Notes on Mathematical Induction.

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Mathematical induction in its simplest form may be stated thus: Suppose there is a set of propositions $p_0, p_1, p_2 \ldots$ which are so related that the truth of p_n implies the truth of p_{n+1}, then if p_0 is true, it follows that all the other propositions of the set are true. For since p_0 is true therefore p_1 is true, therefore p_2 is true, and so on as far as we please.

A well-known example of the application of this method is the proof of the binomial theorem for a positive integral index. It is based on the identity

$$\left(1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \ldots\right)(1+x) = 1 + (n+1)x + \frac{(n+1)n}{1 \cdot 2}x^2 + \ldots . (1)$$

From this we deduce

$$\left\{1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \ldots - (1+x)^n\right\}(1+x)$$

$$= 1 + (n+1)x + \frac{(n+1)n}{1 \cdot 2}x^2 + \ldots - (1+x)^{n+1} \ldots\ldots (2)$$

Denoting by p_n the proposition that the first factor of the left side of this equation is identically zero, the proposition that the right side is identically zero will be denoted by p_{n+1}, and (2) shows that if p_n is true, so is p_{n+1}. The proposition p_0 asserts that $1-(1+x)^0 = 0$, and is true. Hence p_n is true.

It will be convenient in what follows to denote propositions by equations of the form $p = 0$, in which p may be called the *error* of the proposition, so that $p = 0$ asserts that the error of the proposition is zero, or, in other words, that the proposition is true. In using this notation we do not imply that p is an actual algebraical expression (though very many propositions can be made to take the form of an algebraic expression equated to numerical zero). For example, the letter p might stand for the error of the statement "N is a prime," so that $p=0$ would indicate the assertion "N is a prime."

Returning to our topic, we note that a slightly more complex form of mathematical induction is that which rests on a relation such as this: $p_n=0$ and $p_{n+1}=0$ together imply $p_{n+2}=0$, which combined with the assertions $p_0=0$ and $p_1=0$ enables us to assert that $p_2=0$, $p_3=0$, $p_4=0$ and so on, as far as we please.

As an example, let us prove that for all non-fractional values of x the expression $x^4-4x^3+5x^2-2x \equiv f(x)$ is divisible exactly by 12

$$f(x)=x(x-1)(x-2)(x-3)+2x(x-1)(x-2)$$

$$\therefore f(x+1)-f(x)=4x(x-1)(x-2)-6x(x-1)$$

$$\therefore f(x+2)-2f(x+1)+f(x)=12x(x-1)+12x=12x^2 \text{.........} (3).$$

Denoting by $p_x=0$ the proposition that $f(x)$ is divisible by 12, we see at once that (3) shows that $p_x=0$ and $p_{x+1}=0$ together imply $p_{x+2}=0$. But $f(0)=0$ and $f(1)=0$, so that $p_0=0$ and $p_1=0$. It follows that $p_2=0$, $p_3=0$, $p_4=0, \ldots p_x=0$, where x is any positive integer.

We remark that (3) also shows that $p_{x+1}=0$ and $p_{x+2}=0$ together imply $p_x=0$, hence we deduce also $p_{-1}=0$, $p_{-2}=0$, $\ldots p_x=0$ where x is any negative integer.

The next higher type of mathematical induction is when there exists a relation, true for all values of n, or for all integral values, between p_n, p_{n+1}, p_{n+2}, p_{n+3} in virtue of which we can infer that $p_{n+3}=0$ is a consequence of the assertions $p_n=0$, $p_{n+1}=0$ and $p_{n+2}=0$ taken jointly, and it is granted that $p_1=0$, $p_2=0$ and $p_3=0$. The conclusion is that $p_4=0$, $p_5=0 \ldots p_n=0$, when n is any positive integer.

But the term Mathematical Induction may have still wider application. The following general statement would include many special forms. Let $p_1=0$, $p_2=0$, $\ldots$ $p_r=0$ be a set of propositions, and let it be granted

(1) that between certain groups of these propositions relations exist of the type $\phi(p_a, p_b, p_c, \ldots p_h, p_k)=0$ in virtue of which we can infer $p_k=0$ if the "errors" $p_a, p_b \ldots p_h$ are all zero;

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(2) that the relations ϕ can be placed in a certain order, say $\phi_1, \phi_2, \phi_3, \phi_4 \ldots$ such that each relation involves only one new "error" *i.e.* only one p which has not already occurred in the preceding relations, that one being p_k;

(3) that if $p_{a_1}, p_{b_1}, \ldots p_{h_1}, p_{k_1}$ be the errors involved in the first relation $\phi_1 = 0$, it is granted that $p_{a_1} = 0$, $p_{b_1} = 0$, $\ldots p_{h_1} = 0$.

It will then follow that all the propositions of the set under consideration must be admitted as true.

The most usual case is when the ϕ-relations are all of the same type, each referring to the same number of "errors." In this case the successive relations may be written
$\phi(p_0, p_1, p_2 \ldots p_r) = 0$, $\phi(p_1, p_2, p_3 \ldots p_{r+1}) = 0 \ldots \phi(p_s, p_{s+1}, p_{s+2}, \ldots p_{s+r}) = 0$; and the propositions which must be granted as a basis for the induction, are $p_0 = 0$, $p_1 = 0$, $\ldots$ $p_{r-1} = 0$, if we suppose that $p_0, p_1, p_2 \ldots$ are successive terms in a regular series of propositions.

A more general case arises when we have the relations

$$\phi(p_\alpha, p_\beta, p_\gamma \ldots p_\lambda) = 0\,;\ \phi(p_{\alpha+1}, p_{\beta+1}, \ldots p_{\lambda+1}) = 0\,;$$

$$\phi(p_{\alpha+2}, p_{\beta+2}, \ldots p_{\lambda+2}) = 0 \ldots \phi(p_{\alpha+s}, p_{\beta+s}, \ldots p_{\lambda+s}) = 0.$$

In this case the necessary "basis" of the induction will depend on the mutual relations of $\alpha, \beta, \gamma \ldots \lambda$.

If, for example, $\alpha, \beta, \ldots \lambda$ form an arithmetic progression of k terms with common difference d, a sufficient basis would be afforded by the $kd - 1$ propositions $p_1 = 0$, $p_2 = 0 \ldots p_{kd-1} = 0$.

The case in which there is a two dimensional array of propositions to be proved deserves special treatment. A simple case of this is the elementary theorem ${}^nC_r = \dfrac{n\,.\,(n-1)(n-2)\ldots(n-r+1)}{r\,!}$ which has to be proved for all positive integral values of n, and all positive integral values of r which are not greater than n.

From the identities ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$ and

$$\frac{(n+1)n(n-1)\ldots(n-r)}{r\,!} = \frac{n(n-1)\ldots(n-r+1)}{r\,!} + \frac{n(n-1)\ldots(n-r)}{r-1\,!}$$

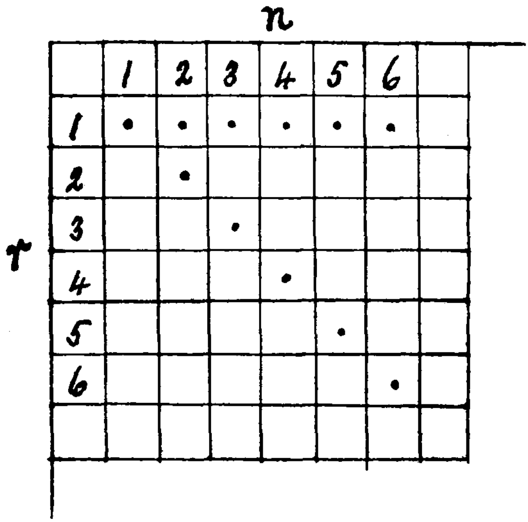
we deduce the relation $p_{n+1,\,r} - p_{n,\,r} - p_{n,\,r-1} = 0$ ....(4),

where $p_{n,\,r}$ stands for ${}^nC_r - \frac{n\,\overline{n-1}\ldots\overline{n-r+1}}{r\,!}$.

Here the ϕ-relation (4) implies not only that $p_{n+1,\,r} = 0$ follows from $p_{n,\,r} = 0$ together with $p_{n,\,r-1} = 0$, but that if *any* two of the errors are both zero, the third also must be zero.

The propositions required to start with in this case are infinite in number, but are all proved by a simpler mathematical induction, and for the present purpose may be assumed to be true. They are the propositions $p_{1,\,1} = 0,\ p_{2,\,2} = 0,\ p_{3,\,3} = 0 \ldots p_{m,\,m} = 0 \ldots$
and $p_{2,\,1} = 0,\ p_{3,\,1} = 0,\ p_{4,\,1} = 0 \ldots p_{m,\,1} = 0 \ldots$

To make it clear that the proposition $p_{n,\,r} = 0$ can be reached by induction for any significant values of n and r let us take a row-and-column table, as shewn in Fig. 1, where each place refers to



certain values of n and of r, and the dots indicate the values of n and r for which $p_{n,\,r} = 0$ is given.

The "graph" of the relation (4) is clearly of the form :., and it is easy to see that by superposing it sufficiently often on the tablespaces beginning with the first and second rows, proceeding with the second and third rows, and so on, we can reach any place to the right of the dotted diagonal of the table.

In order to have no blanks in the square array of the places in the table for this particular application it would be necessary to write $n+r=1$ instead of n in the expression for which $p_{n,r}$ stands, so that we should have $p_{n,r} \equiv {}^{n+r-1}C_r - \dfrac{(n+r-1)!}{(n-1)!\,r!}$ the relation $\phi=0$ will still be $p_{n+1,r} - p_{n,r} - p_{n,r-1} = 0$, but there will now be a meaning for $p_{n,r}$ for any positive integral values of n and r.

The graph of ϕ is still of the form $\begin{smallmatrix}\bullet & \\ \bullet & \bullet\end{smallmatrix}$, and the necessary basis of admitted propositions will consist of those corresponding to all the places in the one row and in one column, *e.g.* in the first row and the first column.

In what immediately follows we will suppose that the array of places is square, and includes places corresponding to all positive integral values of n and of r.

The preceding example suggests a discussion of the "*pousto*" or graph of the basis of admitted propositions which will suffice for different forms of induction-graphs on a two-spread in order that complete induction by means of the corresponding ϕ-relation should be possible.

The simplest graphs will be those consisting of two places only. The graph $\bullet\ \bullet$ requires as "pousto" no more than the first column of places, and if in $\phi(p_{n,r}, p_{n,r+1}) = 0$, $p_{n,r} = 0$, and $p_{n,r+1}$ mutually imply one another, the first column may be replaced by one place taken arbitrarily from each row.

Similarly for the graph $\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}$ a sufficient "pousto" would be one place in each column.

For the graph $\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}$ the first row and the first column would afford a minimum pousto; but for the graph $\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}$ the first row *or* the first column would suffice.

For the graph $\bullet\ \circ\ \bullet$ the first two columns would serve as pousto, for $\begin{smallmatrix}\bullet & \circ & \circ \\ \circ & \bullet & \bullet\end{smallmatrix}$ the first row and the first two columns, for $\begin{smallmatrix}\bullet & \circ & \bullet \\ \bullet & \bullet & \circ\end{smallmatrix}$ the first row alone, *or* the first and second columns, and so on for other binomial graphs.

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For $\begin{smallmatrix}\bullet & \bullet \\ \bullet & \circ\end{smallmatrix}$ *the first row or the first column* would suffice, while for $\begin{smallmatrix}\bullet & \circ \\ \bullet & \bullet\end{smallmatrix}$, for $\begin{smallmatrix}\circ & \bullet \\ \bullet & \bullet\end{smallmatrix}$ and $\begin{smallmatrix}\bullet & \bullet \\ \circ & \bullet\end{smallmatrix}$ both a row and a column would be required. There is thus a distinction between graphs which require only a row or a column for pousto, and those which require two rows, or a row and a column, or other combinations.

In general, if the graph of a ϕ can be contained in a rectangular block of places with μ rows and ν columns, a sufficient "pousto" will be afforded by $\mu - 1$ rows and $\nu - 1$ columns; but in some cases a smaller basis will suffice, some requiring rows only or columns only.

If we have two independent ϕ relations having graphs of fixed shape, then in many cases a finite number of places will serve as "pousto" on the graph-table. For example, any one place is a sufficient pousto for the simultaneous graphs $\begin{smallmatrix}\bullet \\ \bullet\end{smallmatrix}$ and $\begin{smallmatrix}\bullet & \bullet\end{smallmatrix}$, if the implication of the p's is mutual; and if otherwise, the place common to the first row and column is all that is required.

The graphical treatment of mathematical induction can be extended to cases where there are three or more independent integral parameters, by employing arrays of places of three or more dimensions, and similar remarks will apply to the nature of the "pousto" in various cases.

The problem of determining a ϕ-relation which will enable us to prove a general proposition involving one, or more than one, integral parameter is perhaps worthy of investigation. On this topic I content myself by offering one or two remarks.

Suppose the general proposition to be $p(n, r, q\ldots) = 0$. If $p(n, r, q\ldots) \equiv \alpha(n, r, q\ldots) - \alpha'(n, r, q\ldots)$, and if we can prove two related identities

$$A\alpha + B\beta + C\gamma + \ldots = 0$$
$$A\alpha' + B\beta' + C\gamma' + \ldots = 0$$

where $\alpha, \beta, \gamma\ldots$ are cases of $\alpha(n, r, q)$ with certain related values of the parameters, and $\alpha', \beta', \gamma'\ldots$ are the corresponding cases of $\alpha'(n, r, q)$, then we can deduce the identity

$$A(\alpha - \alpha') + B(\beta - \beta') + C(\gamma - \gamma') + \ldots = 0$$

or
$$Ap_\alpha + Bp_\beta + Cp_\gamma + \ldots = 0,$$

where p_a is the case of $p(n, r, q\ldots)$ corresponding to α, and so on. This is a ϕ-relation which may be used to effect an induction, provided a sufficient basis can be established. The proof of the formula for ${}^nC_r$ given above affords an illustration of this procedure, in the case when the number of parameters is two.

The general problem of this section would be: Given a theorem $p(n, r, q\ldots) = 0$ to be proved, to find a relation

$$\phi\{p(n_1 r_1 q_1 \ldots), p(n_2 r_2 q_2 \ldots), p(n_3, r_3, q_3)\ldots\} = 0$$

where n_1, r_1, $q_1 \ldots n_2$, r_2, $q_2 \ldots$, etc., are integral parameters having definite relations to one another. This problem would probably in most cases have an infinite number of solutions.

The converse problem is: Given the form of the relation ϕ, to find a proposition p which satisfies the relation. It also would probably have an indefinite number of different solutions.