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Two-variable laws for a class of finite simple groups

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This paper presents a two-variable basis for the variety generated by the finite simple group $\mathrm{PSL}(2, 2^n)$.

1. Introduction

In a recent paper (Southcott [7]), the author gave a basis for the laws of the variety generated by the finite simple group $\mathrm{PSL}(2, 2^n)$, $n \geq 2$. With one exception, the laws given there involve only two variables, and it was noted that Bryant [1] has shown that $\operatorname{var} \mathrm{PSL}(2, 2^n)$ can be defined by two-variable laws.

The two-variable basis given below consists of the two-variable laws of the basis in [7], and a disjunction type law which says that in a non-abelian simple group all elements are either of order two or of odd order. Laws of this type were used by Bryant and Powell [2] to construct a two-variable basis for the laws of $\mathrm{PSL}(2, 5) \cong \mathrm{PSL}(2, 4)$.

2. Preliminaries

Throughout, standard group theoretic language and results are used without comment, as are the notation, terminology and results relating to varieties of groups, contained in Chapters 1 and 5 of Hanna Neumann [6]. In particular, variables and words are denoted by lower case Roman letters; the variety generated by the group G is denoted by $\operatorname{var} G$.

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The following lemma is the main result established in the proof of Theorem 5.5.4 of Cossey, Macdonald and Street [4].

LEMMA 2.1. *A finite group* G *belongs to* var $PSL(2, 2^n)$ *if and only if it satisfies the following conditions:*

(I) G *is of exponent dividing* $2(2^{2n}-1)$;

(II) *an element of* G *of order dividing* $2^n + 1$ *which normalises a* 2*-subgroup centralises it;*

(III) *subgroups of* G *of odd exponent are abelian.*

The next lemma is needed in Section 4 in the proof of local finiteness of the variety defined by the laws of Theorem 3.1.

LEMMA 2.2 (Kegel and Wehrfritz [5], Theorem 2.9). *Let* G *be an infinite periodic group containing an involution* i *such that the centraliser in* G *of every involution of* G *centralising* i *is finite. Then* G *is a locally finite group and all the involutions of* G *are conjugate.*

3. A two-variable basis

THEOREM 3.1. *Let*

$$a = \left[[x^2, y^2]^{2^{2n}}, x^2\right]^{j+k}[y^2, x^2] ,$$

$$b = \left[a^{-2^{2n}}, y^2\right]^{j+k} a ,$$

$$c = \left[b^{-2^{2n}}, x^2y^2\right]^{j} b ,$$

where

$$j = 2^{2n-1} - 1 , \quad k = 2^{2n} - 1 .$$

Then the following set is a basis for the laws of var $PSL(2, 2^n)$:

(1) $x^{2(2^{2n}-1)} = 1$,

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(2) $\left[x, y^{2(2^n-1)}\right]^{2^{2n}-1} = 1$,

(3) $c^2 = 1$,

(4) $\left[x^{2^{2n}-1}, (x^2)^y\right] = 1$.

4. Proof of the theorem

Law (4) holds in $PSL(2, 2^n)$, since each element is either of order two or of odd order.

In a non-abelian simple group no non-identity element can commute with all distinct conjugates of any other non-identity element. Hence, in this case, we have that all elements are either of order 2 or of odd order.

By [7] the laws (1), (2) and (3) hold in $PSL(2, 2^n)$ and the variety defined by them satisfies conditions (I), (II) and (III) of Lemma 2.1. Hence its finite groups are precisely those of var $PSL(2, 2^n)$. Since a variety is generated by its finitely generated groups ([6], Theorem 15.61) it is sufficient to prove that the laws (1)-(4) define a locally finite variety.

In the variety defined by laws (1), (2) and (3), the order of a finite group on a given number of generators is bounded, and hence if the variety defined by the laws of Theorem 3.1 is not locally finite it must contain an infinite finitely generated non-abelian simple group. In fact, it contains no such group.

Consider an infinite non-abelian simple group G which satisfies the laws (1)-(4) of Theorem 3.1. By laws (1) and (4) all its elements are either of order two or of odd order. From this, we can deduce the following facts about the maximal 2-subgroups of G .

(i) Each maximal 2-subgroup of G is the centraliser of each of its non-identity elements.

(ii) Each maximal 2-subgroup has trivial intersection with any distinct maximal 2-subgroup; in particular with any distinct conjugate.

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(iii) *All involutions of* G *are conjugate; all non-identity elements of a maximal* 2*-subgroup* T *of* G *are conjugate only in the normaliser* $N_G(T)$ *of* T .

Now $N_G(T)/T$ is an abelian group of odd exponent which is represented faithfully by conjugation as a semi-regular group of automorphisms of T . Using the methods of Burnside [3], §248, we deduce that $N_G(T)/T$ is cyclic, and hence finite. Since all non-identity elements of T are conjugate in $N_G(T)$, it follows that T is finite.

The group G thus satisfies the conditions of Lemma 2.2, and hence is locally finite.

5. Concluding remarks

A simple argument shows that by concatenating the left-hand sides of laws (1), (2) and (3) of Theorem 3.1, we obtain a single two-variable law which is equivalent to the set (1), (2), (3).

In view of the fact that the finite groups in the variety defined by the laws (1), (2) and (3) are precisely the finite groups of var $\mathrm{PSL}\left(2, 2^n\right)$ it is worth investigating whether the variety they define is locally finite, that is, whether the disjunction type law is superfluous.

If this is so, then var $\mathrm{PSL}\left(2, 2^n\right)$ is defined by a single two-variable law.

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