S. Mori
Nagoya Math. J.
Vol. 96 (1984), 127–132

ON DEGREES AND GENERA OF CURVES ON SMOOTH QUARTIC SURFACES IN P^3

SHIGEFUMI MORI

Our result is motivated by the results [GP] of Gruson and Peskin on characterization of the pair of degree d and genus g of a non-singular curve in P^3. In the last step, they construct the required curve C on a singular quartic surface when $g \leqq (d-1)^2/8$. Here we consider curves on smooth quartic surfaces.

The author expresses his hearty thanks to Professor Hartshorne who pointed out a mistake in the earlier version of this paper, and to Professor Iitaka who kindly allowed the author to use his word processor to write up this paper.

THEOREM 1. *Let k be an algebraically closed field of characteristic* 0 *and $d > 0$ and $g \geqq 0$ be integers. Then there is a non-singular curve C of degree d and genus g on a non-singular quartic surface X in P^3 if and only if* (1) $g = d^2/8 + 1$, *or* (2) $g < d^2/8$ and $(d, g) \neq (5, 3)$.

Remark 2. Under the notation of Theorem 1, $g = d^2/8 + 1$ if and only if C is a complete intersection of X and a hypersurface of degree $d/4$, which will be proved in the proof below.

Proof of the only-if-part ($\Rightarrow$) *of Theorem* 1. Let $H = \mathcal{O}_X(1)$. Since $(H\cdot H) > 0$, one has

$$(C\cdot H)^2 - (H\cdot H)\cdot(C\cdot C) = d^2 - 8(g-1) \geqq 0\,,$$

by Hodge index theorem, because X is a $K3$ surface and $K_C = \mathcal{O}_C(C)$. One has $d^2 \equiv 0, 1, 4, 1 \pmod 8$ according as $d \equiv 0, 1, 2, 3 \pmod 4$. If $d^2 - 8(g-1) = 0$ then the classes of $\mathcal{O}_X(C)$ and $\mathcal{O}_X(H)$ are proportional. Since X is a $K3$ surface and $(H\cdot H) = 4$, Pic X is torsion-free and H is not divisible, whence $\mathcal{O}_X(C)$ is a multiple of $\mathcal{O}_X(H)$, which implies that C is a complete intersection of X and a hypersurface of degree $d/4$. It

Received November 17, 1983.

https://doi.org/10.1017/S0027763000021188 Published online by Cambridge University Press

remains to show that $d^2-8(g-1)>8$ when $d^2-8(g-1)>0$, and we will treat three cases $d^2-8(g-1)=8, 1, 4$.

Case (1) $d^2-8(g-1)=8$: Let $d=4d'$ ($d'\geqq 1$, $d'\in Z$), then $2(g-1)=2(2d'^2-1)$. Let $E=d'H-C$, then $(E\cdot H)=0$ and $(E^2)=-2$. Since X is a $K3$ surface, one has

$$h^0(E)+h^0(-E)\geqq \chi(\mathcal{O}(E))=2+(E^2)/2=1\,.$$

Thus E or $-E$ gives a curve E' such that $(E'\cdot H)=0$, which contradicts the very ampleness of H.

Case (2) $d^2-8(g-1)=1$: Let $d=2d'-1$ ($d'\geqq 1$, $d\in Z$), then $2(g-1)=(d'^2-d')$. Let $E=d'H-2C$, then $(E\cdot H)=2$, and $(E^2)=0$.

Thus as in Case (1), $h^0(E)+h^0(-E)\geqq 2$. Since $(E\cdot H)=2$ for ample H, one has $h^0(-E)=0$ and $h^0(E)\geqq 2$. Thus $|E|$ has an effective member E_0. If $E'=(E_0)_{\text{red}}$ is irreducible, then $(E'^2)=0$ and $(E'\cdot H)=1, 2$ for very ample H. Thus $E'\cong P^1$ and $(E'^2)=-2$, which contradicts $(E'^2)=0$. Hence $(E_0)_{\text{red}}$ is reducible and by $(E_0\cdot H)=2$ for very ample H, one has $E_0=E_1+E_2$, where $E_1, E_2\cong P^1$, $\sharp(E_1\cap E_2)\leqq 1$, and the intersection of E_1 and E_2 is transverse. Then $(E_0^2)=-4+2(E_1\cdot E_2)\leqq -2$, which contradicts $(E_0^2)=0$.

Case (3) $d^2-8(g-1)=4$: Let $d=4d'-2$ ($d'\geqq 1$, $d'\in Z$), then $2(g-1)=4(d'^2-d')$. If $E=d'H-C$, then $(E\cdot H)=2$ and $(E^2)=0$. Thus one gets a contradiction as in Case (2). If $d=5$ and $g=3$, then $d>2g-2$. Thus $h^0(\mathcal{O}_C(1))=3$, which implies that C is a plane curve, but this contradicts the genus formula for plane curves. Thus "$\Rightarrow$" is proved.

PROPOSITION 3. *Let d and g be integers such that $0\leqq g\leqq d-3$. If* char $k\neq 2$, *then there exist a non-singular Kummer surface X_0 and effective divisors H_0, C_0 on X_0 such that*

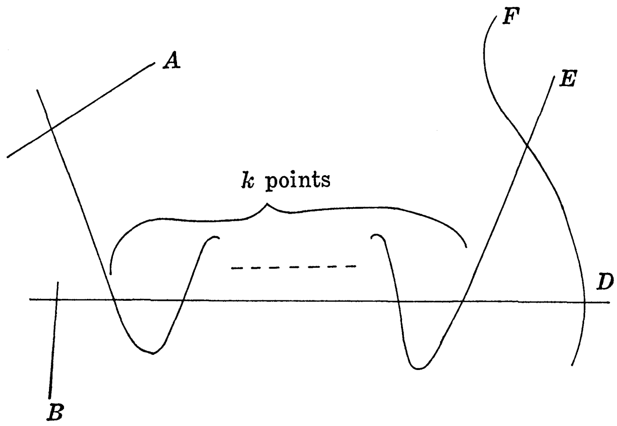
(1) $(H_0^2)=4$, $(H_0\cdot C_0)=d$, $(C_0^2)=2g-2$,

(2) *H_0 is numerically effective,*

(3) *C_0 is numerically effective if $g\geqq 2$,*

(4) *ZH_0+ZC_0 is a direct summand of* Pic X_0.

Proof. Let $k=d-g-3\geqq 0$. Let Y_1 and Y_2 be elliptic curves with an isogeny $f\colon Y_1\to Y_2$ of degree $2k+1$. Let $P, Q\in Y_1$ be non-zero points such that $2P=0$, $f(2Q)\neq 0$. Let X_0 be the non-singular Kummer surface



associated to $Y_1 \times Y_2$. Then $Y_1 \times 0$, $Q \times Y_2$, the graph of f, $P \times f(P)$, and $P \times 0$ give irreducible curves D, F, E, A, and B in X_0 such that $D \cong E \cong A \cong B \cong \boldsymbol{P}^1$, and F is an elliptic curve, with the configuration as in the picture with all the intersections transverse (cf. [MM] or [SI]). Let $H_0 = D + 3F$, and $C_0 = E + gF$. Then (1) is clear; (2) follows from $(H_0 \cdot D) = 1$ and $(H_0 \cdot F) = 1$; (3) follows from $(C_0 \cdot E) = g - 2$ and $(C_0 \cdot F) = 1$; and (4) follows from $(H_0 \cdot B) = 1$, $(H_0 \cdot A) = 0$, $(C_0 \cdot B) = 0$, and $(C_0 \cdot A) = 1$. q.e.d.

Remark 4. Let k be the field of complex numbers. Then, in the local versal deformations space Def of X_0, the locus where H_0 and C_0 lift as line bundles is an 18-dimensional smooth subvariety Pol, and there is a dense subset Pol′ of Pol such that if $q \in \text{Pol}'$, then the surface X and line bundles H and C on X lying over q satisfy the conditions:

(1) $(H^2) = 4$, $(H \cdot C) = d$, $(C^2) = 2g - 2$,

(2) H is numerically effective,

(3) C is numerically effective if $g \geqq 2$, and

(4) $\text{Pic}\, X = ZH + ZC$.

Indeed (1) is clear, whence X is algebraic by [K, Theorem 8], and (4) follows from [K, Theorem 14]. As for (2) and (3), $2H_0$ and $2C_0$ (if $g \geqq 2$) are base point free by (1) of Theorem 5. The obstructions for lifting sections of $\mathcal{O}(2H_0)$ and $\mathcal{O}(2C_0)$ (if $g \geqq 2$) to Pol lie in $H^1(\mathcal{O}(2H_0))$ and $H^1(\mathcal{O}(2C_0))$ which are both 0 by Ramanujam's vanishing theorem.

We now quote results by Saint-Donat:

THEOREM 5 (Saint-Donat [SD] or cf. [MM]). *Let X be a $K3$ surface defined over an algebraically closed field of characteristic $\neq 2$. Let H be* *a numerically effective divisor on X. Then one has*

(1) *H is not base point free if and only if there exist irreducible curves E, Γ, and an integer $k \geqq 2$ such that $H \sim kE + \Gamma$, $(E^2) = 0$, $(\Gamma^2) = -2$, $(E \cdot \Gamma) = 1$. In this case, every member of $|H|$ is of the form $E' + \Gamma$, where E' is a sum of k effective divisors $E_1, \cdots, E_k$ such that $E_i \sim E$ for all i.*

(2) *Let $(H^2) \geqq 4$. Then H is very ample if and only if*

(i) *there is no irreducible curve E such that $(E^2) = 0$, $(E \cdot H) = 1, 2$,*

(ii) *there is no irreducible curve E such that $(E^2) = 2$, $H \sim 2E$, and*

(iii) *there is no irreducible curve E such that $(E^2) = -2$, $(E \cdot H) = 0$.*

PROPOSITION 6. *Let X, H, C be as in Remark 4. Then H is very ample and $|C|$ contains an irreducible smooth member.*

Proof. We will first check that H satisfies the conditions (i)–(iii) in (2) of Theorem 5. We denote by disc(A, B) the determinant of the intersection matrix of divisors A and B. If there is a divisor E such that $(E^2) = -2$, $(E \cdot H) = 0$, then disc$(E, H) = -8$ is divisible by disc$(H, C) = 8(g-1) - d^2$. However, by $g \leqq d - 3$, one has $d^2 \geqq (g+3)^2 > 8g$ and disc$(H, C) < -8$. This is a contradiction. Thus (iii) is checked, (i) is checked in the same way, and (ii) is obvious because H is a part of the basis of Pic X. Hence H is very ample. Assume that $g \geqq 2$. Then we use (1) of Theorem 5 to show that C is base point free. If C is not base point free, then there is a divisor E such that $(E^2) = 0$, $(E \cdot C) = 1$. Then disc$(E, C) = -1$ is divisible by disc(H, C), which is a contradiction, as we have seen above. Thus C is base point free and $|C|$ has an irreducible smooth member because $(C^2) > 0$. Let $g = 1$. Then the equation

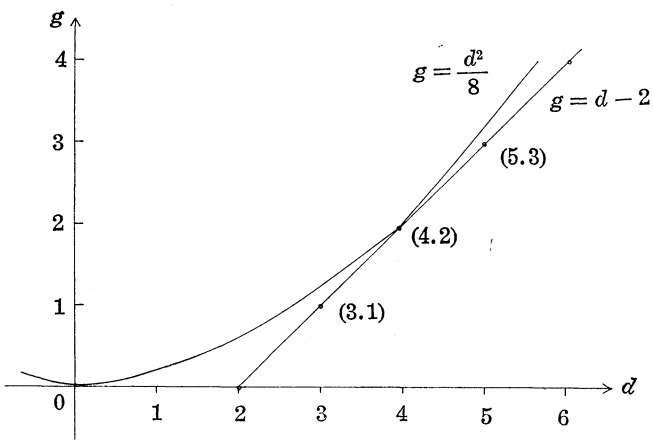
$$(xH + yC)^2 = 2x(2x + dy) = -2$$

does not have integral solutions x, y, because $d \geqq g + 3 = 4$.

Hence X does not contain smooth rational curves by Remark 4, (4). By $(C^2) = 0$, $|C|$ or $|-C|$ contains an effective member. By $(C \cdot H) = d > 0$, $|C|$ contains an effective member C_0. Thus C is numerically effective because otherwise C_0 contains an irreducible curve $Z \cong \boldsymbol{P}^1$, which is a contradiction. Hence by (1) of Theorem 5, C is base point free, and C is a multiple of an elliptic pencil. Since C is a part of the basis of Pic X, $|C|$ is an elliptic pencil, and it contains a smooth elliptic curve. Let $g = 0$. Since $(C^2) = -2$ and $(C \cdot H) > 0$, one has $C \sim E + D$, where

$E \cong \boldsymbol{P}^1$ and D is an effecitve divisor. Since disc(H, C) divides disc(H, E), one has $8 + (C \cdot H)^2 \leqq 8 + (E \cdot H)^2$. Thus $(D \cdot H) \leqq 0$, and $D = 0$. Hence $C = E$, and Proposition 6 is proved.

We can now finish the proof of the if-part ($\Leftarrow$) of Theorem 1. We use induction on d. We omit the proof for $(d, g) = (1, 0)$, $(2, 0)$, $(3, 1)$, since they are well known. We may assume that $g < d^2/8$, otherwise C is given as a complete intersection. We may also assume that $g \geqq d - 2$ by Remark 4 and Proposition 6. Thus as shown by the picture, one sees



$d \geqq 6$. First, we assume that $(d, g) \neq (9, 10)$. Let $d' = d - 4$, and $g' = g - d + 2$. Then $d'^2 - 8g' = d^2 - 8g > 0$ and $(d', g') \neq (5, 3)$. Thus by the induction hypothesis, there exist a non-singular quartic X' and a non-singular curve C' on it of degree d' and genus g'. Let H' be an irreducible hyperplanesection of X', and $C = C' + H'$. Since $d' = d - 4 \geqq 2$, one sees $(C \cdot C') = 2(g' - 1) + d' \geqq 0$, and C is numerically effective. Since $(H'^2) = 4$, C is base point free by (1) of Theorem 5. If we denote by the same C, a smooth member of $|C|$, then C has degree d and genus g. Thus C and X are the required pair for d, g. For $(d, g) = (9, 10)$, let $d' = 1$, $g' = 0$, and C' a straight line on a smooth quartic surface X'. Let H' be an irreducible hyperplanesection of X and $C = C' + 2H'$. Then, one sees that C, X are the required pair as in the above argument. This proves Theorem 1.

References

[GP] L. Gruson and C. Peskin, Genre des courbes algébrique de l'espace projectif (II), Ann. Sci. Ecole Norm. Sup., Paris, 4ᵉ série, t. **15** (1982), 401–418.

[K] K. Kodaira, On the structure of compact complex analytic surfaces, I, Amer. J.

Math., **86** (1964), 751–798.

[MM] S. Mori and S. Mukai, The uniruledness of the moduli space of curves of genus 11, to appear in the proceedings of Japan France symposium on algebraic geometry, 1982.

[SD] B. Saint-Donat, Projective models of K-3 surfaces, Amer. J. Math., **96** (1974), 602–639.

[SI] T. Shioda and H. Inose, On singular $K3$ surfaces, Complex analysis and algebraic geometry, Iwanami Shoten, Cambridge University Press, 1977, 119–136.

Department of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464
Japan