lemma 2.2 then shows

$$\gamma\, P(j; p^e) = \eta_p(p^{-1})p^{3+l} \sum_{\substack{w \\ \epsilon \equiv 1\,(p^e)}} P(1, \epsilon j+\epsilon fw, 0; fp).$$

We write $\epsilon = 1+fo$, where $o \in \mathbb{Z}_p/p\mathbb{Z}_p$. Because $\epsilon \equiv 1 \pmod{p^e}$, $e \geqslant 2$ we find

$$\epsilon j+\epsilon fw \equiv j+f(oj+w) \pmod{pf}.$$

Thus, by the change of summation variable $w \mapsto -oj+w$ we finally obtain

$$\gamma\, P(j; p^e) = \eta_p(p^{-1})p^{4+l} \sum_{w \in \mathbb{Z}_p/p\mathbb{Z}_p} P(1, j+wf, 0; p^{e+1}).$$

This proves the Proposition.

For any integer $e \geqslant 2$ and any pair (η, l) such that $0 < l \leqslant l_0/2$ and $l \equiv \flat \pmod 2$ we define

$$\mu_l^\eta(j+p^e\mathbb{Z}_p) = \mu_{l,w_\infty,\psi_\infty}^\eta(j+p^e\mathbb{Z}_p) := (\eta_p(p)\gamma)^{-e} p^{(4+l)e} P_l^\eta(j; p^e).$$

To simplify the notation we mostly write μ_l instead of μ_l^η. It is immediate by Proposition 2.3 that μ_l defines a distribution on $\mathbb{Z}_p^*$ and using the distribution relation we may extend μ_l to all cosets $j+p^e\mathbb{Z}_p$ with $e \in \mathbb{N}$.

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2.2. THE p-ADIC MELLIN TRANSFORM

We want to integrate μ_l against characters χ and the result is given in the following proposition:

PROPOSITION 2.4. *For any idèle class character χ of conductor $f = p^e$, $e \geqslant 2$ and infinity component $\chi_\infty = \mathrm{id}$ and any integer $0 < l \leqslant l_0/2$ such that $l \equiv \flat \pmod 2$ we have*

$$\int_{\mathbb{Z}_p^*} \chi_p \eta_p^2 \, d\mu_l^\eta = \mathbf{B}\, P(1/2) \cdot L(\pi \otimes \chi\eta, 1-l),$$

where the factor in front of the integral is given by

$$\mathbf{B} = \mathbf{A}^{-1} \frac{2p^3}{(p-1)^2} (\eta_p(p)\gamma)^{-e} p^{(1+l)e}$$

and $\mathbf{A}$ has been defined in Theorem 1. For any character χ with conductor a p-power and infinity component $\chi_\infty = \mathrm{sgn}$ the above integral vanishes identically.

Proof. Using the identity

$$E_{\epsilon,p^e}\left[g\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right] = \eta_p \eta'_p(-1) E_{\epsilon,p^e}(g)$$

(note that $v_p(-1) = 1$), we immediately find

$$\mu_l(-j + f\mathbb{Z}_p) = \mu_l(j + f\mathbb{Z}_p),$$

which proves the vanishing of the integral in the case $\chi_\infty = \mathrm{sgn}$. We now assume $\chi_\infty = \mathrm{id}$. In this case we want to derive the Proposition from Theorem 1 and proceed as follows. Using the inverse Fourier transform we immediately obtain for $\chi \neq 1$

$$E_\chi(g) = \sum_{\epsilon \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^*/\{\pm 1\}} \chi_p(\epsilon)\, E_{\epsilon,p^e}.$$

Plugging this into the integral occurring in Theorem 1 we obtain

$$\begin{aligned} &P(1/2)\, L(\pi \otimes \chi\eta, 1-l) \\ &\quad = \mathbf{A} \sum_{\substack{i,j,y \\ \epsilon \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^*/\{\pm 1\}}} \eta_p(ij^2)\, \chi_p(\epsilon j) \int_{X_\mathbb{A}} \phi\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & ip/f & y/f \\ & 1 & j/f \\ & & 1 \end{pmatrix}\right] E_{\epsilon,p^e}(g)\, dg. \end{aligned}$$

Changing variables like $g \mapsto g\begin{pmatrix} 1 & \\ & \epsilon \end{pmatrix}$ and using (2.3) we obtain for the right-hand side

$$\mathbf{A} \sum_{\substack{i,j,y \\ \epsilon \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^*/\{\pm 1\}}} \eta_p(\epsilon ij^2)\, \chi_p(\epsilon j) \int_{X_\mathbb{A}} \phi\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & ip\epsilon^{-1}/f & y/f \\ & 1 & j\epsilon/f \\ & & 1 \end{pmatrix}\right] E_{1,p^e}(g)\, dg,$$

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which by the obvious change of the summation variables i and j equals

$$\mathbf{A}\,\frac{\varphi(p^e)}{2}\sum_{i,j,y}\eta_p(ij^2)\,\chi_p(j)\,P(i,j,y;p^e).$$

Using Lemma 2.2 we see that this is identical with

$$\mathbf{A}\,\frac{\varphi(p^e)^2p^e}{2p}\sum_{j\in(\mathbb{Z}_p/p^e\mathbb{Z}_p)^*}\eta_p^2\chi_p(j)\,P(1,j,0;p^e).$$

Thus we obtain the equation

$$\begin{aligned}&P(1/2)\,L(\pi\otimes\chi\eta,1-l)\\&\quad=\mathbf{A}\,\frac{\varphi(p^e)^2p^e}{2p}\,\eta_p(p)^e\gamma^e p^{-(4+l)e}\sum_{j\in(\mathbb{Z}/p^e\mathbb{Z})^*}\chi_p\eta_p^2(j)\,\mu_l(j+p^e\mathbb{Z}_p)\end{aligned}$$

and the proof of the proposition is finished.

The conductors of the characters χ and η_0 being relatively prime, a small calculation proves that $G(\tilde{\chi}\tilde{\eta}_0)=\tilde{\chi}(q)\tilde{\eta}_0(p^e)\,G(\tilde{\chi})G(\tilde{\eta}_0)$. Thus, recalling the definition of $\mathbf{A}$ and noting that $\tilde{\chi}=\chi_p^{-1}|_{\mathbb{Z}_p^*}$ and $\tilde{\eta}_0=\eta_{0,q}^{-1}|_{\mathbb{Z}_q^*}$ we deduce that

$$\mathbf{B}=\mathbf{C}_l\;\chi_p(q^{-1})\hat{\zeta}^e\boldsymbol{\pi}^{\flat-3l}\gamma^{-e}G(\chi_p\eta_p^2)L(\tilde{\chi}\tilde{\eta}_0,1-2l)^{-1}$$

where $\hat{\zeta}=\eta_p^{-2}\eta_{0,q}^{-1}\mu_3(p)$, $\mathbf{C}=\mathbf{C}_l\in\mathbb{C}^*$ does not depend on χ and the coset $\mathbf{C}_l\cdot\mathbb{Q}^*$ does not depend on l. On the other hand since $pq|f_{\chi\eta_0}$ (we assume $e\geqslant 2$) it is well known that there is a $\mathbb{Z}$-valued measure $\mu(l)$ on $\mathbb{Z}_p^*$ such that

$$L(\tilde{\chi}\tilde{\eta}_0,1-2l)=\int_{\mathbb{Z}_p^*}\chi_p\,d\mu(l)$$

(cf. [Wa], pp. 239/240; note that the factor $-(1-\chi(c)\langle c\rangle^{2l})$ is analytic and bounded). In addition, the trivial equality $\chi_p(q^{-1})=\int_{\mathbb{Z}_p^*}\chi_p\,d\delta_{q^{-1}}$ holds, where $\delta_{q^{-1}}$ denotes the Dirac distribution at q^{-1}. We define the convolution of distributions $\mu^{\eta,l}:=\mu_l^{\eta}*\mu(l)*\delta_{q^{-1}}$ and immediately deduce from Proposition 2.4 the final form of our p-adic integral:

COROLLARY 1. *Let χ and l be as in Proposition 2.4. If $\chi_\infty=\mathrm{id}$ we have*

$$\int_{\mathbb{Z}_p^*}\chi_p\eta_p^2\,d\mu^{\eta,l}=\mathbf{C}\,P(1/2)\,\boldsymbol{\pi}^{\flat-3l}\,\hat{\zeta}^e\gamma^{-e}\,G(\chi_p\eta_p^2)\cdot L(\pi\otimes\chi\eta,1-l)$$

and if $\chi_\infty\neq\mathrm{id}$ the integral vanishes identically.

The distribution $\mu(l)$ and $\delta_{q^{-1}}$ being $\mathbb{Z}$-valued, we are reduced to investigate the algebraicity and integrality of the distribution μ_l. This will be done in the last three chapters.

3. Cohomology and Rationality

3.1. THE ISOTYPICAL SUBSPACES IN COHOMOLOGY

Let K^f be a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$. We introduce the differentiable manifolds

$$S_n(K^f) := \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f\, K_{n,\infty} Z_n^0(\mathbb{R})$$

and

$$F_n(K^f) := \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f\, K_{n,\infty} \cong \mathbb{R}^*_{>0} \times S_n(K^f).$$

Any finite-dimensional, rational representation $\rho\colon \mathrm{GL}_n \to \mathrm{GL}(V)$ determines a sheaf $\mathcal{V}$ on the spaces $S_n(K^f)$ and we define $H^i(\tilde{S}_n, \mathcal{V}) := \mathrm{inj\,lim}_{K^f}\, H^i(S_n(K^f), \mathcal{V})$.

The results of [Cl] and [Ha 1] in particular imply, that the finite parts of the representations π and $\Pi(\chi)$ appear as direct summands in the cohomology of $\tilde{S}_3$ resp. $\tilde{S}_2$. We want to describe these cohomological realizations and the special cohomology classes corresponding to the automorphic forms ϕ and E_{ϵ,p^e}. We begin with π_f. By assumption the infinity component of π is isomorphic to $\mathrm{Ind}(D_{l_0}, \mathrm{id})$, where $l_0 \in 2\mathbb{Z}$ and the type at infinity of π (cf. [Cl], p. 106) therefore reads $(l_0/2-1, -l_0/2-1, -1) \in \mathbb{Z}^3/S_3$. We define (ρ, V) to be the finite dimensional representation of GL_3 with highest weight $(l_0/2-1, 0, -l_0/2+1)$ with respect to the standard torus in GL_3 and we let $\mathcal{V}$ be the locally constant sheaf attached to ρ. From [Cl], Lemme 3.14 we derive

$$H^i(\mathrm{gl}_3, K_{3,\infty}Z_3^0(\mathbb{R}), \pi_\infty \otimes \rho) = \begin{cases} \mathbb{C}, & \text{for} \quad i = 2, 3, \\ 0, & \text{else.} \end{cases}$$

We choose a basis $\omega_1', \ldots, \omega_5'$ of the dual of the tangent space $\left(\mathrm{gl}_3/\mathrm{so}_3\mathrm{Lie}(Z_3^0(\mathbb{R}))\right)^*$ and a basis $\{v_a\}$ of V. Let

$$\omega_\infty = \sum_{i,j=1,\ldots,5} \sum_a w_{\infty,i,j,a}\, v_a \otimes \omega_i' \wedge \omega_j'$$

be a generator of $H^2(\mathrm{gl}_3, K_{3,\infty}Z_3^0(\mathbb{R}), W(\pi_\infty, \tau_\infty) \otimes V)$. For any Whittaker function $w_f \in W(\pi_f, \tau_f)$ the product $w_f \cdot \omega_\infty$ defines an element of $H^2(\mathrm{gl}_3, K_{3,\infty}Z_3^0(\mathbb{R}), W(\pi, \tau) \otimes V)$. Since π embeds into $L_0^2(\mathrm{GL}_3(\mathbb{Q})\backslash\mathrm{GL}_3(\mathbb{A}))$ this yields an injection

$$\mathcal{F}_\pi\colon\ W(\pi_f, \tau_f) \to H^2_{\mathrm{cusp}}(\tilde{S}_3, \mathcal{V}).$$

We let w_f be the finite part of the Whittaker function defined in Section 1 and denote by $\omega := \mathcal{F}_\pi(w_f)$ the image of w_f in cohomology. The cohomology class ω then reads $\omega = \sum_{i,j,a} \phi_{i,j,a}\, v_a \otimes \omega_i' \wedge \omega_j'$ where $\phi_{i,j,a} \in V(\pi)$ denotes the cusp form attached to the Whittaker function $w_f \cdot w_{\infty,i,j,a}$. We remark that the cuspidal cohomology defines a subspace in the cohomology with compact supports $H^2_{\mathrm{cusp}}(\tilde{S}_3, \mathcal{V}) \leqslant H^2_c(\tilde{S}_3, \mathcal{V})$ (cf. [Cl], p. 123).

Next we describe the embedding of $\Pi_f(\chi)$. To this end let (κ, W) be the finite dimensional representation of GL_2 with trivial central character and highest weight $2l-2$ if restricted to SL_2 and let $\mathcal{W}$ be the locally constant sheaf attached to κ. In the notation of [Ha 1], p. 45 we have $W = M(2l-2, 1-l)$. The representation $\Pi(\chi)$ is non-unitarily induced from the character

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto \left|\frac{t_1}{t_2}\right|^l \eta'(t_1)\chi\eta(t_2),$$

whose type is contained in $\mathrm{Coh}(W)$ in degree 1 (cf. [Ha 1], p. 49) and which also is even. We therefore deduce from Theorem 1 in [Ha 1] that $\Pi_f(\chi)$ appears as a direct summand in the cohomology of the boundary $\partial\bar{S}_2$ of the Borel–Serre compactification $\bar{S}_2$ with coefficients in $\mathcal{W}$

$$\Pi_f(\chi) \hookrightarrow H^1(\partial\tilde{\bar{S}}_2, \mathcal{W}) := \lim_{K^f} H^1(\partial\bar{S}_2(K^f), \mathcal{W}).$$

The cohomology class in $H^1(\partial\tilde{\bar{S}}_2, \mathcal{W})$ attached to an element $\psi \in \Pi_f(\chi)$ is given as follows. We choose a basis ω_1, ω_2 of the dual of the tangent space $\left(\mathrm{gl}_2/\mathrm{so}_2\mathrm{Lie}(Z_2^0(\mathbb{R}))\right)^*$ and a generator ω_3 of $(\mathrm{Lie}(Z_2^0(\mathbb{R})))^*$. The embedding $j:\ \mathrm{GL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \hookrightarrow \mathrm{GL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})Z_3^0(\mathbb{R})$, $g\,\mathrm{SO}_2(\mathbb{R}) \mapsto \mathrm{diag}(g, 1)\mathrm{SO}_3\ (\mathbb{R})$ $Z_3^0(\mathbb{R})$ induces a mapping of the duals of the tangent spaces

$$j^*:\ \left(\mathrm{gl}_3/\mathrm{so}_3\mathrm{Lie}(Z_3^0(\mathbb{R}))\right)^* \to \left(\mathrm{gl}_2/\mathrm{so}_2\right)^*$$

and for later purpose we assume that the basis ω_i' and ω_i are chosen in such a way that $j^*(\omega_i') = \omega_i$ for $i = 1, 2, 3$ and $j^*(\omega_i') = 0$ for $i = 4, 5$. Let $\{w_b\}$ be a basis of W. Using [Ha 1], p. 69 we know that

$$H^1(\mathrm{gl}_2, \mathrm{SO}_2(\mathbb{R})Z_2^0(\mathbb{R}), \Pi_\infty(\chi) \otimes \kappa) = \mathbb{C}$$

and a generator e^1 of this cohomology group reads

$$e^1 = \sum_{i=1,2} \sum_b \psi_{\infty,i,b}\, w_b \otimes \omega_i,$$

where $\psi_{\infty,i,b} \in \eta_\infty \otimes \Pi_\infty^0 < \Pi_\infty(\chi)$ (cf. Remark 3.3). For any $\psi \in \Pi_f(\chi)$ we then obtain an element in the cohomology of the boundary

$$\psi \otimes e^1 \in H^1(\partial\tilde{\bar{S}}_2, \mathcal{W}).$$

The embedding into the cohomology of the symmetric space of GL_2

$$\mathrm{Eis}^*:\ \Pi_f(\chi) \to H^1(\tilde{S}_2, \mathcal{W})$$

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is now given by

$$\mathrm{Eis}^*(\psi) := \sum_{\gamma \in B_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \psi \otimes e^1(\gamma g, D)$$

for $g \in \mathrm{GL}_2(\mathbb{A})$, $D \in \mathrm{gl}_2/\mathrm{so}_2\mathrm{Lie}(Z_2^0(\mathbb{R}))$ (cf. [Ha 1], p. 80). Let $\psi_{p^e,f} := \psi_{1,p^e,f}$ be the finite part of the vector ψ_{1,p^e} defined in Section 2.1. We denote $\omega_{p^e} := \mathrm{Eis}^*(\psi_{p^e,f})$ the image of $\psi_{p^e,f}$ and we identify ω_{p^e} with its image under the canonical map $p^*: H^1(\tilde{S}_2, \mathcal{W}) \to H^1(\tilde{F}_2, p^*\mathcal{W})$ induced by the projection $p: F_2(K^f) \to S_2(K^f)$. The class ω_{p^e} then reads

$$\omega_{p^e} = \sum_{i=1,2} \sum_b E_{p^e,i,b}\, w_b \otimes \omega_i,$$

with Eisenstein series $E_{p^e,i,b} := \mathrm{Eis}(\psi_{p^e,f}\psi_{\infty,i,b})$.

We shall use the differential forms ω and ω_{p^e} to give a cohomological description of the period integrals $P(\epsilon; p^e)$. We denote by $V|_{\mathrm{GL}_2}$ the restriction of the representation V to GL_2 via the embedding $g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$ and we choose a non-trivial, GL_2-equivariant pairing

$$\mathrm{tr}: V|_{\mathrm{GL}_2} \otimes W \to \mathbb{C}.$$

Such a pairing exists due to the following

LEMMA 3.1. *The restriction of the representation* ρ *to* GL_2 *decomposes into a direct sum of representations of* GL_2, $\rho|_{\mathrm{GL}_2} = \check{\kappa} \oplus \rho'$, *where* $\check{\kappa}$ *is the contragredient representation of* κ.

Proof. For any dominant weight $\lambda \in \mathbb{Z}^n$ we denote by F_λ the $\mathrm{GL}_n(\mathbb{C})$–representation of highest weight λ. The representation ρ is then isomorphic to $F_{(l_0-2,l_0/2-1,0)} \otimes \det^{1-l_0/2}$. Using the Schur functor (cf. [F-H], pp. 76, 231/232) we may write

$$F_{(l_0-2,l_0/2-1,0)} = \mathbb{S}_{(l_0-2,l_0/2-1,0)}\, \mathbb{C}^3.$$

Since $l_0/2 \geqslant l > 0$ we have

$$l_0 - 2 \geqslant l_0/2 + l - 2 \geqslant l_0/2 - 1 \geqslant l_0/2 - l \geqslant 0$$

and [F-H], 6.12 then implies that the restriction of $F_{(l_0-2,l_0/2-1,0)}$ to GL_2 is completely reducible and contains the representation $\mathbb{S}_{(l_0/2+l-2,l_0/2-l)}\, \mathbb{C}^2$. This representation has highest weight $(l_0/2 + l - 2, l_0/2 - l)$ (cf. [F-H], 15.15) and hence $\rho|_{\mathrm{GL}_2}$ contains as a direct factor a representation of highest weight $(l-1, 1-l)$. Since $\kappa \cong \check{\kappa}$ and κ has highest weight $(l-1, 1-l)$ this proves the lemma.

We now set

$$\mu^\eta_{\pi,l} := \sum_{i,j,k,a,b} \varepsilon_{i,j,k}\, \mathrm{tr}(v_a \otimes w_b)\, \mu^\eta_{l,w_{\infty,i,j,a},\psi_{\infty,k,b}},$$

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where $\varepsilon_{i,j,k}$ vanishes unless (i, j, k) is equal to $(2, 3, 1)$ resp. $(1, 3, 2)$ in which cases it equals 1 resp. -1. Obviously, $\mu^{\eta}_{\pi,l}$ defines a distribution on $\mathbb{Z}_p^*$ and it is immediate by Proposition 2.4 that by integrating $\mu^{\eta}_{\pi,l}$ against characters we find

$$\int_{\mathbb{Z}_p^*} \chi_p \eta_p^2 \, d\mu^{\eta}_{\pi,l} = \mathbf{B}\, P_l(1/2)\, L(\pi \otimes \chi\eta, 1-l), \tag{3.1}$$

where $P_l \in \mathbb{C}[T]$ denotes the polynomial $P_l := \sum_{i,j,k,a,b} \varepsilon_{i,j,k} \operatorname{tr}(v_a \otimes w_b)\, P_{w_{\infty,i,j,a},\psi_{\infty,k,b}}$.

On the other hand, for every $N \in \mathbb{N}$ let $i(N)$ be the canonical map

$$\begin{array}{ll} i(N): & F_2(K(2,N)) \to S_3(K(3,N)) \\ & \mathrm{GL}_2(\mathbb{Q})\, g\, K(2,N) K_{2,\infty} \mapsto \mathrm{GL}_3(\mathbb{Q}) \begin{pmatrix} g & \\ & 1 \end{pmatrix} K(3,N) K_{3,\infty} Z_3^0(\mathbb{R}), \end{array}$$

where $K(n, N) \leqslant \mathrm{GL}_n(\mathbb{A}_f)$ denotes the principal congruence subgroup of level N. $i(N)$ is a proper map and therefore induces a map on the limit of the cohomology groups with compact support

$$i^*: H_c^2(\tilde{S}_3, \mathcal{V}) \to H_c^2(\tilde{F}_2, i^*\mathcal{V}).$$

The pairing $\operatorname{tr}: V|_{\mathrm{GL}_2} \times W \to \mathbb{C}$ induces a pairing of the associated sheafs $\operatorname{tr}: i^*\mathcal{V} \otimes p^*\mathcal{W} \to \underline{\mathbb{C}}$. Using [Cl], p. 122 we even know that the representations ρ and κ are defined over finite extensions E_ρ and E_κ of $\mathbb{Q}$, i.e. ρ resp. κ act on E_ρ resp. E_κ vector spaces V_{E_ρ} and W_{E_κ}. In particular the above pairing of sheafs is defined over $E_{\rho,\kappa} := E_\rho E_\kappa$

$$\operatorname{tr}: i^*\mathcal{V}_{E_{\rho,\kappa}} \otimes p^*\mathcal{W}_{E_{\rho,\kappa}} \to \underline{E}_{\rho,\kappa}$$

and together with the cup product we obtain the diagram

$$\begin{array}{ccccccc} H_c^2(\tilde{F}_2, i^*\mathcal{V}_{E_{\rho,\kappa}}) & \times & H^1(\tilde{F}_2, p^*\mathcal{W}_{E_{\rho,\kappa}}) & \xrightarrow{\operatorname{tr}\circ\cup} & H_c^3(\tilde{F}_2, \underline{E}_{\rho,\kappa}) & = E_{\rho,\kappa} \\ \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle p^*} & & & \\ H_c^2(\tilde{S}_3, \mathcal{V}_{E_{\rho,\kappa}}) & & H^1(\tilde{S}_2, \mathcal{W}_{E_{\rho,\kappa}}). & & & \end{array}$$

In other words we have a pairing

$$\begin{array}{llll} \langle\ ,\ \rangle: & H_c^2(\tilde{S}_3, \mathcal{V}_{E_{\rho,\kappa}}) \times H^1(\tilde{S}_2, \mathcal{W}_{E_{\rho,\kappa}}) & \to & E_{\rho,\kappa} \\ & (\omega, \omega') & \mapsto & \operatorname{tr} i^*\omega \cup p^*\omega'. \end{array}$$

We want to determine the special values $\langle \omega, \omega_{p^e} \rangle$ of this pairing. We denote by

$$r_u^*: H_c^2(\tilde{S}_3, \mathcal{V}) \to H_c^2(\tilde{S}_3, \mathcal{V}), \quad u \in \mathrm{GL}_3(\mathbb{A}_f)$$

the right translation.

LEMMA 3.2. *For $u = u(1, j, 0; p^e) \in N_3(\mathbb{Q}_p)$ we have*

$$\langle r_u^*\omega, \omega_{p^e} \rangle = p^{-e(4+l)} \eta_p(p^e) \gamma^e \, \mu^{\eta}_{\pi,l}(j + p^e\mathbb{Z}_p).$$

Proof. Let $K^f \leqslant \mathrm{GL}_2(\mathbb{A}_f)$ be a compact subgroup under which $i^* r_u^* \omega$ and ω_{p^e} are invariant. Using [Ha 2] ch. E.4 we find

$$\langle r_u^* \omega, \omega_{p^e} \rangle = \mathrm{vol}(K^f) \int_{F_2(K^f)} \mathrm{tr}\ i^* r_u^* \omega \wedge \omega_{p^e}.$$

It is immediate by the choice of the vectors ω_i' and ω_i that the image $i^* r_u^*(\omega) \in H_c^2(\tilde{F}_2, i^*\mathcal{V})$ is given by

$$i^* r_u^*(\omega) = \sum_{i,j=1,2,3} \sum_a \phi_{i,j,a}\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix} u\right] v_a \otimes \omega_i \wedge \omega_j.$$

Using this and the definition of ω_{p^e} we obtain

$$\begin{aligned}&\langle r_u^* \omega, \omega_{p^e} \rangle \\ &\quad = \mathrm{vol}(K^f) \int_{F_2(K^f)} \sum_{i,j,k} \sum_{a,b} \mathrm{tr}(v_a \otimes w_b) \phi_{i,j,a}\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix} u\right] E_{p^e,k,b}(g)\, \omega_i \wedge \omega_j \wedge \omega_k.\end{aligned}$$

Comparing with the definition of $\mu_l^\eta = \mu^\eta_{l,w_\infty,\psi_\infty}$ in Section 2 and taking into account that an invariant 3-form on $\mathrm{GL}_2(\mathbb{R})/\mathrm{O}_2(\mathbb{R})$ corresponds to a Haar measure dg_∞ on $\mathrm{GL}_2(\mathbb{R})$ we obtain the claim of the lemma.

Remark 3.3. Since $\mathrm{O}_n(\mathbb{R})$ normalizes $\mathrm{SO}_n(\mathbb{R})$ the quotient group $\mathbb{Z}/2\mathbb{Z} = \mathrm{O}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ still acts on $H^i(\tilde{S}_n, \mathcal{V})$ and $H^i(\mathrm{gl}_n, K_{n,\infty} Z_n^0(\mathbb{R}), \pi_\infty \otimes \rho)$ and we want to verify that the classes ω and ω_{p^e} are eigenvectors for this action. The assumption that l_0 is even implies that the central character of π_∞ equals the signum morphism and considering the action of $\mathrm{diag}(-1,-1,-1)$ then shows that ω_∞ and hence ω has eigenvalue -1 under the operation of the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$. To calculate the operation of $\mathbb{Z}/2\mathbb{Z}$ on ω_{p^e} we note that

$$H^1(\mathrm{gl}_2, \mathrm{SO}_2(\mathbb{R}) Z_2^0(\mathbb{R}), D_{2l-1} \otimes \kappa) = \langle v^+, v^- \rangle_{\mathbb{C}},$$

where the generators satisfy $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} v^\pm = \pm v^\pm$. The inclusion $D_{2l-1} \hookrightarrow \Pi_\infty(\chi)$ induces a canonical map in cohomology

$$H^1(\mathrm{gl}_2, \mathrm{SO}_2(\mathbb{R}) Z_2^0(\mathbb{R}), D_{2l-1} \otimes \kappa) \to H^1(\mathrm{gl}_2, \mathrm{SO}_2(\mathbb{R}) Z_2^0(\mathbb{R}), \Pi_\infty(\chi) \otimes \kappa)$$

and an explicit calculation proves that under this map v^- maps to zero whereas v^+ does not. (To see this one has to take into account that $l \equiv \flat$ modulo 2.) We deduce that

$$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} e^1 = e^1$$

and hence, ω_{ϵ,p^e} and $\omega_\chi = \sum_\epsilon \chi_p(\epsilon) \omega_{\epsilon,p^e}$ too are eigenvectors for $\mathrm{diag}(-1, 1)$ with eigenvalue 1 under the operation of this matrix. In addition we also see that the coefficients $\psi_{\infty,i,b}$ are contained in $\eta_\infty \otimes \Pi_\infty^0$, the unique submodule in $\Pi_\infty(\chi)$, which is isomorphic to D_{2l-1} (cf. Section 1). Finally, the action on cohomology also induces an action of $\mathbb{Z}/2\mathbb{Z}$ on our integral representation: Lemma 3.2 immediately implies that the integral appearing in the formula of Theorem 1 coincides with

$$\mathbf{A}\sum_{i,j,y}\chi_p\eta_p^2(j)\eta_p(i)\int_{F_2(K^f)}\omega\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}u(i,j,y;p^e)\right)\wedge\omega_\chi(g)$$

and using the above eigenvalues it is easily seen that this expression is invariant under the change of integration variable $g\mapsto g\,\mathrm{diag}(-1,1)_\infty$, which is orientation reversing. In particular, our cohomological formula does not vanish for parity reasons.

3.2. THE $\bar{\mathbb{Q}}$-STRUCTURE

The non-vanishing of the cohomology implies that π_f and $\Pi_f(\chi)$ are defined over finite extensions $E/\mathbb{Q}$ and we want to show that the differential forms ω and ω_{p^e} too are defined over these extensions. We begin with ω_{p^e}.

For any automorphism $\sigma\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ we define the σ-linear isomorphism

$$\begin{array}{cccc}\sigma: & \Pi_f(\chi) & \to & \Pi_f(\chi^\sigma)\\ & \psi & \mapsto & \psi^\sigma,\end{array}$$

where $\psi^\sigma(g):=\psi(g)^\sigma$. Using [W], ch. I.2 and [Cl], Proposition 3.1 (iii) we know that $\Pi_f(\chi)$ is defined over $\mathbb{Q}(\chi,\eta,\eta')$: for any field $H/\mathbb{Q}$ let $\Pi_{H,f}(\chi)$ denote the H-subspace of H-valued functions in $\Pi_f(\chi)$; we then have

$$\Pi_f(\chi)=\Pi_{\mathbb{Q}(\chi,\eta,\eta'),f}(\chi)\otimes\mathbb{C}.$$

Theorem 2 in [Ha 1] states that the embedding

$$\mathrm{Eis}^*:\ \Pi_{\bar{\mathbb{Q}},f}(\chi)\to H^1(\tilde{S}_2,\mathcal{W}_{\bar{\mathbb{Q}}})$$

is defined over $\bar{\mathbb{Q}}$ and for every $\sigma\in G_{\mathbb{Q}}:=\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we have $\mathrm{Eis}^*(\psi^\sigma)=\mathrm{Eis}^*(\psi)^\sigma$. From the definition of ψ_χ it is obvious that $\psi^\sigma_{\chi,f}=\psi_{\chi^\sigma,f}$ for all $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\eta,\eta'))$. This implies $\psi^\sigma_{\epsilon,p^e,f}=\psi_{\epsilon,p^e,f}$ and using the $G_{\mathbb{Q}}$-equivariance of Eis^*, that $\omega^\sigma_{p^e}=\omega_{p^e}$ for all $\sigma\in\mathrm{Aut}(\mathbb{C}/E_\kappa(\eta,\eta'))$, i.e. ω_{p^e} is contained in $H^1(\tilde{S}_2,\mathcal{W}_{E_\kappa(\eta,\eta')})$.

We consider the cuspidal form ω. Using [Cl], Théorème 3.13 we know that π_f is defined over a finite extension $E=E_\pi$ of $\mathbb{Q}$. In particular the π_f-isotypical component of the cuspidal cohomology is defined over E

$$H^2_{\mathrm{cusp}}(\tilde{S}_3,\mathcal{V})(\pi_f)=H^2_{\mathrm{cusp}}(\tilde{S}_3,\mathcal{V}_E)(\pi_f)\otimes\mathbb{C}$$

(cf. [Cl], Théorème 3.19, note that $E_\rho\leqslant E$). Similarly, there is an E-subspace $W_E(\pi_f,\tau_f)$ of $W(\pi_f,\tau_f)$ such that

$$W(\pi_f,\tau_f)=W_E(\pi_f,\tau_f)\otimes\mathbb{C}.$$

We want to find a field of definition for the complex valued differential form ω

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attached to w_f. Using the uniqueness of E-structures on irreducible, admissible $\mathrm{GL}_3(\mathbb{A}_f)$–-modules up to scalar multiples (cf. [Cl], Proposition 3.1) we deduce that after multiplication by a complex number the isomorphism

$$\tilde{\Omega}(\pi)\cdot\mathcal{F}_\pi:\ W_E(\pi_f,\tau_f)\to H^2_{\mathrm{cusp}}(\tilde{S}_3,\mathcal{V}_E)(\pi_f)$$

respects the E-subspaces as defined above. The following lemma therefore implies the existence of a complex number $\Omega(\pi)\in\mathbb{C}^*$ such that

$$\Omega(\pi)^{-1}\omega\in H^2_{\mathrm{cusp}}(\tilde{S}_3,\mathcal{V}_{E(\zeta_{q(q-1)})}).$$

LEMMA 3.4. *There is an* $\Omega\in\mathbb{C}^*$ *such that* $\Omega\, w_f\in W_{E(\zeta_{q(q-1)})}(\pi_f,\tau_f)$.

Proof. We decompose the Whittaker function $w_f=w_p\otimes w_q\otimes w^{p,q}$, where $w^{p,q}=\otimes_{\ell\neq p,q}w_\ell$. Since $w^{p,q}$ is an essential vector and the space of essential vectors is 1-dimensional, we deduce that there is a complex number $\Omega'\in\mathbb{C}^*$ such that $\Omega'\,w^{p,q}\in W_E(\otimes_{\ell\neq p,q}\pi_\ell,\otimes_{\ell\neq p,q}\tau_\ell)$. Thus, we are left with examining w_p and w_q. We recall from Section 1 that π_p is isomorphic to the non-unitarily induced representation $\pi_p=\mathrm{Indn}(\delta_{3,p}^{1/2}\mu)$. The σ-linear action

$$\mathrm{Indn}(\delta_{3,p}^{1/2}\mu)\to\mathrm{Indn}((\delta_{3,p}^{1/2}\mu)^\sigma),\quad \psi_p\mapsto\psi_p^\sigma$$

commutes with the operation of the Hecke algebra and we see that

$$\mathrm{Indn}(\delta_{3,p}^{1/2}\mu)=\mathrm{Indn}_E(\delta_{3,p}^{1/2}\mu)\otimes\mathbb{C},$$

where $\mathrm{Indn}_E(\delta_{3,p}^{1/2}\mu):=\mathrm{Indn}(\delta_{3,p}^{1/2}\mu)^{\mathrm{Aut}(\mathbb{C}/E)}$. This immediately implies that the character $\mu\delta_{3,p}^{1/2}$ is E-valued and since ψ_p^1 has values in $\mathbb{Q}(\delta_{3,p}^{1/2}\mu)$ we see that $\psi_p^1\in\mathrm{Indn}_E(\delta_{3,p}^{1/2}\mu)$. For later purpose we mention the consequence

$$\gamma=\delta_{3,p}^{1/2}\mu\begin{pmatrix}1&&\\&p&\\&&p^2\end{pmatrix}\in E.\tag{3.2}$$

Again, Proposition 3.2 in [Cl], tells us that after multiplication by a complex number the isomorphism

$$\mathrm{Indn}_E(\delta_{3,p}^{1/2}\mu)\to W_E(\pi_p,\tau_p),\quad \psi_p\mapsto\int_{N_3(\mathbb{Q}_p)}\psi_p(w_0ng)\bar{\tau}_p(n)\,dn$$

respects the E-subspaces and some complex multiple of w_p therefore is contained in $W_E(\pi_p,\tau_p)$.

In order to examine w_q (cf. Section 1 for the definition) we shall use the twisted action of automorphisms $\sigma\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on the Whittaker model

$$\begin{array}{llll}\tilde{\sigma}: & W(\pi_q,\tau_q) & \to & W(\pi_q^\sigma,\tau_q)\\ & w & \mapsto & w^{\tilde{\sigma}}(g):=w(\mathrm{diag}(t_\sigma^{-2},t_\sigma^{-1},1)g)^\sigma,\end{array}$$

where $\sigma\mapsto\sigma|_{\mathbb{Q}(\zeta_{q^\infty})}\mapsto t_\sigma\in\mathbb{Z}_q^*$ is the q-component of the cyclotomic character. Since π_q is defined over E, we deduce that $W_{E'}(\pi_q,\tau_q)=W(\pi_q,\tau_q)^{\mathrm{Aut}(\mathbb{C}/E')}$ for any finite extension E'/E. It therefore suffices to verify that $w_q^{\tilde\sigma} = w_q$ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/E(\zeta_{q(q-1)}))$. In view of (3.2) Proposition in [J-S 2] it is enough to show that $\mathrm{res}_{\mathrm{GL}_2(\mathbb{Q}_q)}(w_q^{\tilde\sigma}) = \mathrm{res}_{\mathrm{GL}_2(\mathbb{Q}_q)} w_q$ (note that $w_q, w_q^{\tilde\sigma} \in W(\pi_q, \tau_q)$ for $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$). Using the equation

$$\begin{pmatrix} t^2 & \\ & t \end{pmatrix}\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} k = \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 1-t & 1 \end{pmatrix} k \begin{pmatrix} t & \\ & t \end{pmatrix}.$$

and taking into account that $t_\sigma \equiv 1 \pmod q$ for $\sigma \in \mathrm{Aut}(\mathbb{C}/E(\zeta_q))$ and $f_{\eta_q} = q$, it is easily verified that

$$\mathrm{res}_{\mathrm{GL}_2(\mathbb{Q}_q)} w_q(\mathrm{diag}(t_\sigma^{-2}, t_\sigma^{-1})g) = \mathrm{res}_{\mathrm{GL}_2(\mathbb{Q}_q)} w_q(g)$$

for all $g \in \mathrm{GL}_2(\mathbb{Q}_q)$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/E(\zeta_{q(q-1)}))$. Applying (the untwisted) σ to this equation and recalling that w_q is $\mathbb{Q}(\zeta_{q-1})$-valued we obtain the invariance of w_q under $\mathrm{Aut}(\mathbb{C}/E(\zeta_{q(q-1)}))$. This finishes the proof of the lemma.

Lemma 3.2 together with the algebraicity of the forms $\Omega(\pi)^{-1}\omega$ and ω_{p^e} now immediately implies

THEOREM 2. *For all pairs (η, l) consisting of a character $\eta : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of finite order and conductor $f_\eta = pq$ and an integer $0 < l \leqslant l_0/2$ such that $l \equiv \flat \pmod 2$ the distributions $\mu_{\pi,l}^\eta$ are $\Omega(\pi) \cdot E_\pi E_\kappa(\eta, \eta', \zeta_{q(q-1)})$-valued, i.e. for any open subset $U \leqslant \mathbb{Z}_p^*$ we have*

$$\frac{\mu_{\pi,l}^\eta(U)}{\Omega(\pi)} \in E_\pi E_\kappa(\eta, \eta', \zeta_{q(q-1)}).$$

Of course, Equation (3.1) calculating the Mellin transform of $\mu_{\pi,l}^\eta$ becomes completely trivial if $P_l(1/2)$ vanishes and the distribution $\mu_{\pi,l}^\eta$ fails to interpolate the automorphic L-function. Hence we have to make the

Assumption. $P_l(1/2)$ does not vanish.

Using Corollary 1 and Equation (3.2) we then obtain

COROLLARY 2. *Under the above assumption, for all characters $\eta: \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of finite order and conductor $f_\eta = pq$ and all integers $0 < l \leqslant l_0/2$, which are congruent to $\flat \pmod 2$ we obtain*

$$\frac{L(\pi \otimes \chi\eta, 1-l)}{P_l^{-1}(1/2)\boldsymbol{\pi}^{3l-\flat}\Omega(\pi)} \in E_\pi E_\kappa(\eta, \eta', \zeta_{q(q-1)}, \chi).$$

Remark 3.5. The case of trivial coefficients. We finally consider the case that π has cohomology with trivial coefficients. This means $l_0 = 2$, hence $l = 1$, i.e. $\Pi(\chi)$ too has cohomology with trivial coefficients and the only critical integers $s = 0, 1$ occur only if $\eta_\infty = \mathrm{sgn}$. Moreover we know: *The value $P_1(1/2)$ does not vanish.*

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Proof. The generator e^1 is contained in the $+1$-eigenspace of the cohomology (cf. Remark 3.3). Our polynomial P_1 therefore coincides up to a nonzero factor with the one chosen in [Sch 2], Theorem 3.8 and the claim follows.

In particular, multiplying the infinity component ω_∞ with some scalar we may assume that $P_1(1/2) = 1$.

In the case of trivial coefficients Corollary 2 has already been obtained in [Ma], Corollary 3.3 using a different method.

4. The Denominators of Eisenstein Classes

The remaining part of this article is devoted to an investigation of the denominators of $\mu^\eta_{\pi,l}/\Omega(\pi)$ in $E_\pi E_\kappa(\eta, \eta', \zeta_{q(q-1)})$. This is a more arithmetic question since it involves the ring of integers of the field $E_\pi E_\kappa(\eta, \eta', \zeta_{q(q-1)})$ and to answer it we will make use of cohomology with coefficients in rings of integers. As we will see in Section 5.1 the essential step is to calculate (bounds for) the denominators of the Eisenstein classes ω_{p^e}. This will be the object of this chapter. In special cases such denominators have been computed in [Ha 2] and [Kai] and in a large part we are relying on their expositions.

For simplicity, from now on we will restrict ourselves to the case of trivial coefficients as described in Remark 3.5, i.e. we assume that $(l_0, l, \eta_\infty) = (2, 1, \mathrm{sgn})$. In particular, there is only one distribution $\mu_\pi = \mu^\eta_{\pi,1}$, which corresponds to the (only) non-positive critical integer $s = 0$.

The idea is to construct a system of generating cycles for $H_1(S_2(K), \mathbb{Z})$, where $K \leqslant \mathrm{GL}_2(\mathbb{A}_f)$ is the (largest) subgroup under which ω_{p^e} is invariant and to evaluate ω_{p^e} on these cycles. In particular, for any number field F with ring of integers $\mathcal{O}_F$ we set

$$H^\bullet(S_n(K), \mathcal{O}_F)_{\mathrm{int}} = \mathrm{Im}\Big(H^\bullet(S_n(K), \mathbb{Z}) \otimes \mathcal{O}_F \to H^*(S_n(K), F)\Big).$$

In the same way we define $H^\bullet(S_n(K), \mathcal{O}_{F,\mathcal{L}})_{\mathrm{int}}$, where $\mathcal{O}_{F,\mathcal{L}}$ denotes the completion of $\mathcal{O}_F$ at the prime ideal $\mathcal{L}$. The following remark yields a slight simplification of the calculation of the denominators.

Using exactly the same reasoning as in Section 3.1 we see that the induced representation $\mathrm{Ind}(\alpha^{1/2}, \eta_0\chi\alpha^{-1/2}) = \eta'^{-1} \otimes \Pi(\chi)$ too occurs in the boundary cohomology of $\bar{S}_2$. We denote by $\psi^0_{p^e,f}$ the finite part of ψ^0_{1,p^e} (cf. Section 2.1). $\psi^0_{p^e,f}$ is invariant under $K_1(p^e, q) := K_1(2, p^e) \times K_1(2, q)$ as well as under the operation of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\eta_0))$ and therefore defines an element in $H^1(\partial\bar{S}_2(K_1^f(p^e, q)), \mathbb{Q}(\eta_0))$, where $K_1^f(p^e, q) = \prod_{\ell \neq p,q,\infty} \mathrm{GL}_2(\mathbb{Z}_\ell)\, K_1(p^e, q)$. We let

$$e^1_0 = \sum_{i=1,2} \psi^0_{\infty,i}\omega_i \in H^1(\mathrm{gl}_2, K_{2,\infty}Z^0_2(\mathbb{R}), \mathrm{Ind}(\alpha^{1/2}_\infty, \alpha^{-1/2}_\infty))$$

be a generator of the cohomology at infinity and we set

$$\omega^0_{p^e}(g, D) := \sum_\gamma \psi^0_{p^e,f}\, e^1_0(\gamma g, D).$$

As in Section 3.1 we then deduce $\omega^0_{p^e} \in H^1(S_2(K^f_1(p^e, q)), \mathbb{Q}(\eta_0))$ and also that $\omega_{p^e} = \eta' \otimes \omega^0_{p^e}$. In particular ω_{p^e} and $\omega^0_{p^e}$ have the same denominators (in $\mathbb{Q}(\eta, \eta')$) and we may (and will) replace ω_{p^e} by $\omega^0_{p^e}$ for the calculation of the denominators.

4.1. SPECIAL CYCLES IN THE HOMOLOGY

We use the following notations: $\mathbb{H} = \mathrm{GL}_2^+(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})Z_2^0(\mathbb{R})$ is the upper half plane, Γ resp. $\Gamma_1(m)$ the full group $\mathrm{GL}_2(\mathbb{Z})$ resp. the congruence subgroup consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying $c \equiv 0$ and $d \equiv 1 \bmod m$ and $\bar{\mathbb{H}}$ is the Borel–Serre compactification. On the level of sets, $\bar{\mathbb{H}}$ is the disjoint union

$$\bar{\mathbb{H}} = \mathbb{H} \cup \bigcup_{s \in \mathbb{P}^1(\mathbb{Q})} \mathbb{H}_{s,\infty},$$

where $\mathbb{H}_{s,\infty}$ is the boundary component at the cusp s. In this section we denote by $\mathcal{O}$ the ring of integers of $\mathbb{Q}(\eta_0)$. We also denote by $\mathcal{O}$ the one-dimensional $\mathcal{O}$-module with trivial $\Gamma_1(p^e q)$-operation; the Shapiro Isomorphism in homology then reads

$$H_i(\Gamma_1(p^e q)\backslash\mathbb{H}, \mathcal{O}) \xrightarrow{\sim} H_i(\Gamma\backslash\mathbb{H}, \mathrm{ind}^{\Gamma}_{\Gamma_1(p^e q)\langle\pm 1\rangle}\mathcal{O}),$$

where an element $c \otimes \alpha$ of the right hand side is being sent to $\sum_{\gamma \in \Gamma_1(p^e q)\langle\pm 1\rangle\backslash\Gamma} \alpha(\gamma)\gamma c$. The same isomorphism also holds for the relative homology. (For a definition of homology with non-trivial coefficients and its properties cf. [Ha 2], ch. E). We put $M := \mathrm{ind}^{\Gamma}_{\Gamma_1(p^e)\langle\pm 1\rangle}\mathcal{O}$. We want to explicitely construct a set of cycles generating $H_1(\Gamma_1(p^e q)\backslash\mathbb{H}, \mathcal{O})$ and we proceed as follows. We let $B(c, s)$, $s \in P^1(\mathbb{Q})$ be the set of all points $g \in \mathbb{H}$, whose distance to s is equal to c. For any $r \in P^1(\mathbb{R})\backslash\{s\}$ the intersection of $B(c, s)$ with the geodesic $\mathcal{Z}_{r,s}$ joining r and s consists of a single point $g_r \in \mathbb{H}$ and the assignment $r \mapsto g_r$ yields a natural identification $\mathbb{H}_{s,\infty} \leftrightarrow P^1(\mathbb{R})\backslash\{s\}$. We write $\{r\}_s$ to denote the point on the boundary component belonging to the cusp s and corresponding to $r \in P^1(\mathbb{R})\backslash\{s\}$ and we let $\mathcal{Z}_{[0,\infty]}$ be the geodesic in $\mathbb{H}$ running from $\{0\}_\infty$ to $\{\infty\}_0$, where by ∞ we understand the cusp belonging to the standard Borel subgroup $\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$. The first relative homology then consists of the cycles $\mathcal{Z}_{[0,\infty]} \otimes \varphi$, $\varphi \in M$. We want to find out, which of the cycles $\mathcal{Z}_{[0,\infty]} \otimes \varphi$ are images of absolute cycles. Using the long exact homology sequence

$$\cdots \to H_i(\Gamma\backslash\mathbb{H}, M) \xrightarrow{\mathrm{rel}} H_i(\Gamma\backslash\bar{\mathbb{H}}, \partial\Gamma\backslash\bar{\mathbb{H}}, M) \xrightarrow{\partial} H_{i-1}(\partial\Gamma\backslash\bar{\mathbb{H}}, M) \to \cdots$$

we see that $\mathcal{Z}_{[0,\infty]} \otimes \varphi$ is contained in the image of rel, precisely if

$\partial(\mathcal{Z}_{[0,\infty]} \otimes \varphi) = \{0\}_\infty \otimes (\varphi - w\varphi) = 0$. Since $\partial\Gamma\backslash\bar{\mathbb{H}} \cong N_2(\mathbb{Z})\backslash\mathbb{H}$ we find

$$H_0(\partial\Gamma\backslash\bar{\mathbb{H}}, M) \cong M/(1-T)M$$

where $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$. Thus, $\mathcal{Z}_{[0,\infty]} \otimes \varphi \in \operatorname{im} \operatorname{rel}$ is equivalent to

$$(\varphi - w\varphi) = (1-T)\varphi' \tag{4.1}$$

for some $\varphi' \in M$. Since $(1-T)\varphi' = \partial([0,1]_\infty \otimes \varphi')$ this is equivalent to $\partial(\mathcal{Z}_{[0,\infty]} \otimes \varphi) = \partial([0,1]_\infty \otimes \varphi')$. Thus, assuming (4.1) we see that $\mathcal{Z}_{[0,\infty]} \otimes \varphi - [0,1]_\infty \otimes \varphi'$ represents an absolute cycle, which maps to $\mathcal{Z}_{[0,\infty]} \otimes \varphi$ under rel. Applying the Shapiro Isomorphism this cycle reads $\mathcal{Z}_\varphi := \mathcal{Z}^i_\varphi - \mathcal{Z}^b_\varphi$, where

$$\mathcal{Z}^i_\varphi := \sum_\gamma \varphi(\gamma)\,\gamma\mathcal{Z}_{[0,\infty]} \quad \text{and} \quad \mathcal{Z}^b_\varphi := \sum_\gamma \varphi'(\gamma)\,\gamma[0,1]_\infty$$

and we conclude: modulo cycles which are supported on the boundary, $H_1(\Gamma_1(p^e q)\backslash\bar{\mathbb{H}}, \mathcal{O})$ is generated by the cycles $\mathcal{Z}_\varphi$ with $\varphi \in M$ satisfying (4.1).

We identify the coset spaces $\Gamma_1(p^e q)\backslash\Gamma = K_1(p^e, q)\backslash \mathrm{GL}_2(\mathbb{Z}_p) \times \mathrm{GL}_2(\mathbb{Z}_q)$. Any (relative) cycle $\mathcal{Z}^i_\varphi$ is a $\mathcal{O}$-linear combination of the chains $g\mathcal{Z}_{[0,\infty]}$ for $g \in K_1(p^e, q)\backslash \mathrm{GL}_2(\mathbb{Z}_p) \times \mathrm{GL}_2(\mathbb{Z}_q)$. Moreover, we have

$$g\mathcal{Z}_{[0,\infty]} = -gw\mathcal{Z}_{[0,\infty]}$$

where $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ and it is therefore sufficient to consider translates by elements $g \in K_1(p^e, q)\backslash \mathrm{GL}_2(\mathbb{Z}_p) \times \mathrm{GL}_2(\mathbb{Z}_q)/\langle w\rangle$ (w is embedded diagonally into $\mathrm{GL}_2(\mathbb{Z}_p) \times \mathrm{GL}_2(\mathbb{Z}_q)$).

LEMMA 4.1. *The union of the following elements forms a system of representatives for the double coset space* $K_1(p^e, q)\backslash \mathrm{GL}_2(\mathbb{Z}_p) \times \mathrm{GL}_2(\mathbb{Z}_q)/\langle w\rangle$*:*

$$\begin{pmatrix} 1 & \\ & d \end{pmatrix} w \begin{pmatrix} 1 & p^k \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ & d \end{pmatrix} w \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix},$$

where $d \in (\mathbb{Z}/p^{e-k}q\mathbb{Z})^*$, $t \in (\mathbb{Z}/p^e q\mathbb{Z})^*$ *and* $k = 0, \ldots, e$,

$$\begin{pmatrix} 1 & \\ & d \end{pmatrix} w \begin{pmatrix} 1 & p^k \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ & d \end{pmatrix} w,$$

where $d \in (\mathbb{Z}/p^{e-k}q\mathbb{Z})^*$, $t \in (\mathbb{Z}/p^e\mathbb{Z})^*$ *and* $k = 0, \ldots, e$ *and*

$$\begin{pmatrix} 1 & \\ & d \end{pmatrix} w \begin{pmatrix} 1 & p^k \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ & d \end{pmatrix},$$

where $d \in (\mathbb{Z}/p^{e-k}q\mathbb{Z})^*$, $t \in (\mathbb{Z}/p^e\mathbb{Z})^*$ *and* $k = 0, \ldots, e$. *Here, we identify* $(\mathbb{Z}/p^e q\mathbb{Z})^* = (\mathbb{Z}/p^e\mathbb{Z})^* \times (\mathbb{Z}/q\mathbb{Z})^*$.

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Proof. We start from the decomposition

$$\mathrm{GL}_2(\mathbb{Z}_p) = \dot{\cup}_{i=0,\ldots,e} K_0(2,p^e)\begin{pmatrix}1 & \\ p^i & 1\end{pmatrix}K_0(2,p^e).$$

For any $i > 0$ we have the inclusion

$$K_0(2,p^e)\begin{pmatrix}1 & \\ p^i & 1\end{pmatrix}K_0(2,p^e)w \subset K_0(2,p^e)\begin{pmatrix}1 & \\ 1 & 1\end{pmatrix}K_0(2,p^e).$$

Furthermore the decomposition

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}b - ad/c & -a\\ & -c\end{pmatrix} w \begin{pmatrix}1 & d/c\\ & 1\end{pmatrix}$$

proves that any element $g \in K_1(2,p^e)\backslash K_0(2,p^e)\left(\begin{smallmatrix}1 & \\ 1 & 1\end{smallmatrix}\right)K_0(2,p^e)$ has a representative

$$g = \begin{pmatrix}1 & \\ & d\end{pmatrix} w \begin{pmatrix}1 & u\\ & 1\end{pmatrix},$$

where $d \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^*$ and $u \in \mathbb{Z}_p/p^e\mathbb{Z}_p$. We write u in the form $u = p^k t$, $t \in \mathbb{Z}_p^*, k = 0, \ldots, e$ and finally obtain that there is a representative of the form

$$g = \begin{pmatrix}1 & \\ & d\end{pmatrix} w \begin{pmatrix}1 & p^k\\ & 1\end{pmatrix}\begin{pmatrix}t & \\ & 1\end{pmatrix}$$

where $t \in (\mathbb{Z}/p^e\mathbb{Z})^*$ and even $d \in (\mathbb{Z}/p^{e-k}\mathbb{Z})^*$. In a quite similar way we see that any coset in $K_1(2,q)\backslash \mathrm{GL}_2(\mathbb{Z}_q)$ has a representative either of the form

$$\begin{pmatrix}1 & \\ & d\end{pmatrix} w \begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\begin{pmatrix}t & \\ & 1\end{pmatrix}$$

with $d, t \in (\mathbb{Z}/q\mathbb{Z})^*$ or of one of the forms

$$\begin{pmatrix}1 & \\ & d\end{pmatrix} w, \quad \begin{pmatrix}1 & \\ & d\end{pmatrix}$$

where $d \in (\mathbb{Z}/q\mathbb{Z})^*$. This implies that any double coset has a representative of the desired form and since it is easily verified that these elements yield different cosets, the proof of the Lemma is complete.

EXAMPLE 4.2. We label the matrices appearing in the above lemma by $g_{d,k,t}$, $g'_{d,k,t}$ and $g''_{d,k,t}$. Since $\Gamma_1(N) = \Gamma(N) \cdot N_2(\mathbb{Z})$ we see that a set of representatives for the cusps of $\Gamma_1(p^e q)\backslash\mathbb{H}$ is given by ∞ and the fractions r/s with $(r,s) = 1$, $0 \leqslant s < N$, $r < s$ and two cusps r/s and r'/s' are equivalent if $r/s - r'/s' \in \mathbb{Z}$ or $(r,s) \equiv (r',s') \pmod N$. In particular we deduce that the chain $g_{d,k,t}\mathcal{Z}_{[0,\infty]}$ runs from the cusp 0 to $1/d^2tp^k$. Hence, for any $d, d' \in (\mathbb{Z}/p^{(e-k)}q\mathbb{Z})^*$, $t, t' \in (\mathbb{Z}/p^e q\mathbb{Z})^*$ such

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that $d^2 t \equiv d'^2 t' \pmod{p^{e-k} q}$ the chain

$$\mathcal{Z} = g_{d,k,t} \mathcal{Z}_{[0,\infty]} - g_{d',k,t'} \mathcal{Z}_{[0,\infty]}$$

is a relative cycle which is the image of an absolute cycle $\mathcal{Z}_{d,d',t,t',k}$ in $H_1(\Gamma_1(p^e q)\backslash\bar{\mathbb{H}}, \mathbb{Z})$.

4.2. THE INTEGRAL ON THE BOUNDARY

We want to evaluate the Eisenstein class $\omega^0_{p^e}$ on the cycles $\mathcal{Z}_\varphi$. We do this by seperately evaluating $\omega^0_{p^e}$ on the inner and the boundary component of $\mathcal{Z}_\varphi$ and we start with the evaluation on the boundary component. We denote by

$$\omega^0_{p^e}|_{\partial\bar{S}_2} \in H^1(\partial\bar{S}_2(K_1^f(p^e, q)), \mathbb{Q}(\eta_0))$$

the restriction of $\omega^0_{p^e}$ to the boundary of $\bar{S}_2(K_1^f(p^e q))$.

LEMMA 4.3. *$\omega^0_{p^e}|_{\partial\bar{S}_2}$ satisfies the properties*

- *$\omega^0_{p^e}|_{\partial\bar{S}_2}$ is contained in $H^1(\partial\bar{S}_2(K_1^f(p^e, q)), \mathcal{O})_{\mathrm{int}}$.*
- *$\omega^0_{p^e}|_{\partial\bar{S}_2}$ is only supported at boundary components belonging to the cusps $\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)\infty$, $d \in (\mathbb{Z}_q/q\mathbb{Z}_q)^*$.*

Proof. The boundary $\partial\bar{S}_2(K_1^f(p^e, q))$ is homotopy equivalent to the space

$$B_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/K_1^f(p^e, q)K_{2,\infty}Z_2^0(\mathbb{R})$$

and under this equivalence $\omega^0_{p^e}|_{\partial\bar{S}_2}$ corresponds to the cocycle $\psi^0_{p^e,f}\, e^1_0$. We will explicitely describe the section $\psi^0_{p^e,f}$. We set

$$\mathcal{M}_e = B_2(\mathbb{Q}) \prod_{\ell \neq p,q,\infty} \mathrm{GL}_2(\mathbb{Z}_\ell) K_0(2, p^e) K_0(2, q)$$

and write any $g \in \mathcal{M}_e$ in the form $g = \left(\begin{smallmatrix}\alpha & * \\ & \beta\end{smallmatrix}\right) k k_p k_q$, where

$$k_p = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \in K_0(2, p^{e_v}), \qquad k_q = \begin{pmatrix} a_q & b_q \\ c_q & d_q \end{pmatrix} \in K_0(2, q).$$

Note that using strong approximation for $(\mathbb{A}, +)$ we always can achieve that $b_p \equiv 0 \pmod{p^k}$ for any $k \in \mathbb{N}$. Using the decomposition

$$\mathrm{GL}_2(\mathbb{A}_f) = B_2(\mathbb{Q}) \prod_{\ell \neq \infty} \mathrm{GL}_2(\mathbb{Z}_\ell)$$

and left $B_2(\mathbb{Q})$-invariance of ψ^0_v as well as the explicit description of the local components $\psi^0_{v,p}$ and $\psi^0_{v,q}$ (cf. Section 1) we see that the finite part $\psi^0_{v;f}$ for $v \neq 1$ is

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supported on $\mathcal{M}_{e_v}$ and for $g \in \mathcal{M}_{e_v}$ as above we have

$$\psi^0_{v;f}(g) = |\alpha/\beta|^{-1}_\infty \eta_{0,q}(d_q)v_p(d_p).$$

On the other hand, for $e_v \geqslant 1$ we have

$$\mathcal{M}_{e_v}\begin{pmatrix} p^m & \\ & 1 \end{pmatrix} = \mathcal{M}_{e_v+m}$$

(choose $b_p \equiv 0 \pmod{p^k}$ with k sufficiently large) and we deduce that the finite part $\psi^0_{v,p^e;f}$ is supported on $\mathcal{M}_e$. Moreover, for $g \in \mathcal{M}_e$ we obtain

$$\psi^0_{v,p^e;f}(g) = |\alpha/\beta|^{-1}_\infty \eta_{0,q}(d_q)v_p(d_p).$$

In the same way we see that $\psi^0_{1,p^e;f}$ is supported on $\mathcal{M}_e$ and for $g \in \mathcal{M}_e$ the section $\psi^0_{1,p^e;f}$ is given by

$$\psi^0_{1,p^e;f}(g) = |\alpha/\beta|^{-1}_\infty \eta_{0,q}(d_q).$$

The Fourier transform defining $\psi^0_{p^e,f} = \psi^0_{1,p^e,f}$ is now easily evaluated: $\psi^0_{p^e,f}$ is supported on

$$B_2(\mathbb{Q}) \prod_{\ell \neq p,q,\infty} \mathrm{GL}_2(\mathbb{Z}_\ell)K_0(2,p^e)K_0(2,q)$$

and for $g \in \mathcal{M}_e$ we have

$$\psi^0_{p^e,f}(g) = \begin{cases} |\alpha/\beta|^{-1}_\infty\, \eta_{0,q}(d_q), & \text{for } d_p \equiv \pm 1 \pmod{p^e}, \\ 0, & \text{else.} \end{cases}$$

In particular we see that the restriction of $\psi^0_{p^e,f}$ to $\mathrm{GL}_2(\hat{\mathbb{Z}})$ is $\mathcal{O}$-valued, which by an easy calculation (cf. [Kai], 2.3.3 Korollar) implies that the cohomology class $\psi^0_{p^e,f}\, e^1_0$ is contained in $H^1(\partial\bar{S}_2(K^f_1(p^e,q)), \mathcal{O})_{\mathrm{int}}$. Moreover since $\psi^0_{p^e,f}$ is supported only on the cusps $\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)\infty$, where $d \in (\mathbb{Z}_q/q\mathbb{Z}_q)^*$, the second claim of the lemma follows and the proof therefore is finished.

Using the lemma we deduce that

$$\int_{\mathcal{Z}^b_\varphi} \omega^0_{p^e} = \sum_{d \in (\mathbb{Z}_q/q\mathbb{Z}_q)^*} \varphi'\begin{pmatrix} 1 & \\ & d \end{pmatrix} \int_{\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)[0,1]_\infty} \omega^0_{p^e}|_{\partial\bar{S}_2}. \tag{4.2}$$

The width of the cusps $\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)\infty$ is 1 and $\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)[0,1]_\infty$ therefore is closed, i.e. contained in $H_1(\partial\bar{S}_2(K^f_1(p^e,q)), \mathbb{Z})$. Hence, the integral of $\omega^0_{p^e}$ on $\mathcal{Z}^b_\varphi$ is contained in $\mathcal{O}$. In other words: *No denominators are coming from the boundary.*

4.3. THE INNER INTEGRAL

In this section we complete the determination of the denominators of $\omega^0_{p^e}$ by calculating the values of $\omega^0_{p^e}$ on the inner components $\mathcal{Z}^i_\varphi$. In the following we distinguish between cocycles in the relative Lie-algebra cohomology $\omega^0_{p^e}$ and their

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associated cocycles $\tilde{\omega}^0_{p^e}$ in the de-Rham cohomology. We will evaluate $\tilde{\omega}^0_{p^e}$ seperately on the chains $g_{d,k,t}\mathcal{Z}_{[0,\infty]}$, $g'_{d,k,t}\mathcal{Z}_{[0,\infty]}$ and $g''_{d,k,t}\mathcal{Z}_{[0,\infty]}$ and we begin with the first one. In fact we will evaluate $\tilde{\omega}^0_{p^e}$ on a slightly more general chain: we denote by T_1 the torus in GL_2 consisting of elements of the form $\begin{pmatrix} * & \\ & 1 \end{pmatrix}$, i.e. $T_1 \cong \mathbb{G}_m$. For $d \in (\mathbb{Z}/p^{e-k}q\mathbb{Z})^*$, $k = 0, \dots, e$ and any map $\lambda\colon (\mathbb{Z}/p^e q\mathbb{Z})^* \to \mathcal{O}$ we define the relative cycles

$$\mathcal{Z}_{d,k,\lambda} := \sum_{t \in T_1(\mathbb{Z}/p^e q)} \lambda(t)\, g_{d,k}\, t\, \mathcal{Z}_{[0,\infty]} \subset \mathbb{H},$$

where

$$g_{d,k} = \begin{pmatrix} 1 & \\ & d \end{pmatrix} (w, w) \left(\begin{pmatrix} 1 & p^k \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \right)$$

and $g_{d,k}\, t$ acts via its image in $\Gamma_1(p^e q)\backslash \mathrm{SL}_2(\mathbb{Z})$. These chains translate into an adelic setting as follows. The adelization of the symmetric space $\Gamma_1(p^e q)\backslash\mathbb{H}$ reads $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/ K_1^f(p^e, q) K_{2,\infty} Z_2^0(\mathbb{R})$ and for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we obtain the canonical map

$$\begin{array}{ccc} \mathbb{R}^*_{>0} & \to & \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/K_1^f(p^e, q) K_{2,\infty} Z_2^0(\mathbb{R}) \\ t_\infty & \mapsto & \left(\gamma \begin{pmatrix} t_\infty & \\ & 1 \end{pmatrix}, 1, \dots, 1 \right). \end{array}$$

Therefore, in the adelic symmetric space the chain $\gamma\mathcal{Z}_{[0,\infty]}$ equals $\mathcal{Z}_{[0,\infty]} \times \mathbf{g}^{-1}$ where $\mathbf{g} = (g_\ell) \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ is any representative of the coset $K_1^f(p^e, q)(\gamma, \dots, \gamma) \in K_1^f(p^e, q)\backslash \mathrm{GL}_2(\hat{\mathbb{Z}})$. We therefore have to calculate

$$\int_{\mathcal{Z}_{d,k,\lambda}} \tilde{\omega}^0_{p^e} = \sum_{t \in T_1(\mathbb{Z}/p^e q)} \lambda(t) \int_{\mathcal{Z}_{[0,\infty]} \times t^{-1} \mathbf{g}_{d,k}^{-1}} \tilde{\omega}^0_{p^e},$$

where $\mathbf{g}_{d,k} = (g_\ell) \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ with $(g_p, g_q) = g_{d,k}$ and all components outside p, q equal to 1 and $t \in T_1(\mathbb{Z}/p^e q\mathbb{Z})$ also stands for the adèlic matrix $(t_\ell) \in T_1(\hat{\mathbb{Z}})$ with $t_p \equiv t \pmod{p^e}$, $t_q \equiv t \pmod{q}$ and $t_\ell = 1$ for $\ell \neq p, q$. We parametrize our cycle $\mathcal{Z}_{[0,\infty]}$ as follows

$$\begin{array}{rl} \sigma\colon\quad T_1(\mathbb{R})^0 = \mathbb{R}^*_{>0} & \to \mathcal{Z}_{[0,\infty]} \subset \mathbb{H} \\ t_\infty & \mapsto \begin{pmatrix} t_\infty & \\ & 1 \end{pmatrix} i \end{array}$$

and obtain

$$\int_{\mathcal{Z}_{[0,\infty]} \times t^{-1} \mathbf{g}_{d,k}^{-1}} \tilde{\omega}^0_{p^e} = \int_{T_1(\mathbb{R})^0} \omega^0_{p^e}\left((t_\infty, t^{-1} \mathbf{g}_{d,k}^{-1}), D_{L_{t_\infty^{-1}}} \circ D_\sigma(t_\infty \frac{\partial}{\partial t_\infty}) \right) \frac{dt_\infty}{t_\infty},$$

where $D_{L_{t_\infty^{-1}}}$ denotes the derivative of the left translation by t_∞^{-1} and dt_∞ resp. $\partial/\partial t_\infty$ is a Haar measure resp. an invariant tangent vector field on $(\mathbb{R}, +)$. Taking into

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account that

$$D_{L_{t_\infty^{-1}}} \circ D_\sigma \left(t_\infty \frac{\partial}{\partial t_\infty} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(cf. [Ha 2], ch. 6.2, p. 28, [Kai], ch. 3.1) we obtain

$$\int_{\mathcal{Z}_{d,k,\lambda}} \tilde{\omega}^0_{p^e} = \sum_{t \in T_1(\mathbb{Z}/p^e q)} \lambda(t) \int_{T_1(\mathbb{R})^0} \omega^0_{p^e} \left((t_\infty, t^{-1} \mathbf{g}^{-1}_{d,k}), \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \frac{dt_\infty}{t_\infty}.$$

We adèlize λ as follows. We set

$$T_{1,p^e q} := \{ x \in T_1(\hat{\mathbb{Z}}) \colon x_p \equiv 1 \pmod{p^e},\ x_q \equiv 1 \pmod{q} \},$$

i.e. we have $T_1(\hat{\mathbb{Z}})/T_{1,p^e q} = (\mathbb{Z}/p^e q \mathbb{Z})^*$. Any $\mathbf{t} \in T_1(\mathbb{A})$ uniquely decomposes as $\mathbf{t} = r t_\infty k$ with $r \in T_1(\mathbb{Q})$, $t_\infty \in T_1^0(\mathbb{R}) = \mathbb{R}^*_{>0}$, $k \in T_1(\hat{\mathbb{Z}})$ and we define

$$\tilde{\lambda} \colon T_1(\mathbb{Q}) \backslash T_1(\mathbb{A}) \to \mathcal{O}$$

by setting $\tilde{\lambda}(\mathbf{t}) := \lambda(k^{-1} \cdot T_{1,p^e q})$. Using adèlic variables the above integral now reads

$$\int_{\mathcal{Z}_{d,k,\lambda}} \tilde{\omega}^0_{p^e} = \operatorname{vol}(1 + p^e \mathbb{Z}_p \times 1 + q \mathbb{Z}_q)^{-1} \int_{T_1(\mathbb{Q}) \backslash T_1(\mathbb{A})} \tilde{\lambda}(\mathbf{t})\, \omega^0_{p^e} \left(\mathbf{t} \cdot \mathbf{g}^{-1}_{d,k}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) d\mathbf{t},$$

where $d\mathbf{t} = \prod_\ell dt_\ell$ with local measures

$$dt_\ell = \begin{cases} dt_\infty / |t_\infty| & \text{if} \quad \ell = \infty \\ \text{Haar measure on } \mathbb{Z}_\ell^* \text{ with } \int_{\mathbb{Z}_\ell^*} dt_\ell = 1 & \text{if} \quad \ell \neq \infty. \end{cases}$$

To proceed further we replace $\omega^0_{p^e} = \sum_{\gamma \in B_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{Q})} \gamma \cdot \psi^0_{p^e, f} e^1_0$ by its defining sum and split the summation over γ according to the decomposition of $\mathrm{GL}_2(\mathbb{Q})$ into disjoint $T_1(\mathbb{Q})$-orbits

$$\mathrm{GL}_2(\mathbb{Q}) = B_2(\mathbb{Q}) \dot{\cup} B_2(\mathbb{Q}) \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \dot{\cup} B_2(\mathbb{Q}) \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} T_1(\mathbb{Q}).$$

The integrals corresponding to the first two $T_1(\mathbb{Q})$-orbits vanish (cf. [Ha 2], ch. 6.2, p. 29, [Kai], ch. 3.1) and for the integral corresponding to the last orbit we obtain

$$\begin{aligned} &\int_{\mathcal{Z}_{d,k,\lambda}} \tilde{\omega}^0_{p^e} \\ &\quad = \operatorname{vol}(1 + p^e \mathbb{Z}_p \times 1 + q \mathbb{Z}_q)^{-1} \int_{T_1(\mathbb{A})} \tilde{\lambda}(\mathbf{t}) \psi^0_{p^e, f} e^1_0 \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{t} \cdot \mathbf{g}^{-1}_{d,k}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) d\mathbf{t}. \end{aligned}$$

Since any $\lambda \colon (\mathbb{Z}/p^e q \mathbb{Z})^* \to \mathbb{Z}$ can be written $\lambda = \lambda^+ + \lambda^-$, where $\lambda^\pm(x) = (\lambda(x) \pm \lambda(-x))/2$, we may assume that λ is either even or odd. For any $t_f \in T_1(\mathbb{A}_f)$ we set $\tilde{\lambda}_f(t_f) := \tilde{\lambda}(t_f, 1)$ and we also define $\tilde{\lambda}_\infty = \mathrm{id}$ if λ is even and $\tilde{\lambda}_\infty = \mathrm{sgn}$ if λ is odd.

$\tilde{\lambda}_f$ defines a map on the quotient space

$$\tilde{\lambda}_f\colon\ T_{1,>0}(\mathbb{Q})\backslash T_1(\mathbb{A}_f)/T_{1,p^eq} = (\mathbb{Z}/p^eq\mathbb{Z})^* \to \mathcal{O},$$

which coincides with $\lambda \circ (\ )^{-1}$ on $(\mathbb{Z}/p^eq\mathbb{Z})^*$ and $\tilde{\lambda}$ decomposes

$$\tilde{\lambda}(\mathbf{t}) = \tilde{\lambda}_f(t_f)\tilde{\lambda}_\infty(t_\infty)$$

for $\mathbf{t} = (t_f, t_\infty)$. Accordingly the adelic integral decomposes

$$\int_{\mathcal{Z}_{d,k,\lambda}} \tilde{\omega}^0_{p^e} = \int_{T_1(\mathbb{R})} \tilde{\lambda}_\infty(t_\infty) e^1_0\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} t_\infty, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) \frac{dt_\infty}{|t_\infty|} \times$$
$$\times\ \mathrm{vol}(1 + p^e\mathbb{Z}_p \times 1 + q\mathbb{Z}_q)^{-1} \int_{T_1(\mathbb{A}_f)} \tilde{\lambda}_f(t_f)\psi^0_{p^e,f}\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} t_f \mathbf{g}^{-1}_{d,k}\right) dt_f.$$

We denote the first resp. the second factor by I_∞ resp. I_f. Concerning I_∞ we have (cf. [Kai], ch. 3.2.3, or with minor changes in the case $\tilde{\lambda}_\infty = \mathrm{sgn}$– [Ha 2], ch. 6.2, p. 31)

$$I_\infty = \begin{cases} 0, & \text{if} \quad \tilde{\lambda}_\infty = \mathrm{id},\\ 2\Gamma(1)\Gamma(1)/\Gamma(2), & \text{if} \quad \tilde{\lambda}_\infty = \mathrm{sgn}.\end{cases}$$

In particular we are reduced to calculating I_f and in doing so to *odd* λ. Each odd λ is a $\mathcal{O}$-linear combination of the maps $\lambda_\epsilon\colon\ (\mathbb{Z}/p^eq\mathbb{Z})^* \to \mathbb{Z}$, where $\epsilon \in (\mathbb{Z}/p^eq\mathbb{Z})^*$ and λ_ϵ is given by

$$\lambda_\epsilon(t) := \begin{cases} 1, & \text{if} \quad t \equiv \epsilon \ (\mathrm{mod}\ p^eq),\\ -1, & \text{if} \quad t \equiv -\epsilon \ (\mathrm{mod}\ p^eq),\\ 0, & \text{else}.\end{cases}$$

Therefore, from now on we may assume $\lambda = \lambda_\epsilon$. Using the unique decomposition $\mathbb{A}^*_f = \mathbb{Q}^*_{>0} \times \hat{\mathbb{Z}}$ we obtain

$$I_f = \sum_{r \in T_{1,>0}(\mathbb{Q})} \int_{T_1(\hat{\mathbb{Z}})/T_{1,p^eq}} \tilde{\lambda}_{\epsilon,f}(rt_f)\psi^0_{p^e,f}\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} rt_f \mathbf{g}^{-1}_{d,k}\right) dt_f.$$

Since $\tilde{\lambda}_f(rt_f) = \lambda(t_f)$ for $r \in T_{1,>0}(\mathbb{Q})$, $t_f \in T_1(\hat{\mathbb{Z}})$ we further obtain $I_f = I_\epsilon - I_{-\epsilon}$, where

$$I_\epsilon = \sum_{r \in T_{1,>0}(\mathbb{Q})} \psi^0_{p^e,f}\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} r \begin{pmatrix}\epsilon^{-1} & \\ & 1\end{pmatrix} \mathbf{g}^{-1}_{d,k}\right).$$

We write $r \in \mathbb{Q}^*_{>0}$ in the form $r = a/b$ with $a, b \in \mathbb{Z}_{>0}$, $(a, b) = 1$. There are integers $x, y \in \mathbb{Z}$ such that $xb - ya = 1$ and we obtain the global Iwasawa decomposition

$$\begin{pmatrix}1 & \\ 1 & 1\end{pmatrix}\begin{pmatrix}r & \\ & 1\end{pmatrix} = \begin{pmatrix}a & \\ & b^{-1}\end{pmatrix}\begin{pmatrix}1 & -y/b\\ & 1\end{pmatrix}\begin{pmatrix}x & y\\ a & b\end{pmatrix}.$$

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The description of $\psi^0_{p^e,f}$, which we obtained in the proof of Lemma 4.2, implies that

$$I_\epsilon = \sum_{a,b\in\mathbb{Z}_{>0},\,(a,b)=1} |ab|_\infty^{-1}\,\psi^0_{p^e,f}\left(\begin{pmatrix} x & y \\ a & b\end{pmatrix}\begin{pmatrix}\epsilon^{-1} & \\ & 1\end{pmatrix} \times \mathbf{g}_{d,k}^{-1}\right). \tag{4.3}$$

Recalling the definition of $\mathbf{g}_{d,k}$ we see that

$$\begin{aligned}&\begin{pmatrix} x & y \\ a & b\end{pmatrix}\begin{pmatrix}\epsilon^{-1} & \\ & 1\end{pmatrix} \times \mathbf{g}_{d,k}^{-1}\\ &\quad = \left(\ldots, \begin{pmatrix} y-\epsilon^{-1}xp^k & -d^{-1}\epsilon^{-1}x \\ b-p^k\epsilon^{-1}a & -\epsilon^{-1}d^{-1}a\end{pmatrix}, \ldots, \begin{pmatrix} y-\epsilon^{-1}x & -d^{-1}\epsilon^{-1}x \\ b-\epsilon^{-1}a & -d^{-1}\epsilon^{-1}a\end{pmatrix}, \ldots\right)\end{aligned}$$

is contained in the support of $\psi^0_{p^e,f}$ if and only if there is an $x_0 \in (\mathbb{Z}/q\mathbb{Z})^*$ such that

$$a \equiv x_0,\ b \equiv \epsilon^{-1}x_0 \pmod q, \quad a \equiv d\epsilon,\ b \equiv dp^k \pmod{p^e}$$

or

$$a \equiv x_0,\ b \equiv \epsilon^{-1}x_0 \pmod q, \quad a \equiv -d\epsilon,\ b \equiv -dp^k \pmod{p^e}. \tag{4.4}$$

Thus we obtain (note that $\eta_{0,q}$ is even)

$$I_\epsilon = \eta_{0,q}^{-1}(\epsilon d) \sum_{x_0\in(\mathbb{Z}/q\mathbb{Z})^*} \eta_{0,q}(x_0) \sum_{a,b} \frac{1}{ab},$$

where $a, b \in \mathbb{Z}_{>0}$, $(a, b) = 1$ run over all pairs as in (4.4). For any $x \in (\mathbb{Z}/p^e q\mathbb{Z})^*$, $y \in (\mathbb{Z}/p^{e-k}q\mathbb{Z})^*$ we set

$$S_{x,y,k} := \frac{1}{p^k} \sum_{a,b} \frac{1}{ab},$$

where $a, b \in \mathbb{Z}$ run over all pairs satisfying $(a, b) = 1$ as well as $a \equiv x \pmod{p^e q}$ $b \equiv y \pmod{p^{e-k}q}$. In particular, since $\tilde{\omega}^0_{p^e}$ vanishes on even cycles we see that $\int_{\mathcal{Z}_{d,k,t}} \tilde{\omega}^0_{p^e} = \int_{\mathcal{Z}_{d,k,\lambda_t}} \tilde{\omega}^0_{p^e}$ and recalling that $I_f = I_\epsilon - I_{-\epsilon}$ we finally obtain

$$\int_{g_{d,k,t}\mathcal{Z}_{[0,\infty]}} \tilde{\omega}^0_{p^e} = \eta_{0,q}^{-1}(td) \sum_{x_0\in(\mathbb{Z}/q\mathbb{Z})^*/\{\pm 1\}} \eta_{0,q}(x_0)(S_{(x_0,dt),(t^{-1}x_0,d)} + S_{(x_0,-dt),(t^{-1}x_0,-d)}).$$

Remark 4.4. The sum defining $S_{x,y,k}$ like the one defining $\omega^0_{p^e}$ does not converge absolutely and to give sense to the definition of $S_{x,y,k}$ as well as to justify our calculations we apply the standard analytic continuation: using a (global) Iwasawa decomposition $g = bk$ we set $\psi^0_{p^e,i,s}(g) = \delta^s_{2,\mathbb{A}}(b)\psi^0_{p^e,i}(g)$, where $\psi^0_{p^e,i} = \psi^0_{p^e,f}\psi^0_{\infty,i}$ and $\omega^0_{p^e,s}(g; D) = \sum_{\gamma\in B_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \sum_{i=1,2} \psi^0_{p^e,i,s}(\gamma g)\,\omega_i(D)$. The sum defining $\omega^0_{p^e,s}$ converges absolutely for $\mathrm{Re}(s) > 0$ and has a holomorphic continuation to the entire complex plane satisfying $\omega^0_{p^e,0} = \omega^0_{p^e}$. Since the cycle $\mathcal{Z}_\varphi$ is compact we deduce that the mapping $F\colon s \mapsto \int_{\mathcal{Z}_\varphi} \tilde{\omega}^0_{p^e,s}$ too is holomorphic and uniqueness of the analytic continuation then yields

$$\int_{\mathcal{Z}_\varphi} \tilde{\omega}^0_{p^e} = \lim_{s\to 0} \int_{\mathcal{Z}_\varphi} \tilde{\omega}^0_{p^e,s} = F(0).$$

To calculate $F(0)$ we split the domain of integration into the boundary and the inner component. Equation (4.2) implies the holomorphy of $\int_{\mathcal{Z}^b_\varphi} \tilde{\omega}^0_{p^e,s}$ in s. Moreover, replacing $\omega^0_{p^e}$ by $\omega^0_{p^e,s}$ in the above calculations we find that $\int_{g_{d,k,t}\mathcal{Z}_{[0,\infty]}} \tilde{\omega}^0_{p^e,s}$ for $\mathrm{Re}(s) > 0$ is given by the same formula as above with $S_{x,y,k}$ replaced by

$$S_{x,y,k,s} = \frac{1}{p^k} \sum_{a,b} \frac{1}{(ab)^{1+s}},$$

where $a, b \in \mathbb{Z}$ are as in the definition of $S_{x,y,k}$ (compare Eq. (4.3)). We note that for $\mathrm{Re}(s) > 0$ this sum converges absolutely. Thus, assuming that $S_{x,y,k,s}$ has a holomorphic continuation to $\mathbb{C}$ we may define $S_{x,y,k} = S_{x,y,k,0}$ and with this definition $\int_{g_{d,k,t}\mathcal{Z}_{[0,\infty]}} \tilde{\omega}^0_{p^e}$ is then given by the above formula. It will follow from (4.5) below that $S_{x,y,k,s}$ can be analytically continued to $\mathbb{C}$.

The evaluation of $\omega^0_{p^e}$ on the chains $g'_{d,k,t}\mathcal{Z}_{[0,\infty]}$ and $g''_{d,k,t}\mathcal{Z}_{[0,\infty]}$ is quite analogous, one simply has to replace the source of λ by $(\mathbb{Z}/p^e\mathbb{Z})^*$. We state the results:

$$\int_{g'_{d,k,t}\mathcal{Z}_{[0,\infty]}} \omega^0_{p^e} = \eta^{-1}_{0,q}(d) \sum_{a,b} \frac{\eta_{0,q}(a)}{ab}$$

where $a, b \in \mathbb{Z}$ run over all pairs satisfying $(a, b) = 1$, $a \equiv \epsilon^{-1} d (\mathrm{mod}\ p^e)$, $b \equiv dp^k\ (\mathrm{mod}\ p^e)$, $a \in \mathbb{Z}^*_q$ and $q|b$ and

$$\int_{g''_{d,k,t}\mathcal{Z}_{[0,\infty]}} \omega^0_{p^e} = \eta_{0,q}(d^{-1}) \sum_{a,b} \frac{\eta_{0,q}(b)}{ab}$$

where $a, b \in \mathbb{Z}$ run over all $(a, b) = 1$ with $a \equiv \epsilon^{-1} d\ (\mathrm{mod}\ p^e)$, $b \equiv dp^k\ (\mathrm{mod}\ p^e)$, $b \in \mathbb{Z}^*_q$ and $q|a$. In particular these integrals are integral linear combinations of terms of the form

$$S'_{x,y,k} := \frac{1}{p^k} \sum_{a,b} \frac{1}{ab}$$

where $x \in (\mathbb{Z}/p^e q\mathbb{Z})^*$, $y \in (\mathbb{Z}/p^{e-k}\mathbb{Z})^*$ and $a, b \in \mathbb{Z}$ run over all pairs, such that $(a, b) = 1$, $a \equiv x\ (\mathrm{mod}\ p^e q)$, $\quad b \equiv y\ (\mathrm{mod}\ p^{e-k})$. Again, this has to be understood as the value at $s = 0$ of $S'_{x,y,k,s} = p^{-k} \sum_{a,b} 1/(ab)^{1+s}$. We may summarize our results so far: *for any cycle* $[\mathcal{Z}] \in H_1(\bar{S}_2(K^f_1(p^e, q)), \mathbb{Z})$ *we have*

$$\int_{[\mathcal{Z}]} \omega^0_{p^e} \equiv \mathcal{O} - \text{linear combination of the terms } S_{x,y,k},\ S'_{x,y,k}\ (\mathrm{mod}\ \mathcal{O}).$$

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4.4. THE COMPLETION AT p

We are left with determining the denominators of $S_{x,y,k}$ and $S'_{x,y,k}$ in the completions of $\mathcal{O}$. Since the case $S'_{x,y,k}$ is analogous, we will only deal with $S_{x,y,k}$. We let $\chi : (\mathbb{Z}/p^e q\mathbb{Z})^* \to \mathbb{C}^*$ be any even and $\psi : (\mathbb{Z}/p^{e-k} q\mathbb{Z})^* \to \mathbb{C}^*$ be any odd Dirichlet character (not necessarily primitive) and we denote by $L(\chi, s)$ and $L(\psi, s)$ their Dirichlet series (i.e. we omit the Euler factors at p and q). A small calculation proves

$$\sum_{\substack{x\in(\mathbb{Z}/p^e q\mathbb{Z})^* \\ y\in(\mathbb{Z}/p^{e-k} q\mathbb{Z})^*}} \chi\psi^{-1}(x)\psi(y)\, S_{x,y,k} = \frac{4}{p^k}\,\frac{L(\chi\psi^{-1}, 1)L(\psi, 1)}{L(\chi, 2)},$$

where we regard $\chi\psi^{-1}$ as Dirichlet character modulo $p^e q$. Since $S_{ax,by,k} = ab\, S_{x,y,k}$ for $a, b \in \{\pm 1\}$, this implies

$$S_{x,y,k} = \frac{1}{\phi(p^e q)\phi(p^{e-k} q)}\frac{1}{p^k}\sum_{\chi,\psi} \bar{\chi}\psi(x)\bar{\psi}(y)\,\frac{L(\chi\psi^{-1}, 1)L(\psi, 1)}{L(\chi, 2)}, \tag{4.5}$$

where χ resp. ψ run over all even, resp. odd, characters as above. We note that by our assumptions on the parity of χ and ψ no poles occur in the above equation. The definition of the L-function yields

$$L(\psi, 1) = \frac{1}{p^e q}\sum_{\epsilon\in(\mathbb{Z}/p^e q\mathbb{Z})^*/\{\pm 1\}} \psi(\epsilon)\sum_{n\in\mathbb{Z}}\frac{1}{n+\epsilon/p^e q}$$

and the partial fraction $\pi\cot(\pi x) = \sum_{n\in\mathbb{Z}}\frac{1}{n+x}$ then yields

$$L(\psi, 1) = \frac{\pi}{p^e q}\sum_{\epsilon\in(\mathbb{Z}/p^e q\mathbb{Z})^*/\{\pm 1\}} \psi(\epsilon)\,\frac{\zeta_{p^e q}^{\epsilon} + \zeta_{p^e q}^{-\epsilon}}{\zeta_{p^e q}^{\epsilon} - \zeta_{p^e q}^{-\epsilon}} \qquad (\zeta_{p^e q}^{p^e q} = -1).$$

Plugging in and taking into account the character relations for ψ we obtain

$$\begin{aligned} S_{x,y,k} = \frac{1}{\phi(p^e q)}\sum_{\chi}\bar{\chi}(x)\frac{\pi^2}{p^{2e}q^2 L(\chi, 2)}\times \\ \times \sum_{\epsilon\in(\mathbb{Z}/p^e q\mathbb{Z})^*/\{\pm 1\}} \chi(\epsilon)\frac{\zeta_{p^e q}^{\epsilon} + \zeta_{p^e q}^{-\epsilon}}{\zeta_{p^e q}^{\epsilon} - \zeta_{p^e q}^{-\epsilon}}\cdot\frac{\zeta_{p^e q}^{\epsilon y x^{-1}} + \zeta_{p^e q}^{-\epsilon y x^{-1}}}{\zeta_{p^e q}^{\epsilon y x^{-1}} - \zeta_{p^e q}^{-\epsilon y x^{-1}}}. \end{aligned}$$

Let f_χ be the conductor of χ. Using the functional equation

$$\frac{p^{2e}q^2}{\pi^2}L(\chi, 2) = -2G(\chi)p^{2e}q^2 f_\chi^{-2}\, L(\chi^{-1}, -1),$$

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the divisibility $G(\chi)|f_\chi$ and the congruence

$$\sum_{\substack{\epsilon\in(\mathbb{Z}/p^eq\mathbb{Z})^* \\ \epsilon\equiv\epsilon_0\,(f_\chi)}} \frac{\zeta_{p^eq}^{\epsilon}+\zeta_{p^eq}^{-\epsilon}}{\zeta_{p^eq}^{\epsilon}-\zeta_{p^eq}^{-\epsilon}}\cdot\frac{\zeta_{p^eq}^{\epsilon\delta}+\zeta_{p^eq}^{-\epsilon\delta}}{\zeta_{p^eq}^{\epsilon\delta}-\zeta_{p^eq}^{-\epsilon\delta}} \in \frac{p^e}{f_\chi}\frac{1}{pq}\mathcal{O}_{\mathbb{Q}(\chi)}, \tag{4.6}$$

which holds for fixed $\epsilon_0 \in (\mathbb{Z}/p^eq\mathbb{Z})^*$ (see Lemma 4.6. for a proof), we see that

$$S_{x,y,k} = \frac{1}{q^3p^{e+1}\phi(p^eq)}\sum_{\chi}\sum_{\epsilon \bmod p^eq/f_\chi}\chi(x^{-1}\epsilon)\frac{\alpha_{\epsilon,yx^{-1},\chi}}{L(\chi^{-1},-1)} \tag{4.7}$$

with integers $\alpha_{\epsilon,yx^{-1},\chi}\in\mathcal{O}_{\mathbb{Q}(\chi)}$. Since $L(\chi,-1)^\sigma = L(\chi^\sigma,-1)$ for all $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ the above expressions imply that $S_{x,y,k}\in\mathbb{Q}$. We want to obtain a bound for $|L(\chi^{-1},-1)|_p$. We denote by L_p be the *Kubota–Leopoldt* p-adic L-function. L_p interpolates the values

$$L_p(\omega^2\chi,-1) = L(\chi,-1)\in\mathbb{Q}(\chi)$$

(note that we omit the local factor at p from the definition of $L(\chi,s)$).

LEMMA 4.5. *For all even characters χ of $(\mathbb{Z}/p^eq\mathbb{Z})^*$, the absolute values of the p-adic L-function are bounded, i.e. there are constants M_1 and M_2 such that*

$$M_1 < |L_p(\chi,-1)|_p < M_2.$$

This is independent of the imbedding i_p.

Proof. Let $u_1\in 1+p\mathbb{Z}_p$ be a topological generator. We put $\chi_0 = \chi|_{(\mathbb{Z}/pq\mathbb{Z})^*}$. It is known from Iwasawa theory that for any character $\kappa\colon(\mathbb{Z}/pq\mathbb{Z})^*\to\mathbb{C}_p^*$ there are power series $f(\kappa,T)\in\mathbb{Z}_p[\zeta_{q-1}][[T]]$, such that

$$L_p(\chi,-1) = \begin{cases} f(\chi_0,\chi(u_1)u_1^2-1), & \text{if } \chi_0\neq \mathrm{id},\\ \dfrac{f(\mathrm{id},\chi(u_1)u_1^2-1)}{\chi(u_1)u_1^2-1}, & \text{if } \chi_0=\mathrm{id}.\end{cases}$$

Using the Weierstrass Preparation Theorem we obtain a factorization

$$f(\kappa,T) = a_\kappa P_\kappa(T)f_\kappa(T),$$

where $a_\kappa\in\mathbb{Z}_p[\zeta_{q-1}]$, $P_\kappa(T)\in\mathbb{Z}_p[\zeta_{q-1}][T]$ is a distinguished polynomial and $f_\kappa(T)=\sum_{i\geqslant 0}a_{\kappa,i}T^i\in\mathbb{Z}_p[\zeta_{q-1}][[T]]^*$. The algebraic integer $\chi(u_1)u_1^2-1$ is even contained in the maximal ideal $\mathcal{P}$ of the completion of $i_p(\mathcal{O}_{\mathbb{Q}(\chi(u_1))})$ (since $\mathbb{Q}_p(\chi(u_1))/\mathbb{Q}_p$ is purely ramified this is independent of the embedding i_p), which implies that for any κ not necessarily equal to χ_0

$$f_\kappa(\chi(u_1)u_1^2-1)\equiv a_{\kappa,0} \pmod{\mathcal{P}}.$$

Hence, $|f_\kappa(\chi(u_1)u_1^2-1)|_p = 1$. We now assume that the conductor f_χ is large enough so that $(|\chi(u_1)u_1^2-1|_p)^{\deg(P_\kappa)} > |p|_p$ for all characters κ. This only excludes finitely many characters χ and since $P_\kappa(T)$ is distinguished we see that

$$P_\kappa(\chi(u_1)u_1^2 - 1) \equiv (\chi(u_1)u_1^2 - 1)^{\deg(P_\kappa)} \pmod{p},$$

i.e. $|P_\kappa(\chi(u_1)u_1^2 - 1)|_p = |(\chi(u_1)u_1^2 - 1)^{\deg(P_\kappa)}|_p$ (note that $\mathbb{Z}_p[\zeta_{q-1}]/\mathbb{Z}_p$ is unramified). Thus, for all characters χ having conductor so large that $|p|_p < |\chi(u_1)u_1^2 - 1|_p^{\deg(P_\kappa)} (< 1)$ holds for all characters κ, we obtain

$$|p|_p \min_\kappa |a_\kappa|_p < |L_p(\chi, -1)|_p < |p|_p^{-1} \max_\kappa |a_\kappa|_p.$$

This proves the lemma.

Since $S_{x,y,k} \in \mathbb{Q}$ the lemma together with Equation (4.7) immediately implies that

$$p^{2e} M_p\, S_{x,y,k} \in \mathbb{Z}_p$$

for some constant $M_p \in \mathbb{Z}$ independent of the embedding i_p.

We still have to prove (4.6). This will follow from

LEMMA 4.6. *Let $N \in \mathbb{N}$ and let $N'|N$ be any divisor which is divisible by the same prime numbers as N. Then for any $\delta, \epsilon_0 \in (\mathbb{Z}/N\mathbb{Z})^*$ the following congruence is true:*

$$\sum_\epsilon \frac{\zeta_N^\epsilon + \zeta_N^{-\epsilon}}{\zeta_N^\epsilon - \zeta_N^{-\epsilon}} \cdot \frac{\zeta_N^{\epsilon\delta} + \zeta_N^{-\epsilon\delta}}{\zeta_N^{\epsilon\delta} - \zeta_N^{-\epsilon\delta}} \equiv 0 \pmod{N/N'\, \mathcal{O}_{\mathbb{Q}(\zeta_{N'})}},$$

where $\epsilon \in (\mathbb{Z}/N\mathbb{Z})^*$ runs over all elements satisfying $\epsilon \equiv \epsilon_0 \bmod N'$.

Proof. Let $p_1, \ldots, p_s$ be the prime numbers dividing N. Let $\Phi_d \in \mathbb{Z}[T]$ denote the dth cyclotomic polynomial. Since $\Phi_{p_1\cdot\ldots\cdot p_s}$ and $T - 1$ have leading coefficient 1 and are coprime (Φ_d is irreducible) there are polynomials $h, P \in \mathbb{Z}[T]$ such that

$$(T - 1)P(T) = 1 - h(T)\Phi_{p_1\cdot\ldots\cdot p_s}(T).$$

We set $N_0 = N/(p_1 \cdot \ldots \cdot p_s)$. Substituting $T \mapsto T^{N_0}$ we obtain

$$(T - 1)P_N(T) = 1 - h(T^{N_0})\Phi_{p_1\cdot\ldots\cdot p_s}(T^{N_0}),$$

where $P_N(T) = P(T^{N_0}) \cdot \sum_{i<N_0} T^i \in \mathbb{Z}[T]$. Since $\Phi_{p_1\cdot\ldots\cdot p_s}(T^{N_0}) = \Phi_N(T)$ (cf. [L], p. 280) we see that P_N is the inverse of $T - 1$ modulo Φ_N. In particular, specializing $T \mapsto \zeta_N, \zeta_N^\delta$ we deduce that

$$\frac{\zeta_N^{2\epsilon} + 1}{\zeta_N^{2\epsilon} - 1} \cdot \frac{\zeta_N^{2\epsilon\delta} + 1}{\zeta_N^{2\epsilon\delta} - 1} = Q_N(\zeta_N^\epsilon),$$

with a certain polynomial $Q_N = \sum_i a_{N,i} T^i \in \mathbb{Z}[T]$. Since

$$\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_{N'})) = \{\zeta_N \mapsto \zeta_N^\epsilon, \epsilon \equiv 1 \bmod N'\}$$

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we obtain

$$\sum_{\epsilon \equiv \epsilon_0} \frac{\zeta_N^{\epsilon} + \zeta_N^{-\epsilon}}{\zeta_N^{\epsilon} - \zeta_N^{-\epsilon}} \cdot \frac{\zeta_N^{\epsilon\delta} + \zeta_N^{-\epsilon\delta}}{\zeta_N^{\epsilon\delta} - \zeta_N^{-\epsilon\delta}} = \sum_i a_{N,i} \mathrm{Tr}(\zeta_N^i),$$

where $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{Q}(\zeta_{N'})}^{\mathbb{Q}(\zeta_N)}$. But for any, not necessarily primitive Nth root of unity ζ and any divisor $N'|N$ satisfying the assumptions we have $\mathrm{Tr}(\zeta) \in N/N' \mathcal{O}_{\mathbb{Q}(\zeta_{N'})}$, which yields the claim of the lemma.

4.5. THE COMPLETION AT $\ell \neq p$

Let ℓ be a prime number, which is different from p and let $i_\ell \colon \bar{\mathbb{Q}} \to \mathbb{C}_\ell$ be any embedding. We denote by $|\cdot|_\ell$ the normalized absolute valuation on $\mathbb{C}_\ell$ as well as the valuation on $\bar{\mathbb{Q}}$ induced by i_ℓ.

LEMMA 4.7. *Let χ run over all even characters of $(\mathbb{Z}/p^e q\mathbb{Z})^*$. There is a constant M, which does not depend on the embedding i_ℓ, such that $|L(\chi, -1)|_\ell > M$.*

Proof. We will use the ℓ-adic L-function

$$L_\ell(\chi\omega^{1-n}, n) = \frac{1}{1 - \chi(\ell)\ell^{-n}} L(\chi, n)$$

to show that $\ell \nmid L(\chi, -1)$ for almost all $\chi \neq 1$ (ω denotes the ℓ-adic Teichmüller character). Specializing $n = -1$ we see that $\ell | L(\chi, -1)$ is equivalent to $\ell | L_\ell(\chi\omega^2, -1)$. Using [Wa], Corollary 5.13 we further see that this is equivalent to $\ell | L_\ell(\chi\omega^2, 0)$. Since $\chi\omega(\ell)$ is either equal to 0 or a $(p-1)p^{f_\chi - 1}$th root of unity (this is only possible if $\ell = q$) we see that $1 - \chi\omega(\ell)$ is not divisible by ℓ for f_χ large enough (use [Wa], Lemma 1.4, Proposition 2.8). Hence $\ell | L(\chi\omega, 0)$. But Theorem 4.1 in [Si] states that $L(\chi, 0)$ is a ℓ-adic unit for all but finitely many characters χ, which proves the lemma.

Using the lemma, Equation (4.7) immediately implies that $M_\ell \, S_{x,y,k} \in \mathbb{Z}_\ell$ for some constant $M_\ell \in \mathbb{Z}$. Recalling that ω_{p^e} and $\omega_{p^e}^0$ have the same denominators, we have proved the following result on the denominators of our Eisenstein classes.

THEOREM 3. *For any prime number ℓ there is a constant $M_\ell \in \mathbb{Z}$ such that for all $e \geqslant 2$ and any prime ideal $\mathcal{L} \subset \mathcal{O}_{\mathbb{Q}(\eta,\eta')}$ lying above ℓ we have*

$$* \, M_\ell \, \omega_{p^e} \in H^1(S_2(K), \mathcal{O}_{\mathbb{Q}(\eta,\eta'),\mathcal{L}})_{\mathrm{int}},$$

where $* = p^{2e}$ if $\ell = p$ and $* = 1$ else.

5. Cohomology and Integrality

5.1. THE p-ADIC GROWTH OF THE DISTRIBUTION

In this last section we finally determine, in the case of trivial coefficients, the bounds for the denominators of the distribution $\mu_\pi/\Omega(\pi) = \mu_{\pi,1}^{\eta}/\Omega(\pi)$, whose p-adic Mellin

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transform interpolates the critical values at $s = 0$. We start from the formula given in Lemma 3.2, which we evaluate as follows. As in Lemma 3.2 we set $u = u(1, \epsilon, 0; p^e)$, $\epsilon \in \mathbb{Z}_p^*$. The cohomology class ω_{p^e} is invariant under the compact open subgroup

$$K_1^1(2, p^e) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : \ a \equiv 1 \ (\mathrm{mod}\ p), \ c \equiv 0, \ d \equiv 1 \ (\mathrm{mod}\ p^e) \right\}$$

and the form $i^* r_u^* \omega \wedge \omega_{p^e}$ therefore is invariant under the subgroup $\mathcal{K}(p^e) \leqslant K_1^1(2, p^e)$, which consists of all elements k satisfying

$$u^{-1} \begin{pmatrix} k & \\ & 1 \end{pmatrix} u \in \mathcal{I}.$$

LEMMA 5.1. *The subgroup* $\mathcal{K}(p^e)$ *does not depend on* $\epsilon \in \mathbb{Z}_p^*$ *and has volume* $p^4 (p-1)^{-1} (p^2 - 1)^{-1} p^{-4e}$.

Proof. A straightforward calculation proves that $\mathcal{K}(p^e)$ consists of all elements $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying $c \equiv 0$, $d \equiv 1$, $b \equiv (d-1)/p^{e-1} \bmod p^e$, $a \equiv 1 + c/p^{e-1} \bmod p^{e-1}$. In particular, using $(\mathrm{GL}_2(\mathbb{Z}_p) : K_1(2, p^e)) = p^{2e}(1 - p^{-2})$ we deduce

$$(\mathrm{GL}_2(\mathbb{Z}_p) : \mathcal{K}(p^e)) = p^{-4}(p-1)(p^2-1)\, p^{4e},$$

which proves the lemma.

We denote by $\mathcal{O}$ the ring of integers of $E_\pi(\eta, \eta', \zeta_{q(q-1)})$. Combining Lemmas 3.2 and 5.1 we obtain

$$\frac{\mu_\pi(\epsilon + p^e \mathbb{Z}_p)}{\Omega(\pi)} = \frac{p^4}{(p-1)(p^2-1)} (\eta_p(p)\gamma)^{-e} p^e \int_{F_2(\mathcal{K}(p^e))} \frac{1}{\Omega(\pi)}\, i^* r_u^* \omega \wedge \omega_{p^e}. \tag{5.1}$$

The right translation $i^* r_u^*$ is defined over $\mathcal{O}$ and Theorem 3 then yields the final result on the denominators of μ_π.

THEOREM 4. *For any* $\epsilon \in \mathbb{Z}_p^*$ *and any embedding* $i_p \colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ *the following holds:*

$$\left| \frac{\mu_\pi(\epsilon + p^e \mathbb{Z}_p)}{\Omega(\pi)} \right|_p \leqslant M_p \, |\gamma^{-1}|_p^e \, p^e,$$

where the constant M_p *does not depend on the embedding* i_p. *If* ℓ *is any prime number different from* p *and* $i_\ell \colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_\ell$ *is any embedding, then* $\left| \mu_\pi(\epsilon + p^e \mathbb{Z}_p)/\Omega(\pi) \right|_\ell < M$ *is bounded by some constant* $M = M_\ell$ *only depending on* ℓ.

Since $\delta_{q^{-1}}$ and $\mu(1)$ are $\mathbb{Z}$-valued the same statement is true for the distribution $\delta_{q^{-1}} * \mu(1) * \mu_\pi/\Omega(\pi)$ (cf. Section 2.2). We note that the value of $|\gamma|_p$ depends on i_p.

5.2. h-ADMISSIBLE MEASURES

We want to give the application of Theorem 4 to the construction of p-adic analytic L-functions. We will use the p-adic integration theory of [V]. Let $\mathcal{C}^h$ be the space of functions $f: \mathbb{Z}_p^* \to \mathbb{C}_p$ which are locally given by polynomials of degree at most h. An *h-admissible measure* is a $\mathbb{C}_p$-linear functional $\tilde{\mu}$: $\mathcal{C}^h \to \mathbb{C}_p$, which satisfies the growth condition

$$\sup_{a \in \mathbb{Z}_p^*} \left| \tilde{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p} \cdot (x-a)^i) \right|_p = o(|p^e|_p^{i-h})$$

for all $0 \leqslant i < h$ and $e \to \infty$ (cf. [V], p. 217) ($\mathbf{ch}_X$ denotes the characteristic function of the set X). We let $F/\mathbb{Q}_p$ be any local field. We denote by $\mathcal{M}_F^h$ the vector space of all F-valued, h-admissible measures. We also denote by $\mathcal{D}_F^h$ the space of all F-valued distributions satisfying the growth condition

$$\sup_{a \in \mathbb{Z}_p^*} |\mu(a + p^e\mathbb{Z}_p)|_p = o(|p^e|_p^{-h}).$$

EXAMPLE 5.2. Let $h^* \in \mathbb{N}$ be large enough so that $p^{h^*}\gamma^{-1} \in \mathcal{O}$. Theorem 4 then shows that $\mu_\pi/\Omega(\pi) \in \mathcal{D}^{h^*+2}_{i_p(E(\eta,\eta',\zeta_{q(q-1)}))}$. In particular, the same is true for $\delta_{q^{-1}} * \mu(1) * \mu_\pi/\ \Omega(\pi)$.

For $\tilde{\mu} \in \mathcal{M}_F^h$ we define a distribution $\mathrm{res}(\tilde{\mu})$ through the equation

$$\mathrm{vol}_{\mathrm{res}(\tilde{\mu})}(a + p^e\mathbb{Z}_p) = \tilde{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p}).$$

This induces an F-linear map

$$\mathrm{res}: \mathcal{M}_F^h \to \mathcal{D}_F^h.$$

LEMMA 5.3. *The map* res: $\mathcal{M}_F^h \to \mathcal{D}_F^h$ *is surjective, i.e. any distribution* $\mu \in \mathcal{D}_F^h$ *can be lifted to an h-admissible measure* $\tilde{\mu}$.

Proof. Let $\mu \in \mathcal{D}_F^h$. Multiplying μ by some scalar we may assume that $\mu(a + p^e\mathbb{Z}_p) \in p^{-(e-1)h}\mathcal{O}_F$. We want to construct an element in $\mathcal{M}_F^h$, which maps to μ. Since $\mathcal{C}^h$ is generated over F by the functions $\mathbf{ch}_{a+p^e\mathbb{Z}_p} \cdot x^i$, where $a \in \mathbb{Z}_p^*$, $e \in \mathbb{N}$ and $i = 0, \ldots, h$ it is sufficient in order to define an h-admissible measure to define its values on these functions. We choose a natural number $h' < h$ and we first define an F-linear functional $\tilde{\mu}: \mathcal{C}^h \to F$ satisfying the properties

(1) $\tilde{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p}) = \mathrm{vol}_\mu(a + p^e\mathbb{Z}_p)$,
(2) $\tilde{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p} \cdot x^k) \equiv \sum_{i=0}^{k-1}(-1)^{k-i}\binom{k}{i}a^{k-i}\,\tilde{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p} \cdot x^i) \pmod{p^{-(e-1)(h'-k)}\mathcal{O}_F}$.

We will construct $\tilde{\mu}$ using induction on k and in view of 1. we may start with $0 < k \leqslant h$ and assume that the values $\tilde{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p} \cdot x^i)$, $i < k$ have already been defined. We first define a map $\bar{\mu}$ on the F-span of the functions $\mathbf{ch}_{a+p^e\mathbb{Z}_p} \cdot x^k$, $a \in \mathbb{Z}_p^*$, $e \in \mathbb{N}$, by the equation in 2. This means that $\bar{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p} \cdot x^k)$ is defined modulo $p^{-(e-1)(h'-k)}\mathcal{O}_F$.

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We want to verify that $\bar{\mu}$ is well defined, i.e. that it fulfills the distribution relation

$$\sum_{u\in\mathbb{Z}/p\mathbb{Z}} \bar{\mu}(\mathbf{ch}_{a+up^e+p^{e+1}\mathbb{Z}_p}\cdot x^k) \equiv \bar{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p}\cdot x^k) \pmod{p^{-e(h'-k)}\mathcal{O}_F}. \tag{5.2}$$

Using the binomial formula and the identity

$$\binom{k}{i}\binom{k-i}{j} = \binom{k}{j}\binom{k-j}{i}$$

we find after a little calculation

$$\begin{aligned}\bar{\mu}(\mathbf{ch}_{a+up^e+p^{e+1}\mathbb{Z}_p}\cdot x^k) \equiv{}& \sum_{j=0}^{k}(-1)^j\binom{k}{j}(p^e u)^j {\sum}_{i=0}^{*\,k-j}(-1)^{k-i-j}\binom{k-j}{i}a^{k-i-j}\times\\ &\times\tilde{\mu}(\mathbf{ch}_{a+up^e+p^{e+1}\mathbb{Z}_p}\cdot x^i) \pmod{p^{-e(h'-k)}\mathcal{O}_F},\end{aligned}$$

where $\sum^*$ means, that we omit the index $(i,j)=(k,0)$ from the summation. The induction hypothesis on the values $\tilde{\mu}(\mathbf{ch}_{a+up^e+p^{e+1}\mathbb{Z}_p}\cdot x^i)$, $i<k$ implies that for $j\neq 0$ the inner sum is congruent 0 modulo $p^{-e(h'-k+j)}$. Therefore we obtain

$$\bar{\mu}(\mathbf{ch}_{a+up^e+p^{e+1}\mathbb{Z}_p}\cdot x^k) \equiv \sum_{i=0}^{k-1}(-1)^{k-i}\binom{k}{i}a^{k-i}\tilde{\mu}(\mathbf{ch}_{a+up^e+p^{e+1}\mathbb{Z}_p}\cdot x^i) \pmod{p^{-e(h'-k)}\mathcal{O}_F}.$$

Summing this equation over $u\in\mathbb{Z}/p\mathbb{Z}$ and taking into account that by our induction hypothesis

$$\sum_{u\in\mathbb{Z}/p\mathbb{Z}} \tilde{\mu}(\mathbf{ch}_{a+up^e+p^{e+1}\mathbb{Z}_p}\cdot x^i) = \tilde{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p}\cdot x^i)$$

for $i<k$ we see that (5.2) is true. It is easy to verify that there is an F-linear functional $\tilde{\mu}$ from the span of $\mathbf{ch}_{a+p^e\mathbb{Z}_p}\cdot x^k$, $\epsilon\in\mathbb{Z}_p^*$, $e\in\mathbb{N}$ to F, which modulo $p^{-(e-1)(h'-k)}\mathcal{O}_F$ coincides with $\bar{\mu}$. This finishes the proof of the existence of an F-linear functional $\tilde{\mu}$ satisfying 1. and 2. We show that $\tilde{\mu}$ is even h-admissible: again, using the binomial formula we find

$$\mu(\mathbf{ch}_{a+p^e\mathbb{Z}_p}\cdot(x-a)^k) = \sum_{i=0}^{k}\binom{k}{i}a^{k-i}\tilde{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p}\cdot x^i)$$

and 2 then immediately implies that

$$\mu(\mathbf{ch}_{a+p^e\mathbb{Z}_p}\cdot(x-a)^k) \in p^{-(e-1)(h'-k)}\mathcal{O}_F.$$

Thus $\tilde{\mu}$ is an h-admissible measure and since we trivially have $\mathrm{res}(\tilde{\mu})=\mu$ the lemma is proved.

We note that res is not injective and the h-admissible distributions restricting to some given distribution μ under res may be seen as p-adic deformations of μ.

We are now able to apply the p-adic integration theory from [V] to construct p-adic analytic functions, which interpolate the automorphic L-function. We note in advance that these p-adic L-functions satisfy some logarithmic growth condition. We let $\tilde{\mu}_\pi$ be a lift of the distribution $\delta_{q^{-1}} * \mu(1) * \mu_\pi/\Omega(\pi) \in D^{h^*+2}_{i_p(E(\eta,\eta',\zeta_{q(q-1)}))}$. We set

$$X_p := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p)$$

and we define the function $L_p\colon X_p \to \mathbb{C}_p$ by

$$L_p(\chi_p) := \int_{\mathbb{Z}_p^*} \chi_p \eta_p^2 \, d\tilde{\mu}_\pi,$$

the integral being defined as in Section 1 of [V]. Theorem 2.3 in [V] then shows that L_p is analytic and its growth is at most $o(\log^{h^*+2}(\cdot))$ for any h^* as in Example 5.2. We summarize our results (cf. Corollary 1).

COROLLARY 3. *There is a p-adic analytic function $L_p\colon X_p \to \mathbb{C}_p$ such that for all characters $\chi\colon \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of conductor a p-power and infinity component $\chi_\infty = \mathrm{id}$ we have*

$$L_p(\chi_p) = \hat{\zeta}^e \gamma^{-e} G(\chi_p \eta_p^2) \frac{L(\pi \otimes \chi\eta, 0)}{\Omega(\pi)},$$

where $\hat{\zeta}$ is a fixed root of unity. For all characters χ with conductor a p-power and infinity component $\chi_\infty = \mathrm{sgn}$ the function L_p vanishes identically. Moreover L_p has no poles and equals $\mathrm{o}(\log^{h^*+2}(\cdot))$.

Of course, $L_p(\cdot)$ is given by some power series $f \in \mathrm{Quot}(i_p(\mathcal{O}))[[T]]$, but is not uniquely determined.

Remark 5.4. The p-ordinary case. Let π be a cuspidal representation of $\mathrm{GL}_3(\mathbb{A})$ with unramified p-component $\pi_p \cong \mathrm{Ind}(\mu_1, \mu_2, \mu_3)$. We call π *p-ordinary* with respect to i_p, if $|\mu_i(p)|_p = p^{i-2}$. In this case the complex number γ defined in Equation (2.4) is a p-adic unit and we immediately obtain

$$\left| \frac{\mu_\pi(\epsilon + p^e\mathbb{Z}_p)}{\Omega(\pi)} \right|_p \leqslant M_p p^e.$$

In different words, the growth of the distribution $\mu_\pi/\Omega(\pi)$ equals at most the growth of the *Haar distribution* μ_{Haar} on $\mathbb{Z}_p^*$. Using the *Lefschetz–Poincaré isomorphism*

$$\mathcal{P}\colon\ H^2(\bar{F}_2(\mathcal{K}(p^e)), \partial\bar{F}_2(\mathcal{K}(p^e)), \mathbb{Z}) \to H_1(\bar{F}_2(\mathcal{K}(p^e)), \mathbb{Z})$$

we may rewrite (5.1) as

$$\frac{\mu_\pi(\epsilon + p^e\mathbb{Z}_p)}{\Omega(\pi)} = \text{local unit} \cdot p^e \int_{\Omega(\pi)^{-1}\mathcal{P}(i^* r_u^* \omega)} \omega_{p^e},$$

where $u = u(1, \epsilon, 0; p^e)$. Thus, boundedness of $\mu_\pi/\Omega(\pi)$ is equivalent to

$$\int_{\Omega(\pi)^{-1}\mathcal{P}(i^* r_u^* \omega)} \omega_{p^e} \in \frac{1}{p^e}\, \overline{i_p(\mathcal{O})}$$

for all $\epsilon \in \mathbb{Z}_p^*$ and $e \geqslant 2$.

On the other hand using the expressions of Section 4 for the integral $\int_{\mathcal{Z}_{d,t,k}} \omega_{p^e}$ we calculated for small e (and p) the integral of ω_{p^e} over the absolute cycles $\mathcal{Z}_{d,d',t,t',k}$ (cf. Example 4.2). The results seem to indicate that for each level e there is a cycle such that

$$\int_{\mathcal{Z}_{d,d',t,t',k}} \omega_{p^e} \equiv \text{local unit} \cdot \frac{1}{p^{2e}} \pmod{\mathcal{O}}.$$

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