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RESEARCH ARTICLE

The existence of the Kähler–Ricci soliton degeneration

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Abstract

We prove an algebraic version of the Hamilton–Tian conjecture for all log Fano pairs. More precisely, we show that any log Fano pair admits a canonical two-step degeneration to a reduced uniformly Ding stable triple, which admits a Kähler–Ricci soliton when the ground field $\mathbb{k} = \mathbb{C}$.

Contents

Throughout this paper, we work over an algebraically closed field $\mathbb{k}$ of characteristic 0.

1. Introduction

A Kähler–Einstein metric is arguably 'the most canonical' metric that one can find on a Fano variety. However, not every Fano variety admits a Kähler–Einstein metric. So it is natural to ask what kind of structure one should look for on a general Fano variety. In fact, there are several candidates. In this note, we will study one structure, namely the *Kähler–Ricci soliton*. This kind of metric has been investigated in many previous works. While not every Fano variety itself admits a Kähler–Ricci soliton, it is expected that any Fano variety has a unique degeneration to one with a Kähler–Ricci soliton (see, e.g., [Tia97, Section 9]).

For a smooth Fano manifold X, the approach of using Kähler–Ricci flow to study Kähler–Ricci solitons has been intensively studied in complex geometry literature and leads to the solution of the Hamilton–Tian conjecture (see [TZ16, Bam18, CW20]), which says that the Gromov–Hausdorff limit X_∞ of X under the Kähler–Ricci flow admits a Kähler–Ricci soliton. What is more relevant to us is that

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in [CSW18] it is shown that X_∞ can be realized as a (two-step) degeneration of X, and in [DS20] that the first degeneration minimizes the H-functional among all possible $\mathbb{R}$-degenerations (note that our sign on H is opposite to the one in [DS20]).

In this paper, we will pursue a purely algebraic study of the above degeneration process by studying the geometry of the minimizer of the non-Archimedean type functional $\mathbf{H}^{\mathrm{NA}}$, which in particular can be applied to a general (possibly singular) log Fano pair (X, Δ). Such algebraic study, including developing the non-Archimedean theory of the $\mathbf{H}^{\mathrm{NA}}$-functional, was initiated in [HL20b]. There it is shown the *uniqueness* of the above degeneration process. Our first main theorem of this paper is the *existence* of such a degeneration for general log Fano pairs. It can be considered to be an algebraic version of the Hamilton–Tian conjecture, though there is no metric involved. More precisely, we have the following result.

Theorem 1.1. *Let (X, Δ) be a log Fano pair. Then it admits a two-step degeneration to a K-polystable triple (Y, Δ_Y, ξ_Y), which is indeed reduced uniformly Ding stable. In particular, it admits a Kähler–Ricci soliton if $\mathbb{k} = \mathbb{C}$.*

In the above theorem, $\xi_Y = 0$ if and only if (X, Δ) is K-semistable. The proof of Theorem 1.1 will be separated into two parts, contained, respectively, in Theorems 1.2 and 1.3.

By [HL20b, Theorem 1.3], it is already known that any K-semistable triple (X_0, Δ_0, ξ) admits a unique K-polystable degeneration (Y, Δ_Y, ξ_Y) (whose proof is based on [LWX21]). Therefore, it suffices to establish the first-step degeneration which degenerates (X, Δ) to a K-semistable triple (X_0, Δ_0, ξ). To construct such a degeneration, we follow [HL20b] and study the valuation which computes $h(X, \Delta)$ (i.e., the minimizer of the $\tilde{\beta}$). As a result, we prove the following statement, which establishes the first half of Theorem 1.1 and gives an affirmative answer to [HL20b, Conjecture 4.10].

Theorem 1.2. *Let (X, Δ) be a log Fano pair. Let r be a positive integer such that $r(K_X + \Delta)$ is Cartier. Then (X, Δ) has a unique valuation v computing $h(X, \Delta)$.*

Moreover, the associated graded ring $\mathrm{gr}_v R$ for $R = \bigoplus_{m \in \mathbb{N}} H^0(-mr(K_X + \Delta))$ is finitely generated, whose Proj *together with the degeneration of Δ and the induced vector ξ_v yields the first-step degeneration to a K-semistable triple (X_0, Δ_0, ξ_v).*

It was shown in [HL20b] that the uniqueness statement in the above theorem follows from the finite generation of the $\mathrm{gr}_v R$. In this note, we first establish stronger convexity results for various non-Archimedean functionals (see Theorem 3.7). Then we will obtain uniqueness as a consequence, without using finite generation.

The second step to proving Theorem 1.1 is to establish the following Yau–Tian–Donaldson (YTD) conjecture for Kähler–Ricci solitons.

Theorem 1.3 (YTD Conjecture for Kähler-Ricci Solitons). *A triple (X, Δ, ξ) is K-polystable if and only if it is reduced uniformly Ding stable. In particular, when $\mathbb{k} = \mathbb{C}$, (X, Δ, ξ) admits a Kähler–Ricci soliton if and only if it is K-polystable.*

In fact, in [HL20], it is proven that the reduced uniform Ding stability of (X, Δ, ξ) is equivalent to the existence of a Kähler–Ricci soliton, by using variational methods. Here, we verify that reduced uniform Ding stability is equivalent to K-polystability. When X is smooth and $\Delta = 0$, the second part of Theorem 1.3 is proved in [DS16, CSW18].

Remark 1.4. As we already mentioned, when X is smooth, the Cheeger–Colding–Tian theory can be used to establish Theorem 1.1, Theorem 1.2 and Theorem 1.3. In fact, one can obtain the optimal degeneration from the study of the Hamilton–Tian conjecture on the long time behavior of Kähler–Ricci flows on X. See [DS16, TZ16, Bam18, CW20, CSW18]. However, it seems to us it is hard to extend these types of arguments to the more general (possibly singular) case.

Recent work of Han and Li builds an algebraic framework for studying the two-step degeneration process. Specifically, in [HL20b], they developed the non-Archimedean theory for the $\mathbf{H}^{\mathrm{NA}}$-functional (based on the H-function(al) defined in [TZZZ13, He16]) and interpret the existence of the optimal

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degeneration, which is essentially equivalent to Theorem 1.2, in terms of geometric properties of the minimizer of the $\tilde{\beta}_{X,\Delta}$-function, which is a variant of the $\mathbf{H}^{\mathrm{NA}}$-functional but defined on $\mathrm{Val}^{\circ}_X \cup \{v_{\mathrm{triv}}\}$.

From the algebro-geometric viewpoint, the study of $\tilde{\beta}_{X,\Delta}$ in [HL20b] is entirely parallel to the study of the stability threshold of a log Fano pair or in the local setting the normalized volume function of a Kawamata log terminal (klt) singularity. Therefore, it is natural to apply the arguments in the former problems to the current case. Indeed, [HL20b] has made significant progress in carrying out this study, and the main remaining step is the finite generation of the graded ring induced by minimizers of various functions, for example, $\tilde{\beta}_{X,\Delta}$ and $\delta_g(X,\Delta,\xi)$.

In [LXZ22], the finite generation of the associated graded ring for the valuation computing the stability threshold is solved. In this note, we solve the finite generation in Theorem 1.2 and Theorem 1.3 by a similar method. We also give a more straightforward argument of the uniqueness without using the finite generation (which is needed in [HL20b]) but by establishing convexity of various functionals based on the arguments in [XZ21].

We will also investigate a moduli approach to study general log Fano pairs with fixed h-invariant.

Theorem 1.5 (=Theorem 6.1). *For a fixed dimension n, volume V, a positive integer N and a constant h_0, families of n-dimensional log Fano pairs (X,Δ) with $(-K_X-\Delta)^n = V$, $N\Delta$ integral and $h(X,\Delta) \geq h_0$ are parameterized by an Artin stack $\mathcal{M}^{\mathrm{Fano}}_{n,V,N,h_0^+}$ of finite type.*

In the upcoming work, we aim to show that there is a finite type Artin stack $\mathcal{M}^{\mathrm{Kss}}_{n,V,N,h_0}$ which parametrizes families of n-dimensional K-semistable triples (X,Δ,ξ) with $(-K_X-\Delta)^n = V$, $N\Delta$ integral and $h(\xi) = h_0$. Moreover, $\mathcal{M}^{\mathrm{Kss}}_{n,V,N,h_0}$ admits a proper good moduli space $M^{\mathrm{Kps}}_{n,V,N,h_0}$. Then we will study the two-step degeneration from a moduli theoretic viewpoint.

1.1. *Outline of the proof*

In recent years, there have been two functions on the space of valuations which have been intensively studied in algebraic geometry. The first one is the function $\frac{A_{X,\Delta}(\cdot)}{S(\cdot)}$ of a log Fano pair (X,Δ), and the second one is the normalized volume function on a klt singularity $x \in (X,\Delta)$. Many of their fundamental properties were proved in a sequence of works. The general framework for the proofs of the theorems in this paper is largely parallel to the previous works on the study of these two functions, especially the first one.

Step 1: The first step is to show the strict convexity of the $\mathbf{H}^{\mathrm{NA}}$-functional. For any pair of filtrations $\mathcal{F}_0$ and $\mathcal{F}_1$, there is a natural family $(\mathcal{F}_t)_{t\in[0,1]}$ of filtrations connecting them called the *geodesic* (see Section 3.1.2). To study it, we define a measure $\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1}$ over $\mathbb{R}^2$ (called *the compatible Duistermaat–Heckman measure*) that encodes $\mathrm{DH}_{\mathcal{F}_t}$ for $t \in [0,1]$ (see Section 3.1.3). Then convexity results for various functionals, for example, $\mathbf{E}^{\mathrm{NA}}$ and $\widetilde{S}$ along geodesics can be proved by doing integration over this measure. And for $\mathbf{L}^{\mathrm{NA}}$, the convexity is proved by interpreting it as the log canonical slope μ and then applying results from [XZ21] to compare log canonical thresholds. As a result, this yields the convexity of $\mathbf{D}^{\mathrm{NA}}$ and the strict convexity of $\mathbf{H}^{\mathrm{NA}}$ along geodesics (analogous results in the Archimedean setting were proved in [Ber15]). The latter will imply the uniqueness of the minimizer.

Step 2: To prove Theorem 1.3, we will take a similar strategy to the solution of the usual YTD conjecture for log Fano pairs. First, we will extend the usual definition of the S-invariant function to the weighted setting with respect to a quasi-monomial valuation v_0 (see Section 4.1) and then we can define the corresponding $\delta(X,\Delta,v_0)$.

We are first interested in the special case when v_0 is a valuation coming from a vector field ξ induced by a torus action. In this case, as seen in [HL20b], many criteria for testing the the K-semistability or (reduced) uniformly K-stability of a pair (X,Δ) can be extended in to the setting of triples (X,Δ,ξ). In particular, we extend results from [Li22] and [XZ20, Appendix] to this setting in Sections 4.2 and 4.3.

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In a similar but slightly different setting, we consider the minimizer v_0 of the nonhomogeneous function $\tilde{\beta}_{X,\Delta}$, and we show it computes $\delta(X,\Delta,v_0)$ for the weight function $g = e^{-x}$. See Section 5.1.

Step 3: In the last step, we will show in the above two cases v_0 is a monomial log canonical (lc) place of a special $\mathbb{Q}$-complement (in the sense of [LXZ22, Definition 3.3]) constructed from a weighted basis type divisor. Then we can apply the finite generation result in [LXZ22] to show the associated graded ring of v_0 is finitely generated (see Corollary 5.7 and 5.8). This completes the proof of Theorem 1.2 and 1.3.

Finally, to prove Theorem 1.5, that is, to verify $\mathcal{M}^{\mathrm{Fano}}_{n,V,N,h_0^+}$ is an Artin stack of finite type, we first need to show that the set of all n-dimensional log Fano pairs (X,Δ) with $(-K_X-\Delta)^n = V$, $N\Delta$ integral and $h(X,\Delta) \geq h_0$ is bounded, and then we conclude by showing, for any $\mathbb{Q}$-Gorenstein family of $(X,\Delta) \to B$ over a finite type base B, the function $B \ni t \mapsto h(X_t,\Delta_t)$ is constructible and lower semicontinuous.

2. Preliminaries

Notation and Conventions: We follow the standard notation as in [KM98, Kol13, Laz04].

A variety is a separated integral scheme of finite type over $\mathbb{k}$. A pair (X,Δ) consists of a normal variety X and an effective $\mathbb{Q}$-divisor Δ on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. A pair (X,Δ) is called *log Fano* if X is projective, (X,Δ) is klt and $-K_X-\Delta$ is ample. A log smooth model (Y,E) of a pair (X,Δ) consists of a projective birational morphism $\pi\colon Y \to X$ and a reduced divisor E on Y such that $(Y, \mathrm{Supp}(E+\mathrm{Ex}(\pi)+\pi_*^{-1}\Delta))$ has simple normal crossing support.

Let X be a normal variety. We denote by Val_X the space of real valuations $K(X)^\times \to \mathbb{R}$ centered on X whose restriction over the ground field $\mathbb{k}$ is trivial. We endow Val_X with the weak topology. We denote the trivial valuation on X by v_{triv}.

For the definitions of divisorial valuations, quasi-monomial valuations and log discrepancy of valuations; see, for example, [JM12, LLX20, Xu20].

Definition 2.1. Let (X,Δ) be a pair. We denote by

$$\mathrm{Val}_X^\circ := \{v \in \mathrm{Val}_X \mid A_{X,\Delta}(v) < +\infty \text{ and } v \neq v_{\mathrm{triv}}\}.$$

If (X,Δ) admits a torus $\mathbb{T}$-action, then we denote by $\mathrm{Val}_X^{\mathbb{T}}$ the subset of Val_X consisting of all $\mathbb{T}$-invariant valuations, and let $\mathrm{Val}_X^{\mathbb{T},\circ} := \mathrm{Val}_X^\circ \cap \mathrm{Val}_X^{\mathbb{T}}$.

2.1. Filtrations

Let (X,Δ) be an n-dimensional log Fano pair. Fix $r>0$ so that $L := -r(K_X+\Delta)$ is an ample Cartier divisor. We write

$$R(X,L) := R := \bigoplus_{m\in\mathbb{N}} R_m := \bigoplus_{m\in\mathbb{N}} H^0(X,\mathcal{O}_X(mL))$$

for the section ring of L and $N_m := \dim R_m$.

Definition 2.2. A *filtration* $\mathcal{F}$ of R is a collection of vector subspaces $\mathcal{F}^\lambda R_m \subset R_m$ for each $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$ satisfying

(F1) $\mathcal{F}^\lambda R_m \subset \mathcal{F}^{\lambda'} R_m$ for $\lambda \geq \lambda'$;
(F2) $\mathcal{F}^\lambda R_m = \bigcap_{\lambda'<\lambda} \mathcal{F}^{\lambda'} R_m$;
(F3) $\mathcal{F}^\lambda R_m = R_m$ for $\lambda \ll 0$ and $\mathcal{F}^\lambda R_m = 0$ for $\lambda \gg 0$;
(F4) $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subset \mathcal{F}^{\lambda+\lambda'} R_{m+m'}$.

A filtration $\mathcal{F}$ is a *$\mathbb{Z}$-filtration* if $\mathcal{F}^{\lceil\lambda\rceil} R_m = \mathcal{F}^\lambda R_m$ for all $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$. A filtration is *finitely generated* if the associated graded $\mathbb{k}$-algebra $\mathrm{gr}_{\mathcal{F}} R := \bigoplus_{(m,\lambda)\in\mathbb{N}\times\mathbb{R}} \mathrm{gr}^\lambda_{\mathcal{F}} R_m$, where $\mathrm{gr}^\lambda_{\mathcal{F}} R_m = \mathcal{F}^\lambda R_m / (\cup_{\mu>\lambda} \mathcal{F}^\mu R_m)$, is finitely generated.

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The *translation* of $\mathcal{F}$ by $c \in \mathbb{R}$ is the filtration defined by $\mathcal{G}^\lambda R_m := \mathcal{F}^{\lambda - mrc} R_m$. The *scaling* by $a \in \mathbb{R}_{>0}$ is the filtration defined by $\mathcal{H}^\lambda R_m := \mathcal{F}^{\lambda/a} R_m$.

A filtration $\mathcal{F}$ is *linearly bounded* if there exists $C > 0$ so that $\mathcal{F}^{mC} R_m = 0$ for all $m > 0$. Note that there always exists $C > 0$ so that $\mathcal{F}^{-mC} R_m = R_m$ for all $m > 0$ by the finite generation of R combined with (F3) and (F4).

For an element $s \in R_m \setminus \{0\}$, we set $\mathrm{ord}_\mathcal{F}(s) := \max\{\lambda \in \mathbb{R} \mid s \in \mathcal{F}^\lambda R_m\}$. We set $\mathrm{ord}_\mathcal{F}(0) = +\infty$ by convention. A basis $(s_1, \ldots, s_{N_m})$ of R_m is said to be *compatible* with $\mathcal{F}$ if $\mathcal{F}^\lambda R_m = \mathrm{span}\langle s_j \mid \mathrm{ord}_\mathcal{F}(s_j) \geq \lambda\rangle$ for each $\lambda \in \mathbb{R}$.

Example 2.1. The following filtrations play an important role in this paper.

1. Given $v \in \mathrm{Val}_X$, there is an induced filtration $\mathcal{F}_v$ of R defined by

$$\mathcal{F}_v^\lambda R_m := \{s \in R_m \mid v(s) \geq \lambda\}.$$

   When $A_{X,\Delta}(v) < +\infty$, $\mathcal{F}_v$ is linearly bounded [BJ20, Lemma 3.1].
2. Similarly, any effective $\mathbb{Q}$-divisor G on X induces a filtration $\mathcal{F}_G$ of R by setting

$$\mathcal{F}_G^\lambda R_m := \{s \in R_m \mid \{s = 0\} \geq \lambda G\}.$$

3. A *test configuration* $(\mathcal{X}, \mathcal{L})$ of (X, L) induces a finitely generated $\mathbb{Z}$-filtration of R. See [BHJ17, Section 2.5] for details.

2.1.1. Successive minima

Given a basis $(s_1, \ldots, s_{N_m})$ of R_m compatible with $\mathcal{F}$, the numbers

$$\lambda_j^{(m)} := \mathrm{ord}_\mathcal{F}(s_j) \quad \text{for } 1 \leq j \leq N_m$$

are called the *successive minima* of $\mathcal{F}$ along R. Since $\frac{d}{d\lambda} \dim \mathcal{F}^\lambda R_m = -\sum_j \delta_{\lambda_j^{(m)}}$ the values $(\lambda_j^{(m)})_j$ are independent of the choice of compatible basis up to reordering. We write $\lambda_{\max}^{(m)} = \max\{\lambda \in \mathbb{R} \mid \mathcal{F}^\lambda R_m \neq 0\}$ and set $\lambda_{\max} := \sup_{m \geq 1} \frac{\lambda_{\max}^{(m)}}{mr}$.

2.1.2. Graded linear series

A graded linear series $V_\bullet = (V_m)_{m \in \mathbb{N}}$ of L is a collection of vector subspaces $V_m \subset R_m$ satisfying $V_m \cdot V_{m'} \subset V_{m+m'}$. The *volume* of $V_\bullet$ is the value

$$\mathrm{vol}(V_\bullet) := \limsup_{m \to \infty} \frac{\dim V_m}{m^n/n!}.$$

Given a filtration $\mathcal{F}$ of R and $s \in \mathbb{R}$, we define a graded linear series $V_\bullet^{(s)}$ by setting $V_m^{(s)} := \mathcal{F}^{mrs} R_m$. When the choice of filtration is not clear from context, we will denote it by $V_\bullet^{\mathcal{F},(s)}$.

The following result is a consequence of [BC11]. See [BHJ17, Theorem 5.2].

Proposition 2.3. *Let $\mathcal{F}$ be a linearly bounded filtration of R.*

(i) *For each $s < \lambda_{\max}$, $\mathrm{vol}(V_\bullet^{(s)})$ is a limit.*
(ii) *The function $s \mapsto \mathrm{vol}(V_\bullet^{(s)})^{1/n}$ is concave on $(-\infty, \lambda_{\max})$ and vanishes on $(\lambda_{\max}, \infty)$ (hence, it is continuous away from $s = \lambda_{\max}$).*

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2.1.3. Base ideals

Given a linearly bounded filtration $\mathcal{F}$ of R, we set

$$I_{m,\lambda} := \mathrm{im}(\mathcal{F}^{\lambda} R_m \otimes_k \mathcal{O}_X(-mL) \to \mathcal{O}_X),$$

which is the base ideal of the linear series $\mathcal{F}^{\lambda} R_m$. Note that $I_{m,\lambda} \cdot I_{m',\lambda'} \subset I_{m+m',\lambda+\lambda'}$.

For each $\lambda \in \mathbb{R}$, we write $I_{\bullet}^{(\lambda)}$ for the graded sequence of ideals on X defined by setting $I_m^{(\lambda)} := I_{m,\lambda mr}$. For each $m \in \mathbb{Z}_{>0}$, we set

$$\mathcal{I}_m := \sum_{i \in \mathbb{Z}} I_{m,i} t^{-i} \subset K(X)(t) \simeq K(X \times \mathbb{A}^1).$$

Note that $t^{\lceil \lambda_{\max}^{(m)} \rceil} \cdot \mathcal{I}_m \subset \mathcal{O}_{X_{\mathbb{A}^1}}$ since $I_{m,i} = 0$ for $i > \lambda_{\max}^{(m)}$. Therefore, $\mathcal{I}_m$ is a fractional ideal on $X_{\mathbb{A}^1} := X \times \mathbb{A}^1$.

2.1.4. Duistermaat–Heckman measure

Given a linearly bounded filtration $\mathcal{F}$ of R and an integer $m > 0$, we consider the probability measure on $\mathbb{R}$ defined by

$$\nu_m^{\mathcal{F}} := \frac{1}{N_m} \sum_{i=1}^{N_m} \delta_{(mr)^{-1}\lambda_i^{(m)}} = -\frac{d}{d\lambda} \frac{\dim \mathcal{F}^{mr\lambda} R_m}{N_m}.$$

By [BC11] (see [BHJ17, §5.1]), $\nu_m^{\mathcal{F}}$ converges weakly as $m \to \infty$ to the probability measure

$$\mathrm{DH}_{\mathcal{F}} := -\frac{d}{d\lambda} \frac{\mathrm{vol}(V_{\bullet}^{(\lambda)})}{L^n}.$$

The measure satisfies $\mathrm{supp}(\mathrm{DH}_{\mathcal{F}}) = [\lambda_{\min}, \lambda_{\max}]$, where

$$\lambda_{\min} := \inf\{\lambda \in \mathbb{R} \mid \mathrm{vol}(V_{\bullet}^{(\lambda)}) < (L^n)\}$$

and $\lambda_{\max}$ is as defined previously.

The following statement is an extension of [BHJ17, Lemma 5.13] from divisorial valuations to valuations with finite log discrepancy.

Lemma 2.4. *If $v \in \mathrm{Val}_X$ and $A_{X,\Delta}(v) < +\infty$, then $\lambda_{\min}(\mathcal{F}_v) = 0$.*

Proof. It is clear that $\lambda_{\min}(\mathcal{F}_v) \geq 0$ since $\mathcal{F}_v^{\lambda} R = R$ for $\lambda \leq 0$. For the reverse inequality, fix a log resolution $Y \to X$ of (X, Δ) and let $\xi := c_Y(v)$ be the center of v on Y. Note that $A_{Y,0}(v)$ is finite since $A_{X,\Delta}(v) < +\infty$ by assumption. An Izumi type inequality [JM12, Proposition 5.10] implies

$$v(f) \leq c \cdot \mathrm{ord}_{\xi}(f) \qquad \text{for all } f \in \mathcal{O}_{Y,\xi},$$

where $c := A_{Y,0}(v) > 0$, and, hence, $\mathcal{F}_v^{\lambda} \subset \mathcal{F}_{c \cdot \mathrm{ord}_{\xi}}^{\lambda} R$ for all $\lambda \in \mathbb{R}$. Therefore, $\lambda_{\min}(\mathcal{F}_v) \leq \lambda_{\min}(\mathcal{F}_{c \cdot \mathrm{ord}_{\xi}}) = 0$, where the equality holds by [BHJ17, Lemma 5.13], since $c \cdot \mathrm{ord}_{\xi}$ is a divisorial valuation. □

2.2. *Non-Archimedean functionals*

2.2.1. Energy functional

Following [BHJ17], the Monge–Ampère energy of $\mathcal{F}$ is given by

$$\mathbf{E}^{\mathrm{NA}}(\mathcal{F}) := \int_{\mathbb{R}} \lambda \, \mathrm{DH}_{\mathcal{F}}(d\lambda)$$

which is the barycenter of $\mathrm{DH}_{\mathcal{F}}$. When $\mathcal{F} = \mathcal{F}_v$ for a valuation $v \in \mathrm{Val}_X$ with $A_{X,\Delta}(v) < \infty$, $\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_v) = S(v)$ and $\lambda_{\max}(\mathcal{F}_v) = T(v)$, where $S(\cdot)$ and $T(\cdot)$ are the expected and maximal vanishing order appearing in [BJ20].

2.2.2. Ding-functional

The *Ding invariant* of a linearly bounded filtration $\mathcal{F}$ is defined by

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{F}) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}) - \mathbf{E}^{\mathrm{NA}}(\mathcal{F}),$$

where $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) = \lim_{m\to\infty} \mathrm{lct}(X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1}, \mathcal{I}_m^{\frac{1}{mr}}; (t)) - 1$ and

$$\mathrm{lct}(X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1}, \mathcal{I}_m^{\frac{1}{mr}}; (t)) := \sup\{c \in \mathbb{R} \mid (X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1}, \mathcal{I}_m^{\frac{1}{mr}} \cdot (t)^c) \text{ is sub-lc}\}.$$

This invariant was introduced in [Ber16] for test configurations and [BHJ17, Fuj19] for general filtrations.

2.2.3. $\mathbf{H}^{\mathrm{NA}}$-functional

Following [HL20b], for a linearly bounded filtration $\mathcal{F}$ we set

$$\mathbf{H}^{\mathrm{NA}}(\mathcal{F}) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}) - \widetilde{S}(\mathcal{F}),$$

where $\mathbf{L}^{\mathrm{NA}}(\mathcal{F})$ is defined above and $\widetilde{S}(\mathcal{F}) := -\log \int_{\mathbb{R}} e^{-\lambda}\, \mathrm{DH}_{\mathcal{F}}(d\lambda)$. This invariant was introduced in [TZZZ13] for holomorphic vector fields and then extended to $\mathbb{R}$-test configurations in [DS20] and linearly bounded filtrations in [HL20b]. We set

$$h(X,\Delta) := \inf_{\mathcal{F}} \mathbf{H}^{\mathrm{NA}}(\mathcal{F}),$$

where the infimum runs through all linearly bounded filtrations of $\mathcal{F}$. By [HL20b, Corollary 4.7], $h(X,\Delta) \le 0$ and equality holds iff (X,Δ) is K-semistable.

For a valuation $v \in \mathrm{Val}_X^{\circ} \cup \{v_{\mathrm{triv}}\}$, we define

$$\tilde{\beta}_{X,\Delta}(v) := A_{X,\Delta}(v) - \widetilde{S}(v),$$

where $\widetilde{S}(v) = \widetilde{S}(\mathcal{F}_v)$. Note that $\tilde{\beta}_{X,\Delta}(v_{\mathrm{triv}}) = 0$. By [HL20b, Theorem 1.5],

$$h(X,\Delta) = \inf_{v \in \mathrm{Val}_X^{\circ} \cup \{v_{\mathrm{triv}}\}} \tilde{\beta}(v).$$

We say that $v \in \mathrm{Val}_X^{\circ} \cup \{v_{\mathrm{triv}}\}$ computes $h(X,\Delta)$ if it achieves the above infimum. By [HL20b, Theorem 4.9], there always exists a quasi-monomial valuation computing $h(X,\Delta)$.

3. Convexity and uniqueness

In this section, we will obtain the uniqueness of the valuation computing $h(X,\Delta)$. In [HL20b], this was proved to follow from the finite generation, that is, Theorem 1.2. In this section, instead of using the finite generation, we will take the approach of establishing more general convexity results. In fact, for two filtrations $\mathcal{F}_0$ and $\mathcal{F}_1$, we consider a segment in the space of filtrations $(\mathcal{F}_t)_{t\in[0,1]}$, which we call the *geodesic* between the two filtrations. We then introduce a probability measure on $\mathbb{R}^2$ that encodes $\mathrm{DH}_{\mathcal{F}_t}$ for $t \in [0,1]$. This will allow us to deduce the convexity of a number of functionals which take the form of integrating over the Duistermaat–Heckman measure (DH). For $\mathbf{L}^{\mathrm{NA}}$, the proof of its convexity uses the ideas from [XZ21] in the local setting.

Throughout, (X,Δ) is a log Fano pair, $r > 0$ a rational number so that $L := -r(K_X + \Delta)$ is a Cartier divisor, and $R := R(X,L)$.

3.1. Geodesics and DH measures

Fix two linearly bounded filtrations $\mathcal{F}_0$ and $\mathcal{F}_1$ of R. For each integer $m > 0$, choose a basis $(s_1, \dots, s_{N_m})$ of R_m that is compatible with both $\mathcal{F}_0$ and $\mathcal{F}_1$ simultaneously; see [AZ22, Lemma 3.1] or [BE21, Proposition 1.14] for the existence of such a basis. For $1 \le i \le N_m$, we set

$$\lambda_i^{0,(m)} := \mathrm{ord}_{\mathcal{F}_0}(s_i) \quad \text{and} \quad \lambda_i^{1,(m)} = \mathrm{ord}_{\mathcal{F}_1}(s_i).$$

The pairs $(\lambda_i^{0,(m)}, \lambda_i^{1,(m)})$ are unique up to reordering. For example, this follows from the observation that $-\frac{\partial^2}{\partial x \partial y} \dim(\mathcal{F}_0^x R_m \cap \mathcal{F}_1^y R_m) = \sum_i \delta_{\lambda_i^{0,(m)}, \lambda_i^{1,(m)}}$. The above basis and notation will be used in the constructions below.

3.1.1. Relative limit measure

For each integer $m > 0$, we define a probability measure on $\mathbb{R}$ by

$$\nu_m^{\mathcal{F}_0,\mathcal{F}_1} := \frac{1}{N_m} \sum_{i=1}^{N_m} \delta_{(mr)^{-1}(\lambda_i^{0,(m)} - \lambda_i^{1,(m)})}.$$

It was proven in [CM15] that $\nu_m^{\mathcal{F}_0,\mathcal{F}_1}$ converges weakly as $m \to \infty$ to a compactly supported probability measure that we denote by $\mathrm{RLM}_{\mathcal{F}_0,\mathcal{F}_1}$. See [BJ21, Theorem 3.3] for the statement and proof written in our setting.

The L^1*-distance* [BJ21, Section 3.4] between $\mathcal{F}_0$ and $\mathcal{F}_1$ is defined by

$$d_1(\mathcal{F}_0, \mathcal{F}_1) := \int_{\mathbb{R}} |\lambda| \mathrm{RLM}_{\mathcal{F}_0,\mathcal{F}_1}(d\lambda).$$

We say $\mathcal{F}_0$ and $\mathcal{F}_1$ are *equivalent* if $d_1(\mathcal{F}_0, \mathcal{F}_1) = 0$.

Proposition 3.1 [BJ21, Corollary 3.13]. *If $\mathcal{F}_0$ and $\mathcal{F}_1$ are equivalent, then* $\mathrm{DH}_{\mathcal{F}_0} = \mathrm{DH}_{\mathcal{F}_1}$.

3.1.2. Geodesics

Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be linearly bounded filtrations of R. For $t \in (0,1)$, we define a filtration $\mathcal{F}_t$ of R by setting

$$\mathcal{F}_t^\lambda R_m := \sum_{\mu(1-t)+\nu t \ge \lambda} \mathcal{F}_0^\mu \mathcal{F}_m \cap \mathcal{F}_1^\nu R_m. \tag{3.1}$$

It is straightforward to check that $\mathcal{F}_t$ is a filtration of R and is linearly bounded. We will call $(\mathcal{F}_t)_{t \in [0,1]}$ the *geodesic* connecting $\mathcal{F}_0$ and $\mathcal{F}_1$.

An alternative way to describe $\mathcal{F}_t$ is in terms of the basis $(s_1, \dots, s_{N_m})$ of R_m fixed earlier. Indeed, since $\mathcal{F}_0^\mu R_m \cap \mathcal{F}_1^\nu R_m = \mathrm{span}\langle s_i \mid \lambda_i^{0,(m)} \ge \mu \text{ and } \lambda_i^{1,(m)} \ge \nu \rangle$, it follows that

$$\mathcal{F}_t^\lambda R_m = \mathrm{span}\langle s_i \mid \lambda_i^{0,(m)}(1-t) + \lambda_i^{1,(m)} t \ge \lambda \rangle.$$

Therefore, the basis $(s_1, \dots, s_{N_m})$ is compatible with $\mathcal{F}_t$ and $\mathrm{ord}_{\mathcal{F}_t}(s_i) = (1-t)\lambda_i^{0,(m)} + t\lambda_i^{1,(m)}$. As a consequence of this observation,

$$\nu_m^{\mathcal{F}_t} = \frac{1}{N_m} \sum_{i=1} \delta_{(mr)^{-1}(\lambda_i^{0,(m)}(1-t) + \lambda_i^{1,(m)} t)}. \tag{3.2}$$

In the following section, will analyze a measure on $\mathbb{R}^2$ that encodes equation (3.2) for each $t \in [0,1]$.

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Remark 3.2. In the language of graded norms, the definition of $(\mathcal{F}_t)_{t\in[0,1]}$ appears in the work of Reboulet in a more general setting and plays a key role in his theory of geodesics in the space of non-Archimedean metrics on a line bundle [Reb22].

3.1.3. Duistermaat—Heckman measures

For each $m > 0$, we consider the probability measure on $\mathbb{R}^2$ defined by

$$\boldsymbol{\nu}_m := \frac{1}{N_m}\sum_{i=1}^{N_m} \delta_{((mr)^{-1}\lambda_i^{0,(m)},(mr)^{-1}\lambda_i^{1,(m)})} = \frac{\partial^2}{\partial x \partial y}\frac{\dim(\mathcal{F}_0^{mrx}R_m \cap \mathcal{F}_1^{mry}R_m)}{N_m}.$$

Since $\mathcal{F}_0$ and $\mathcal{F}_1$ are assumed to be linearly bounded, we may fix $C > 0$ so that $\mathcal{F}_i^{Cmr}R_m = 0$ and $\mathcal{F}_i^{-Cmr}R_m = R_m$ for both $i = 0, 1$. Hence, $\mathrm{supp}(\boldsymbol{\nu}_m)$ is contained in the bounded set $[-C, C] \times [-C, C]$.

Theorem 3.3. *The sequence $\boldsymbol{\nu}_m$ converges weakly as $m \to \infty$ to the compactly supported probability measure*

$$\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1} := \frac{\partial^2}{\partial x\partial y}\frac{\mathrm{vol}(W_\bullet^{(x,y)})}{L^n},$$

where $W_\bullet^{(x,y)}$ is the graded linear series defined by $W_m^{(x,y)} = \mathcal{F}_0^{mrx}R_m \cap \mathcal{F}_1^{mry}R_m$.

We will call $\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1}$ the *compatible DH measure* of the two filtrations. The use of the measure is that it encodes $\mathrm{DH}_{\mathcal{F}_t}$ for $t \in [0, 1]$, as well as $\mathrm{RLM}_{\mathcal{F}_0,\mathcal{F}_1}$ (see Proposition 3.6).

To prove Theorem 3.3, we analyze the following functions $\mathbb{R}^2 \to [0, 1]$ that are nonincreasing in both variables:

$$f_m(x, y) = \frac{\dim(W_m^{(x,y)})}{N_m} \quad \text{and} \quad f(x, y) := \limsup_{m\to\infty} f_m(x, y) = \frac{\mathrm{vol}(W_\bullet^{(x,y)})}{(L^n)},$$

as well as the locus $P = \overline{\bigcup_{m\geq 1} P_m}$ where $P_m = \mathrm{Supp}(f_m)$.

Proposition 3.4. *The set P is convex and* $\mathrm{Int}(P) = \cup_m \mathrm{Int}(P_m)$

Proof. Using property (F4) of a filtration, it follows that

$$cmP_m + dqP_q \subset (cm + dq)P_{mc+qd} \quad \text{for all } c, d, m, q \in \mathbb{Z}_{>0}. \tag{3.3}$$

Indeed, if $(x, y) \in cmP_m$ and $(x', y') \in dqP_q$, then there exist nonzero sections

$$s \in \mathcal{F}_0^{rx/c}R_m \cap \mathcal{F}_1^{ry/c}R_m \quad \text{and} \quad s' \in \mathcal{F}_0^{rx'/d}R_q \cap \mathcal{F}_1^{ry'/d}R_q.$$

Hence,

$$s^c s'^d \in \mathcal{F}_0^{r(x+x')}R_{mc+md} \cap \mathcal{F}_1^{r(y+y')}R_{mc+md}$$

which implies $(x+x', y+y') \in (mc+qd)P_{mc+qd}$ as desired. Now, equation (3.3) implies: if $p, q \in \cup_m P_m$ and $t \in [0, 1] \cap \mathbb{Q}$, then $p(1-t) + tq \in \cup_m P_m$. Therefore, the closure of $\cup_m P_m$ is convex.

To show $\mathrm{Int}(P) = \cup_m \mathrm{Int}(P_m)$, first note that the inclusion $\supset$ clearly holds. To see $\subset$ holds, fix $(a, b) \in \mathrm{Int}(P)$. Since $\mathrm{Int}(P)$ is open, we may choose $\epsilon > 0$ so that $(a+\epsilon, b+\epsilon) \in \mathrm{Int}(P)$. Since P is the closure of $\cup_m P_m$ and $(a+\epsilon, b+\epsilon) \in P$, there exists $(x, y) \in \cup_m P_m$ so that $a < x$ and $b < y$. Using that each f_m is ≥ 0 and nonincreasing in both variables, the latter implies $(a, b) \in \cup_m \mathrm{Int}(P_m)$ as desired. □

Proposition 3.5. *On the locus $\mathbb{R}^2 \setminus \partial P$, $f = \lim_{m\to\infty} f_m$ and f is continuous.*

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Proof. The statement clearly holds on $\mathbb{R}^2 \setminus P$ since f_m and f are both zero on that locus. It remains to verify the statement on $\mathrm{Int}(P)$.

Fix $(a, b) \in \mathrm{Int}(P)$. Let $\mathcal{G}$ denote the filtration of R defined by

$$\mathcal{G}^\lambda R_m := \mathcal{F}_0^{\lambda+mra} R_m \cap \mathcal{F}_1^{\lambda+mrb} R_m,$$

which is linearly bounded since both $\mathcal{F}_0$ and $\mathcal{F}_1$ are linearly bounded. Let $V_m^{\mathcal{G},(t)}$ and $V_\bullet^{\mathcal{G},(t)}$ be defined as in Section 2.1.2. If we set

$$g_m(t) = \frac{\dim V_m^{\mathcal{G},(t)}}{N_m} \quad \text{and} \quad g(t) = \limsup_{m\to\infty} \frac{\mathrm{vol}(V_\bullet^{\mathcal{G},(t)})}{(L^n)},$$

then $g_m(t) = f_m(a+t, b+t)$ and $g(t) = f(a+t, b+t)$ since $V_m^{\mathcal{G},(t)} = W_m^{(a+t,b+t)}$. Note that, for $t < \lambda_{\max}(\mathcal{G})$, $g(t) = \lim_{m\to\infty} g_m(t)$ exists and g is continuous at t by Proposition 2.3.

We claim that $\lambda_{\max}(\mathcal{G}) > 0$. Indeed, using that $g_m(t) = f_m(a+t, b+t)$, we see

$$(mr)^{-1}\lambda_{\max}^{(m)}(\mathcal{G}) = \sup\{t \in \mathbb{R} \mid (a+t, b+t) \in P_m\}$$

Since $(a, b) \in \mathrm{Int}(P)$, Proposition 3.4 implies there exists $m' > 0$ so that $(a, b) \in \mathrm{Int}(P_{m'})$. Therefore, $\lambda_{\max}^{(m')}(\mathcal{G}) > 0$ and, hence, $\lambda_{\max}(\mathcal{G}) > 0$ as desired.

Using the above claim, we see that $\lim_{m\to\infty} f_m(a, b) = f(a, b)$, and $f(a+t, b+t)$ is continuous at $t = 0$. Since f is nonincreasing in both variables, the latter implies that f is continuous at (a, b). □

Theorem 3.3 is now an easy consequence of the previous propositions.

Proof of Theorem 3.3. As $m \to \infty$, f_m converge pointwise to f away from a set of measure zero by by Propositions 3.4 and 3.5. Since $0 \le f_m \le 1$, the dominated converges theorem implies $f_m \to f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Therefore, $f_m \to f$ as distributions and, hence, $\boldsymbol{\nu}_m = \frac{\partial^2}{\partial x \partial y} f_m \to \frac{\partial^2}{\partial x\, \partial y} f$ as distributions, as well. Since each distribution $\boldsymbol{\nu}_m$ is a measure, [Hor03, Theorem 2.1.9] implies $\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1} := \frac{\partial^2}{\partial x\, \partial y} f$ is a measure and $\boldsymbol{\nu}_m \xrightarrow{\text{weak}} \mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1}$ as measures. Furthermore, the measure $\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1}$ is a compactly supported probability measure since it is a weak limit of probability measures with uniformly bounded support. □

Proposition 3.6. *Fix $t \in [0, 1]$, $c \in \mathbb{R}_{>0}$, and $d \in \mathbb{R}$. Consider the maps $p, q : \mathbb{R}^2 \to \mathbb{R}$ defined by $p(x, y) = (1-t)x + ty$ and $q(x, y) = x - c(y + d)$. The following hold:*

1. $p_*(\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1}) = \mathrm{DH}_{\mathcal{F}_t}$, *and*
2. $q_*(\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1}) = \mathrm{RLM}_{\mathcal{F}_0,\mathcal{G}}$, *where $\mathcal{G}$ is filtration given by $\mathcal{G}^\lambda R_m := \mathcal{F}_1^{(\lambda-dmr)/c} R_m$.*

Proof. Observe that $p_*(\boldsymbol{\nu}_m) = \nu_m^{\mathcal{F}_t}$ and $q_*(\boldsymbol{\nu}_m) = \boldsymbol{\nu}_m^{\mathcal{F}_0,\mathcal{G}}$. Therefore, $p_*(\boldsymbol{\nu}_m) \xrightarrow{\text{weak}} \mathrm{DH}_{\mathcal{F}_t}$ and $q_*(\boldsymbol{\nu}_m) \xrightarrow{\text{weak}} \mathrm{RLM}_{\mathcal{F}_0,\mathcal{G}}$. By Theorem 3.3 and the continuity of p and q, we also have $p_*(\boldsymbol{\nu}_m) \xrightarrow{\text{weak}} p_*(\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1})$ and $q_*(\boldsymbol{\nu}_m) \xrightarrow{\text{weak}} q_*(\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1})$. Since weak limits of measure on $\mathbb{R}^2$ are unique, the result follows. □

3.2. Convexity

In this section, we prove the following result on the convexity of the non-Archimedean Ding and H-functionals.

Theorem 3.7. *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be linearly bounded filtrations of R and $(\mathcal{F}_t)_{t\in[0,1]}$ be the geodesic connecting them. For $t \in (0, 1)$, the following hold:*

1. $\mathbf{D}^{\mathrm{NA}}(\mathcal{F}_t) \le (1-t)\mathbf{D}^{\mathrm{NA}}(\mathcal{F}_0) + t\mathbf{D}^{\mathrm{NA}}(\mathcal{F}_1)$;
2. $\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_t) \le (1-t)\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_0) + t\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_1)$.

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Furthermore, the inequality in (ii) is strict unless there exists $d \in \mathbb{R}$ so that $d_1(\mathcal{F}_0, \mathcal{G}) = 0$, where $\mathcal{G}$ is the filtration defined by $\mathcal{G}^\lambda R_m := \mathcal{F}_1^{\lambda - dmr} R_m$.

The result is an algebraic analogue of a theorem of Berndtsson on the convexity of the Ding-functional along geodesics in the space of Kähler potentials [Ber15]. In forthcoming work, Reboulet shows that Theorem 3.7 can in fact be deduced from Berndtsson's convexity result when X is smooth [Reb21]. The proof below is self-contained and purely algebraic.

To prove Theorem 3.7, we first show the convexity of $\mathbf{L}^{\mathrm{NA}}$ along geodesics. For this, we first compare $\mathbf{L}^{\mathrm{NA}}$ with the log canonical slope of a filtration $\mathcal{F}$, defined as (c.f. [XZ20, Definition 1.3 and Lemma 4.13])

$$\mu(\mathcal{F}) := \mu_{X,\Delta}(\mathcal{F}) := \sup\left\{ s \in \mathbb{R} \mid \mathrm{lct}(X, \Delta; I_\bullet^{(s)}) \geq \frac{1}{r} \right\} = \sup\left\{ s \in \mathbb{R} \mid \mathrm{lct}(X, \Delta; I_\bullet^{(s)}) > \frac{1}{r} \right\}.$$

Lemma 3.8. *For any linearly bounded filtration $\mathcal{F}$ of R we have $\mu(\mathcal{F}) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F})$, and there exists some valuation $v \in \mathrm{Val}_X^\circ \cup \{v_{\mathrm{triv}}\}$ such that*

$$r^{-1} v(I_\bullet^{(\lambda)}) \geq \lambda + A_{X,\Delta}(v) - \mathbf{L}^{\mathrm{NA}}(\mathcal{F}) \tag{3.4}$$

for all $\lambda \in \mathbb{R}$. Moreover, if $\mathcal{F}$ is a finitely generated $\mathbb{Z}$-filtration and $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) < \lambda_{\max}(\mathcal{F})$, then v can be chosen to be a divisorial lc place of some $\mathbb{Q}$-complement.

Recall that a valuation v is said to be an lc place of some $\mathbb{Q}$-complement if there exists some effective $\mathbb{Q}$-divisor $\Gamma \sim_{\mathbb{Q}} -(K_X + \Delta)$ such that $(X, \Delta + \Gamma)$ is lc and $A_{X,\Delta+\Gamma}(v) = 0$.

Proof. By [XZ20, Theorem 4.3], we already have $\mu(\mathcal{F}) \geq \mathbf{L}^{\mathrm{NA}}(\mathcal{F})$, thus it suffices to show $\mu(\mathcal{F}) \leq \mathbf{L}^{\mathrm{NA}}(\mathcal{F})$. By [JM12, Theorem 7.3], $\mathrm{lct}(X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1} + \mathcal{I}_\bullet^{\frac{1}{r}}; (t)) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}) + 1$ is computed by some $\mathbb{G}_m$-invariant valuation $w \in \mathrm{Val}^\circ_{X \times \mathbb{A}^1}$ (the $\mathbb{G}_m$-equivariant version is not proved in [JM12] but is not hard to achieve from the proof). By [BHJ17, Lemma 4.2], up to rescaling w is the Gauss extension of a valuation $v \in \mathrm{Val}_X^\circ \cup \{v_{\mathrm{triv}}\}$, that is, $w(ft^i) = v(f) + i$ for any $0 \neq f \in K(X)$ and $i \in \mathbb{Z}$. Since w computes the lct and $w(t) = 1$, we have

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) + 1 = A_{X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1}}(w) - w(\mathcal{I}_\bullet^{\frac{1}{r}}) = A_{X,\Delta}(v) + 1 - w(\mathcal{I}_\bullet^{\frac{1}{r}}).$$

Thus, $A_{X,\Delta}(v) - \mathbf{L}^{\mathrm{NA}}(\mathcal{F}) = w(\mathcal{I}_\bullet^{\frac{1}{r}}) \leq w(\mathcal{I}_m^{\frac{1}{mr}}) \leq \frac{v(I_{m,i}) - i}{mr}$ for all $m \in \mathbb{N}$ and $i \in \mathbb{Z}$. It follows that

$$r^{-1} v(I_\bullet^{(\lambda)}) \geq \lambda + A_{X,\Delta}(v) - \mathbf{L}^{\mathrm{NA}}(\mathcal{F})$$

for all $\lambda \in \mathbb{R}$. In particular, $\mathrm{lct}(X, \Delta; I_\bullet^{(\lambda)}) < r^{-1}$ for any $\lambda > \mathbf{L}^{\mathrm{NA}}(\mathcal{F})$. By the definition of log canonical slope, this implies $\mu(\mathcal{F}) \leq \mathbf{L}^{\mathrm{NA}}(\mathcal{F})$ and proves the first part of the lemma.

If $\mathcal{F}$ is finitely generated, then $\mathcal{I}_{pm} = \mathcal{I}_m^p$ for any sufficiently divisible $m, p \in \mathbb{N}$ and thus w can be chosen as a divisorial valuation. Since $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) < \lambda_{\max}(\mathcal{F})$, the valuation v cannot be the trivial one, otherwise equation (3.4) becomes $\lambda \leq \mathbf{L}^{\mathrm{NA}}(\mathcal{F})$ for all $\lambda < \lambda_{\max}(\mathcal{F})$ and therefore $\lambda_{\max}(\mathcal{F}) \leq \mathbf{L}^{\mathrm{NA}}(\mathcal{F})$, a contradiction. By [BHJ17, Lemma 4.1], we know that v is divisorial. Let $\mathfrak{a}_m = I_{m,\mu m r}$. Again, $\mathfrak{a}_{pm} = \mathfrak{a}_m^p$ for any sufficiently divisible $m, p \in \mathbb{N}$ as $\mathcal{F}$ is finitely generated. By equation (3.4), we have $r^{-1} v(\mathfrak{a}_\bullet) \geq A_{X,\Delta}(v)$; thus, from the definition of log canonical slope, we see that v necessarily computes

$$\mathrm{lct}(\mathfrak{a}_\bullet) = m \cdot \mathrm{lct}(\mathfrak{a}_m) = m \cdot \mathrm{lct}(X, \Delta; \{s = 0\})$$

for sufficiently divisible m and general $s \in \mathcal{F}^{\mu m r} R_m$. As $K_X + \Delta + \frac{1}{mr}\{s = 0\} \sim_{\mathbb{Q}} 0$, this easily implies that v is an lc place of some $\mathbb{Q}$-complement. □

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Remark 3.9. From the above proof, it is clear that if the filtration $\mathcal{F}$ is $\mathbb{T}$-equivariant for some torus $\mathbb{T} < \mathrm{Aut}(X, \Delta)$, then the valuation v can be chosen to be $\mathbb{T}$-invariant as well.

Remark 3.10. Lemma 3.8 immediately implies that $\beta(\mathcal{F}) = \mathbf{D}^{\mathrm{NA}}(\mathcal{F})$ (see [XZ20, Definition 4.1] for the definition $\beta(\mathcal{F})$) for any linearly bounded multiplicative filtration $\mathcal{F}$.

Corollary 3.11. *For any $v \in \mathrm{Val}^{\circ}_X$, we have $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_v) \le A_{X,\Delta}(v)$ and $\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_v) \le \tilde{\beta}_{X,\Delta}(v)$.*

Proof. It is not hard to see from the definition that $\mu(\mathcal{F}_v) \le A_{X,\Delta}(v)$ (c.f. [XZ20, Proposition 4.2]), thus the first inequality follows from Lemma 3.8. The second inequality follows from the first and the definition of $\mathbf{H}^{\mathrm{NA}}$ and $\tilde{\beta}$. □

Given the equality $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) = \mu(\mathcal{F})$ (see Lemma 3.8), we can establish the convexity of $\mathbf{L}^{\mathrm{NA}}$ using [XZ21].

Proposition 3.12. *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be linearly bounded filtrations of R and $(\mathcal{F}_t)_{t\in[0,1]}$ be the geodesic connecting them. For $t \in (0, 1)$, $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_t) \le (1-t)\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_0) + t\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_1)$.*

Proof. It is not hard to see that the statement is unaffected by translation of the filtrations. Thus, by Lemma 3.8, we may assume that, after shifting the filtrations, there exists valuations v_0, v_1 on X with $A_{X,\Delta}(v_i) < \infty$ such that $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_i) = \mu(\mathcal{F}_i) = A_{X,\Delta}(v_i)$ and $v_i(I^{(\lambda)}_{\mathcal{F}_i,\bullet}) \ge r\lambda$ for all $\lambda \in \mathbb{R}$ and $i = 0, 1$. In particular, $\mathcal{F}^{\lambda}_i R \subseteq \mathcal{F}^{\lambda}_{v_i} R$ for all $\lambda \in \mathbb{R}$.

Let $(Y = \mathrm{Spec}(R), \Gamma)$ denote the affine cone over (X, Δ) with respect to the polarization $L = -r(K_X + \Delta)$. Let w_i be the $\mathbb{G}_m$-invariant valuation on Y given by $w_i(s) = mr + v_i(s)$ for $s \in R_m$ (informally $w_i = r \cdot \mathrm{ord}_o + v_i$). Let $\mathfrak{b}_{t,\bullet}$ be the graded sequence of ideals on Y defined by

$$\mathfrak{b}_{t,m} = \mathfrak{a}_\bullet((1-t)w_0) \boxplus \mathfrak{a}_\bullet(tw_1) := \sum_{i=0}^{m} \mathfrak{a}_{m-i}((1-t)w_0) \cap \mathfrak{a}_i(tw_1).$$

In other words, $\mathfrak{b}_{t,m}$ is generated by those $s \in R$ with $\lfloor (1-t)w_0(s) \rfloor + \lfloor tw_1(s) \rfloor \ge m$. For any $k \in \mathbb{Z}$ and any $s \in \mathcal{F}^{k+2}_t R_m$, we have $(1-t)w_0(s) + tw_1(s) \ge mr + k + 2$ by equation (3.1). It follows that s is a section of $\mathfrak{b}_{t,mr+k}$ (for if $a + b \ge k + 2$ then $\lfloor a \rfloor + \lfloor b \rfloor \ge k$). Therefore, elements in $\mathcal{F}^{k+2}_t R_m$ yield sections of $\mathfrak{b}_{t,mr+k}$ on Y for any $k \in \mathbb{Z}$. By [XZ21, Theorem 3.11], we have

$$\begin{aligned}\mathrm{lct}(\mathfrak{b}_{t,\bullet}) &\le \mathrm{lct}(\mathfrak{a}_\bullet((1-t)w_0)) + \mathrm{lct}(\mathfrak{a}_\bullet(tw_1)) \le (1-t)A_{Y,\Gamma}(w_0) + tA_{Y,\Gamma}(w_1)\\ &= 1 + (1-t)A_{X,\Delta}(v_0) + tA_{X,\Delta}(v_1) = 1 + (1-t)\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_0) + t\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_1).\end{aligned}$$

Thus, for any rational $c > (1-t)\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_0) + t\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_1)$ and any $s \in \mathcal{F}^{cmr+2}_t R_m$ (where m is sufficiently divisible), as it yields a section of $\mathfrak{b}_{t,(1+c)mr}$ on Y, the pair $(Y, \Gamma + \frac{1}{mr}\{s = 0\})$ is not lc. Using [Kol13, Lemma 3.1(5)], it follows that the base $(X, \Delta + \frac{1}{mr}\{s = 0\})$ is not lc for any $m \in \mathbb{N}$ and any $s \in \mathcal{F}^{cmr+2}_t R_m$. By the definition of log canonical slope, this implies that $\mu(\mathcal{F}_t) \le c$. By Lemma 3.8 and the fact that $c > (1-t)\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_0) + t\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_1)$ was arbitrary, the result follows. □

Using the measure $\mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1}$ constructed in Section 3.1.3, we next describe the behavior of the Monge–Ampère Energy and $\widetilde{S}$ functionals along the geodesic.

Proposition 3.13. *For $t \in [0, 1]$, $\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_t) = (1-t)\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_0) + t\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_1)$.*

Proof. Set $\boldsymbol{\nu} := \mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1}$. We compute

$$\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_t) = \int_{\mathbb{R}} \lambda\, \mathrm{DH}_{\mathcal{F}_t}(\mathrm{d}\lambda) = \int_{\mathbb{R}^2} ((1-t)x + ty)\, \mathrm{d}\boldsymbol{\nu} = (1-t)\int_{\mathbb{R}^2} x\, \mathrm{d}\boldsymbol{\nu} + t\int_{\mathbb{R}^2} y\, \mathrm{d}\boldsymbol{\nu}, \tag{3.5}$$

where the second equality is by Proposition 3.6. From this, the result follows. □

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Proposition 3.14. *For $t \in (0,1)$, $\widetilde{S}(\mathcal{F}_t) \geq (1-t)\widetilde{S}(\mathcal{F}_0) + t\widetilde{S}(\mathcal{F}_1)$. Furthermore, the inequality is strict unless there exists $d \in \mathbb{R}$ so that $d_1(\mathcal{F}_0, \mathcal{G}) = 0$, where $\mathcal{G}$ is the filtration defined by $\mathcal{G}^\lambda R_m := \mathcal{G}^{\lambda - mrd} R_m$.*

Proof. Set $\boldsymbol{\nu} := \mathrm{DH}_{\mathcal{F}_0,\mathcal{F}_1}$, $f(x,y) := e^{-x}$, and $g(x,y) := e^{-y}$. For $t \in [0,1]$,

$$\widetilde{S}(\mathcal{F}_t) = -\log \int_{\mathbb{R}^2} e^{-\lambda}\, \mathrm{DH}_{\mathcal{F}_t}(\mathrm{d}\lambda) = -\log \int_{\mathbb{R}^2} e^{-(1-t)x - ty}\, \mathrm{d}\boldsymbol{\nu} = -\log \| f^{1-t} g^t \|_{1,\boldsymbol{\nu}},$$

where the second equality is by Proposition 3.6. Hölder's inequality implies

$$\begin{aligned} -\log \| f^{1-t} g^t \|_{1,\boldsymbol{\nu}} \geq -\log \left(\| f \|_{1,\boldsymbol{\nu}}^{1-t} \| g \|_{1,\boldsymbol{\nu}}^{t} \right) &= -(1-t) \log \| f \|_{1,\boldsymbol{\nu}} - t \log \| g \|_{1,\boldsymbol{\nu}} \\ &= (1-t)\widetilde{S}(\mathcal{F}_0) + t\widetilde{S}(\mathcal{F}_1) \end{aligned}$$

Furthermore, the inequality is strict unless (i) $f = 0$ or $g = 0$ $\boldsymbol{\nu}$-a.e. or (ii) there exists $c > 0$ so that $f - cg = 0$ $\boldsymbol{\nu}$-a.e.

Condition (i) cannot occur since f and g are > 0. Condition (ii) is equivalent to saying $x - y - d = 0$ $\boldsymbol{\nu}$-a.e., where $d := -\ln(c)$. Now, if we write $\mathcal{G}$ for the filtration of R defined by $\mathcal{G}^\lambda R_m := \mathcal{F}_1^{\lambda - mrd} R_m$, then

$$\| x - y - d \|_{1,\boldsymbol{\nu}} = \int_{\mathbb{R}^2} |x - y - d|\, \mathrm{d}\boldsymbol{\nu} = \int_{\mathbb{R}} |\lambda|\, \mathrm{RLM}_{\mathcal{F}_0,\mathcal{G}}(\mathrm{d}\lambda) = d_1(\mathcal{F}, \mathcal{G}),$$

where the second is by Proposition 3.6. Therefore, (ii) holds iff $d_1(\mathcal{F}, \mathcal{G}) = 0$. □

Proof of Theorem 3.7. The result follows immediately from Propositions 3.12, 3.13 and 3.14. □

3.3. *Uniqueness of valuations computing $h(X, \Delta)$*

As a consequence of the convexity results in the previous section, we prove that the minimizer of $\mathbf{H}^{\mathrm{NA}}$ is unique.

Theorem 3.15. *Assume v and w are valuations in $\mathrm{Val}_X^\circ \cup \{v_{\mathrm{triv}}\}$. If v, w both compute $h(X, \Delta)$, then $v = w$.*

In [HL20b], the previous theorem was shown under the assumption that there exists a special $\mathbb{R}$-test configuration computing $h(X, \Delta)$. The latter assumption will be verified in Corollary 5.7.

Proof. Consider the geodesic $(\mathcal{F}_t)_{t \in [0,1]}$ connecting $\mathcal{F}_0 := \mathcal{F}_v$ and $\mathcal{F}_1 := \mathcal{F}_w$. For $t \in (0,1)$,

$$\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_t) \leq (1-t)\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_0) + t\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_1) \leq (1-t)\tilde{\beta}_{X,\Delta}(v) + t\tilde{\beta}_{X,\Delta}(w) = h(X, \Delta),$$

where first inequality is Theorem 3.7 and the second Corollary 3.11. Since $h(X,\Delta) \leq \mathbf{H}^{\mathrm{NA}}(\mathcal{F}_t)$, the first inequality cannot be strict. Therefore, Theorem 3.7 further implies there exists $d \in \mathbb{R}$ so that $d_1(\mathcal{F}_0, \mathcal{G}) = 0$, where $\mathcal{G}^\lambda R_m := \mathcal{F}_1^{\lambda - mrd} R_m$.

Next, note that $d = 0$, since

$$0 = \lambda_{\min}(\mathcal{F}_0) = \lambda_{\min}(\mathcal{G}) = \lambda_{\min}(\mathcal{F}_1) + d = d,$$

where first and last inequality is by Lemma 2.4 and the second by Proposition 3.1. Therefore, $d_1(\mathcal{F}_0, \mathcal{F}_1) = 0$. By Lemma 3.16, we conclude $v = w$. □

Lemma 3.16 [HL20b, Proposition 2.27]. *Assume v and w are valuations in $\mathrm{Val}_X^\circ \cup \{v_{\mathrm{triv}}\}$. If $\mathcal{F}_v$ and $\mathcal{F}_w$ are equivalent, then $v = w$.*

This result was first proved in [HL20b] using the machinery of non-Archimedean metrics from [BJ21]. For the sake of completeness, we give a proof which only uses the terminology introduced in this paper.

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Proof. It is enough to show $v(f) = w(f)$ for all $f \in R_m$ and $m \in \mathbb{N}$. Indeed, for any $\lambda \in \mathbb{R}$, we may choose some integer $m \gg 0$ such that $\mathcal{O}_X(mL) \otimes \mathfrak{a}_\lambda(v)$ is globally generated, where $\mathfrak{a}_\lambda(v) := \{f \in \mathcal{O}_X \mid v(f) \geq \lambda\}$ denotes the valuation ideal. If $v(f) = w(f)$ for all $f \in H^0(X, \mathcal{O}_X(mL) \otimes \mathfrak{a}_\lambda(v)) \subseteq R_m$, then $\mathfrak{a}_\lambda(v) \subseteq \mathfrak{a}_\lambda(w)$. Switching the role of v and w gives the reverse containment. Thus, $\mathfrak{a}_\lambda(v) = \mathfrak{a}_\lambda(w)$ for all $\lambda \in \mathbb{R}$, and, hence, $v = w$.

Suppose now that $a = v(f) \neq w(f) = b$ for some $f \in R_m$. Without loss of generality, we may assume $a > b$. Let $\lambda = \lambda_{\max}(\mathcal{F}_w)$. Fix some $\varepsilon \in (0, a-b)$, and let p be a sufficiently large integer such that $\lambda r \leq (a-b-\varepsilon)p$. Consider the subspace $V_k := f^{kp} \cdot R_k \subseteq \mathcal{F}_v^{akp} R_{k(mp+1)}$. For any $g \in V_k$, we have $w(g) \leq bkp + k\lambda r \leq (a-\varepsilon)kp$ by our choice of p. Thus, for any basis $(s_1, \cdots, s_N)$ of $R_{k(mp+1)}$ that is compatible with both $\mathcal{F}_v$ and $\mathcal{F}_w$, the part that spans $\mathcal{F}_v^{akp} R_{k(mp+1)}$ contains at least $\dim V_k = \dim R_k$ elements s_i with $w(s_i) \leq (a-\varepsilon)kp$. In particular for these s_i, we have $v(s_i) - w(s_i) \geq \varepsilon kp$. It follows that

$$\int_{\mathbb{R}} |\lambda| \mathrm{d}\nu_{k(mp+1)}^{\mathcal{F}_v, \mathcal{F}_w} \geq \frac{\frac{\varepsilon p}{(mp+1)r} \cdot \dim R_k}{\dim R_{k(mp+1)}}$$

for all $k \gg 0$. Letting $k \to \infty$, we get $d_1(\mathcal{F}_v, \mathcal{F}_w) > 0$, contradicting our assumption. Thus, we must have $v(f) = w(f)$ for all $m \in \mathbb{N}$ and all $f \in R_m$ and therefore $v = w$. □

Corollary 3.17. *Let (X, Δ) be a log Fano pair, there is a unique valuation computing $h(X, \Delta)$ in $\mathrm{Val}_X^\circ \cup \{v_{\mathrm{triv}}\}$ and it is quasi-monomial.*

Proof. By [HL20b, Corollary 4.9], there is a quasi-monomial valuation $v \in \mathrm{Val}_X^\circ \cup \{v_{\mathrm{triv}}\}$ computing $h(X, \Delta)$. The uniqueness is by Theorem 3.15. □

4. Weighted stability

In this section, we provide a common ground to study the stability of both Kähler–Ricci solitons and triples (X, Δ, v_0) (where v_0 is the unique minimizer of $\tilde{\beta}$ from the previous section) in a suitably weighted sense. The results will be applied in the next section to study the finite generation property for various minimizers.

4.1. Weighted δ-invariants

We first introduce a weighted version of the stability threshold and then generalize results from [BJ20] to this setting (see also [RTZ21, Section 6]).

Definition 4.1. Let $v_0 \in \mathrm{Val}_X$ be a quasi-monomial valuation and $g \colon \mathbb{R} \to \mathbb{R}_+$ a continuous function. A *g-weighted (m, v_0)-basis type divisor* is a divisor of the form

$$D = \frac{1}{mrQ_m} \sum_{i=1}^{N_m} g\left(\frac{v_0(s_i)}{mr}\right) \cdot \{s_i = 0\},$$

where $(s_1, \cdots, s_{N_m})$ is a basis of R_m that is compatible with v_0 and $Q_m = \sum_{i=1}^{N_m} g\left(\frac{v_0(s_i)}{mr}\right)$. Note that $D \in |-K_X - \Delta|_{\mathbb{R}}$. We say D is compatible with a filtration $\mathcal{F}$ on R if the basis $(s_1, \cdots, s_{N_m})$ is compatible with $\mathcal{F}$. In particular, we say D is compatible with a valuation $v \in \mathrm{Val}_X^\circ$ (resp. an effective $\mathbb{Q}$-divisor G on X) if it is compatible with the induced filtration $\mathcal{F}_v$ (resp. $\mathcal{F}_G$).

For any linearly bounded filtration $\mathcal{F}$ on R, let $\boldsymbol{\nu}_m$ and $\boldsymbol{\nu} = \mathrm{DH}_{\mathcal{F}_{v_0},\mathcal{F}} = \lim_{m\to\infty} \boldsymbol{\nu}_m$ denote the measures on $\mathbb{R}^2$ associated to $\mathcal{F}_{v_0}$ and $\mathcal{F}$ as constructed in Section 3.1.3. We set

$$S_{g,m}(v_0;\mathcal{F}) := \frac{\int_{\mathbb{R}^2} g(x)y\mathrm{d}\boldsymbol{\nu}_m}{\int_{\mathbb{R}^2} g(x)\mathrm{d}\boldsymbol{\nu}_m} \quad \text{and} \quad S_g(v_0;\mathcal{F}) := \lim_{m\to\infty} S_{g,m}(v_0;\mathcal{F}) = \frac{\int_{\mathbb{R}^2} g(x)y\mathrm{d}\boldsymbol{\nu}}{\int_{\mathbb{R}^2} g(x)\mathrm{d}\boldsymbol{\nu}}.$$

It is clear that

$$S_{g,m}(v_0;\mathcal{F}) = \frac{1}{mrQ_m}\sum_{i=1}^{N_m} g\left(\frac{v_0(s_i)}{mr}\right)\cdot \mathrm{ord}_{\mathcal{F}}(s_i)$$

for any basis $(s_1,\cdots,s_{N_m})$ that is compatible with both v_0 and $\mathcal{F}$.

Lemma 4.2. *For any $\varepsilon > 0$, there exists a positive integer $m_0 = m_0(\varepsilon)$ such that*

$$S_{g,m}(v_0;\mathcal{F}) \le (1+\varepsilon)S_g(v_0;\mathcal{F})$$

for any linearly bounded filtration $\mathcal{F}$ of R satisfying $\mathcal{F}^0R = R$ and any $m \ge m_0$.

This follows from essentially the same argument as [BJ20, Corollary 2.10] except we need to use an Okounkov body that is induced by a suitable valuation.

Definition 4.3. Let $w\colon K(X)^\times \to \mathbb{Z}^n$ be a valuation with values in the group $\mathbb{Z}^n$ (equipped with some total ordering). Following [KK12], we say that w is faithful if its image equals $\mathbb{Z}^n$; we say that w has one-dimensional leaves if $\dim \hat{V}_\alpha \le 1$ for every $\alpha \in \mathbb{Z}^n$, where $\hat{V}_\alpha := \{f \in K(X) \mid w(f) \ge \alpha\}/\{f \in K(X) \mid w(f) > \alpha\}$. Finally, we say that w is a good valuation if it is faithful, has one-dimensional leaves and $n = \dim X$.

Lemma 4.4. *Let v_0 be a quasi-monomial valuation on X. Then there exists a good valuation $w_0\colon K(X)^\times \to \mathbb{Z}^n$ and $u_0 \in \mathbb{R}^n_{\ge 0}$ such that*

$$v_0(f) = \langle u_0, w_0(f)\rangle \qquad \textit{for all } f \in K(X)^\times.$$

Proof. Let r be the rational rank of v_0. Since v_0 is quasi-monomial, there exists a log resolution $\pi\colon Y \to X$, a regular system of parameters $y_1,\dots,y_r$ at a point $\eta \in Y$, and $\alpha \in \mathbb{R}^r_+$ such that $v_0 = v_\alpha$. Let $W = C_Y(v_0)$. Possibly after further blowups, we may choose a flag $W_\bullet\colon W = W_0 \supseteq \cdots \supseteq W_{n-r} = \{\text{point}\}$ of smooth subvarieties such that each W_{i+1} is a divisor in W_i. Let $\nu = \nu_{W_\bullet}\colon K(W)^\times \to \mathbb{Z}^{n-r}$ be the induced valuation as in [LM09]. Any nonzero $f \in \mathcal{O}_{Y,W}$ can be written as $f = c_\beta y^\beta + f_1$ for some (uniquely determined) $\beta \in \mathbb{N}^r$ and $c_\beta, f_1 \in \mathcal{O}_{Y,W}$ such that $\langle\alpha,\beta\rangle = v_0(f)$ and $v_0(f_1) > v_0(f)$. Moreover, $0 \ne \bar{c}_\beta \in K(W)$ is well defined and does not depend on the choice of c_β. Now, consider the valuation $w_0\colon K(X)^\times \to \mathbb{Z}^n$ given by setting $w_0(f) = (\beta, \nu(\bar{c}_\beta))$ for $f \in \mathcal{O}_{Y,W}$. It is not hard to check from the construction that w_0 is faithful and has one-dimensional leaves. Clearly, $v_0(f) = \langle(\alpha,0,\cdots,0), w_0(f)\rangle$. Thus, w_0 is the good valuation we want. □

Proof of Lemma 4.2. Since the argument is very similar to those in [BJ20], we only sketch the proof. By Lemma 4.4, there exists some good valuation $w_0\colon K(X) \to \mathbb{Z}^n$ and some $u_0 \in \mathbb{R}^n_{\ge 0}$ such that $v_0(f) = \langle u_0, w_0(f)\rangle$. Let $\Sigma \subseteq \mathbb{R}^n$ be the corresponding Okounkov body (see [KK12]), that is, the closed convex hull of $\bigcup_{m\ge 1}\{\frac{w_0(s)}{mr} \mid s \in R_m \setminus \{0\}\}$. We regard the function g also as a positive function on Σ by $g(\alpha) = g(\langle u_0,\alpha\rangle)$. Let ρ denote the Lebesgue measure on Σ and ρ_m the atomic probability measure supported on $\Sigma \cap \frac{1}{m}\mathbb{Z}^n$ as defined in [BJ20, Section 2.2]. Note that $\lim_{m\to\infty}\rho_m = \rho$ in the weak topology of measures (see [BJ20, Theorem 2.1]). Using the argument of [BJ20, Lemma 2.2] and the uniform continuity of g on Σ, we see that for each $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon)$ such that

$$\int_\Sigma fg\mathrm{d}\rho_m \le \int_\Sigma fg\mathrm{d}\rho + \varepsilon$$

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for every $m \geq m_0$ and every concave function $f : \Sigma \to \mathbb{R}$ satisfying $0 \leq f \leq 1$. We may then apply the proof of [BJ20, Corollary 2.10] to the concave transform of $\mathcal{F}$ and conclude that (after possibly enlarging m_0) $S_{g,m}(v_0; \mathcal{F}) \leq (1+\varepsilon) S_g(v_0; \mathcal{F})$ for all linearly bounded filtrations $\mathcal{F}$ and all $m \geq m_0$. □

When $\mathcal{F}$ is the filtration induced by some valuation $v \in \mathrm{Val}_X^\circ$ (resp. effective divisor $G \neq 0$ on X), we will simply write $S_g(v_0; v)$ (resp. $S_g(v_0; G)$) for $S_g(v_0, \mathcal{F})$. Let $\mathbb{T} = \mathbb{G}_m^s < \mathrm{Aut}(X, \Delta)$ be a torus subgroup of the automorphism group (we allow $\mathbb{T} = \{1\}$). For any quasi-monomial valuation $v_0 \in \mathrm{Val}_X^{\mathbb{T},\circ}$ we set

$$\delta_{g,\mathbb{T}}(X, \Delta, v_0) := \inf_{v \in \mathrm{Val}_X^{\mathbb{T},\circ}} \frac{A_{X,\Delta}(v)}{S_g(v_0; v)}.$$

We say that $v \in \mathrm{Val}_X^{\mathbb{T},\circ}$ computes $\delta_{g,\mathbb{T}}(X, \Delta, v_0)$ if it achieves the above infimum. For each positive integer m, we also set

$$\delta_{g,\mathbb{T},m}(X, \Delta, v_0) := \min\{\mathrm{lct}(X, \Delta; D) \mid D \text{ is a } \mathbb{T}\text{-invariant } g\text{-weighted } (m, v_0)\text{-basis type divisor}\}.$$

When $\mathbb{T} = \{1\}$, we will suppress the subscript $\mathbb{T}$ and write $\delta_g(X, \Delta, v_0)$ and $\delta_{g,m}(X, \Delta, v_0)$.

Lemma 4.5. *In the above setup, we have*

$$\delta_{g,\mathbb{T}}(X, \Delta, v_0) = \lim_{m \to \infty} \delta_{g,\mathbb{T},m}(X, \Delta, v_0).$$

Proof. It is not hard to check from the definition that

$$S_{g,m}(v_0; v) = \max\{v(D) \mid D \text{ is a } g\text{-weighted } (m, v_0)\text{-basis type divisor}\} \tag{4.1}$$

and in fact $S_{g,m}(v_0; v) = v(D)$ for any g-weighted (m, v_0)-basis type divisor D that's also compatible with v. Moreover, when $v_0, v \in \mathrm{Val}_X^{\mathbb{T},\circ}$, such a divisor D can be chosen to be $\mathbb{T}$-invariant by choosing compatible basis in each component of the weight decomposition under the torus action. Hence,

$$\delta_{g,\mathbb{T},m}(X, \Delta, v_0) = \inf_{v \in \mathrm{Val}_X^{\mathbb{T},\circ}} \frac{A_{X,\Delta}(v)}{S_{g,m}(v_0; v)}. \tag{4.2}$$

Combined with Lemma 4.2, the argument in the proof of [BJ20, Theorem 4.4] then yields $\delta_{g,\mathbb{T}}(X, \Delta, v_0) = \lim_{m\to\infty} \delta_{g,\mathbb{T},m}(X, \Delta, v_0)$. □

4.2. Reduced uniform stability

In this section, we define stability notions for triples (X, Δ, v_0) where (X, Δ) is a log Fano pair, v_0 is a quasi-monomial valuation on X, and $g : \mathbb{R} \to \mathbb{R}_+$ is a continuous function.

We first define the weighted version of the non-Archimedean functional. For any linearly bounded filtration on R, we set

$$\mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}) := \mathbf{L}^{\mathrm{NA}}(\mathcal{F}) - S_g(v_0; \mathcal{F}),$$
$$\mathbf{J}_g^{\mathrm{NA}}(\mathcal{F}) := \lambda_{\max}(\mathcal{F}) - S_g(v_0; \mathcal{F}).$$

Note that $\mathbf{J}_g^{\mathrm{NA}}(\mathcal{F}) \geq 0$. We also set $\mathbf{E}_g^{\mathrm{NA}}(\mathcal{F}) := S_g(v_0; \mathcal{F})$. If $(\mathcal{X}, \Delta_{\mathcal{X}}; \mathcal{L})$ is a normal test configuration of (X, Δ), then we set $\mathbf{D}_g^{\mathrm{NA}}(\mathcal{X}, \Delta_{\mathcal{X}}; \mathcal{L}) := \mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_{(\mathcal{X},\Delta;\mathcal{L})})$ and $\mathbf{J}_g^{\mathrm{NA}}(\mathcal{X}, \Delta; \mathcal{L}) := \mathbf{J}_g^{\mathrm{NA}}(\mathcal{F}_{(\mathcal{X},\Delta_{\mathcal{X}};\mathcal{L})})$. Denote by $\mathrm{Aut}(X, \Delta, v_0)$ the subgroup of $\mathrm{Aut}(X, \Delta)$ that leaves the valuation v_0 invariant, and let $\mathbb{T} < \mathrm{Aut}(X, \Delta, v_0)$ be a torus subgroup.

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Definition 4.6. We say that the triple (X, Δ, v_0) is $\mathbb{T}$-equivariantly g-Ding semistable (or simply g-Ding semistable when $\mathbb{T} = \{1\}$) if $\mathbf{D}_g^{\mathrm{NA}}(\mathcal{X}, \Delta; \mathcal{L}) \geq 0$ for all $\mathbb{T}$-equivariant normal test configurations $(\mathcal{X}, \Delta; \mathcal{L})$ of (X, Δ).

Denote by $M := \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ the weight lattice and $N := \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ the co-weight lattice. Then there is a weight decomposition $R_m = \oplus_{\alpha \in M} R_{m,\alpha}$. Recall that, for any $\mathbb{T}$-equivariant filtration $\mathcal{F}$ and each $\eta \in N_{\mathbb{R}}$, there is an η-twist $\mathcal{F}_\eta$ of $\mathcal{F}$ given by

$$\mathcal{F}_\eta^\lambda R_m = \bigoplus_{\alpha \in M} \mathcal{F}^{\lambda - \langle \alpha, \eta \rangle} R \cap R_{m,\alpha}.$$

Set $\mathrm{Fut}_g(\eta) := \mathbf{E}_g^{\mathrm{NA}}(\mathcal{F}) - \mathbf{E}_g^{\mathrm{NA}}(\mathcal{F}_\eta)$. It is not hard to see from the definition that Fut_g does not depend on the choice of the filtration $\mathcal{F}$ and is linear on $N_{\mathbb{R}}$. We define the *reduced* $\mathbf{J}_g^{\mathrm{NA}}$*-norm* of $\mathcal{F}$ as

$$\mathbf{J}_{g,\mathbb{T}}^{\mathrm{NA}}(\mathcal{F}) := \inf_{\eta \in N_{\mathbb{R}}} \mathbf{J}_g^{\mathrm{NA}}(\mathcal{F}_\eta).$$

Definition 4.7. We say a triple (X, Δ, v_0) is *reduced uniformly g-Ding stable* if there exists a maximal torus $\mathbb{T} < \mathrm{Aut}(X, \Delta, v_0)$ and some $\varepsilon > 0$ such that

$$\mathbf{D}_g^{\mathrm{NA}}(\mathcal{X}, \Delta_{\mathcal{X}}; \mathcal{L}) \geq \varepsilon \mathbf{J}_{g,\mathbb{T}}^{\mathrm{NA}}(\mathcal{X}, \Delta_{\mathcal{X}}; \mathcal{L})$$

for all $\mathbb{T}$-equivariant normal test configurations $(\mathcal{X}, \Delta_{\mathcal{X}}; \mathcal{L})$ of (X, Δ).

Note that the above definition is independent of the choice of $\mathbb{T}$ since any two maximal tori are conjugate.

Lemma 4.8. *Let* $\mathbb{T} < \mathrm{Aut}(X, \Delta, v_0)$ *be a torus. Assume that* $\mathbf{D}_g^{\mathrm{NA}} \geq 0$ *for any product test configurations that is induced by a one parameter subgroup of* $\mathbb{T}$. *Then* $\mathrm{Fut}_g \equiv 0$ *on* $N_{\mathbb{R}}$.

Proof. Let $\mathcal{F}$ be the trivial filtration of R. By Lemma 3.8 and [XZ20, Lemma A.6], we have $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}_\eta)$ for any $\eta \in N_{\mathbb{R}}$. Thus, $\mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_\eta) = \mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}) + \mathrm{Fut}_g(\eta) = \mathrm{Fut}_g(\eta)$. By assumption, $\mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_\eta) \geq 0$ for any $\eta \in N$. By linearity, $\mathrm{Fut}_g \equiv 0$ on N and, hence, the same holds on $N_{\mathbb{R}}$. □

We are most interested in the case $v_0 = \mathrm{wt}_\xi$ for some torus $\mathbb{T} < \mathrm{Aut}(X, \Delta)$ and some $\xi \in N_{\mathbb{R}}$. In this case, we write (X, Δ, ξ) instead of $(X, \Delta, \mathrm{wt}_\xi)$. We note that while $\mathbb{T}$ is not explicitly written out in the notion (X, Δ, ξ), it is indeed part of the data.

Theorem 4.9 [HL20]. *Let* (X, Δ, ξ) *be a triple over* $\mathbb{k} = \mathbb{C}$. *Then it admits a Kähler–Ricci g-soliton if and only if it is reduced uniformly g-Ding stable.*

Definition 4.10. Let (X, Δ, ξ) be a triple. We say that (X, Δ, ξ) is *g-Ding semistable* if it is $\mathbb{T}$-equivariantly g-Ding semistable. We say that (X, Δ, ξ) is *g-Ding polystable* if it is g-Ding semistable and $\mathbf{D}_g^{\mathrm{NA}}(\mathcal{X}, \Delta_{\mathcal{X}}; \mathcal{L}) = 0$ for a weakly special $\mathbb{T}$-equivariant test configuration $(\mathcal{X}, \Delta_{\mathcal{X}}; \mathcal{L})$ only if it is a product test configuration.

We say that (X, Δ, ξ) is *K-semistable* (resp. *K-polystable*, or *reduced uniformly Ding stable*) if it is g-Ding semistable (resp. g-Ding polystable, or reduced uniformly g-Ding stable) for $g(x) = e^{-x}$.

Remark 4.11. The above definition of K-polystability agrees with the notion in [BWN14] when $\mathbb{T}$ is the torus of smallest dimension such that $\xi \in N_{\mathbb{R}}$; see [HL20b, Remark 2.47]. Note that K-polystability of Kähler–Ricci solitons is proved in [BWN14, Theorem 1.5]. Later, we will see that the definition indeed does not depend on the choice of $\mathbb{T}$ (see Remark 5.10).

The following is the main result of this subsection.

Lemma 4.12. *Let* v_0 *be a quasi-monomial valuation on X, and let* $\mathbb{T} < \mathrm{Aut}(X, \Delta, v_0)$ *be a torus. Let* $g : \mathbb{R} \to \mathbb{R}_+$ *be a continuous function and* $c \in [0, 1)$. *Then the following are equivalent:*

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1. $\mathbf{D}^{\mathrm{NA}}_g(\mathcal{X},\Delta_{\mathcal{X}};\mathcal{L}) \geq c\cdot \mathbf{J}^{\mathrm{NA}}_{g,\mathbb{T}}(\mathcal{X},\Delta_{\mathcal{X}};\mathcal{L})$ *for any* $\mathbb{T}$*-equivariant normal test configuration* $(\mathcal{X},\Delta_{\mathcal{X}};\mathcal{L})$ *of* (X,Δ).
2. $\mathbf{D}^{\mathrm{NA}}_g(\mathcal{X},\Delta_{\mathcal{X}};\mathcal{L}) \geq c\cdot \mathbf{J}^{\mathrm{NA}}_{g,\mathbb{T}}(\mathcal{X},\Delta_{\mathcal{X}};\mathcal{L})$ *for any* $\mathbb{T}$*-equivariant weakly special test configuration* $(\mathcal{X},\Delta_{\mathcal{X}};\mathcal{L})$ *of* (X,Δ).
3. $\mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}_v) \geq c\cdot \mathbf{J}^{\mathrm{NA}}_{g,\mathbb{T}}(\mathcal{F}_v)$ *for any divisorial valuation* $v\in \mathrm{Val}_X^{\mathbb{T},\circ}$ *that is an lc place of a* $\mathbb{Q}$*-complement.*

Proof. When $v_0 = \mathrm{wt}_\xi$ this is treated in [HL20, Section 7] using [LX14]. Here, we present a proof that is independent of [LX14]. It is clear that (1) implies (2). By [BLX22, Theorem A.2], (2) implies (3). Thus, it remains to show (3) implies (1). To see this, let $\mathcal{F}$ be a finitely generated $\mathbb{T}$-equivariant $\mathbb{Z}$-filtration of R. If $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) \geq \lambda_{\max}(\mathcal{F})$, then we already have $\mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}) \geq \mathbf{J}^{\mathrm{NA}}_g(\mathcal{F}) \geq \mathbf{J}^{\mathrm{NA}}_{g,\mathbb{T}}(\mathcal{F})$. Thus, to prove (1) we may assume that $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) < \lambda_{\max}(\mathcal{F})$. By the second part of Lemma 3.8 and Remark 3.9, there exists some divisorial lc place of a $\mathbb{Q}$-complement $v\in \mathrm{Val}_X^{\mathbb{T},\circ}$ such that

$$r^{-1}v(I_\bullet^{(\lambda)}) - A_{X,\Delta}(v) \geq \lambda - \mathbf{L}^{\mathrm{NA}}(\mathcal{F})$$

for all $\lambda\in\mathbb{R}$. Since $\mathbf{D}^{\mathrm{NA}}_g(\mathcal{F})$ and $\mathbf{J}^{\mathrm{NA}}_{g,\mathbb{T}}(\mathcal{F})$ are both translation invariant, we may shift $\mathcal{F}$ so that $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) = A_{X,\Delta}(v)$. The above inequality then becomes $v(I_\bullet^{(\lambda)}) \geq \lambda r$ and therefore $\mathcal{F}^\lambda R \subseteq \mathcal{F}_v^\lambda R$ for all $\lambda\in\mathbb{R}$.

Let $\eta\in N_{\mathbb{R}}$, and let $\mathcal{G}$ be the η-twist of $\mathcal{F}_v$. Then we also have $\mathcal{F}_\eta^\lambda R\subseteq \mathcal{G}^\lambda R$ for all λ, and this clearly implies

$$\mathbf{E}^{\mathrm{NA}}_g(\mathcal{F}_\eta) \leq \mathbf{E}^{\mathrm{NA}}_g(\mathcal{G}) \quad\text{and}\quad \lambda_{\max}(\mathcal{F}_\eta)\leq \lambda_{\max}(\mathcal{G}).$$

Since $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) = A_{X,\Delta}(v) \geq \mathbf{L}^{\mathrm{NA}}(\mathcal{F}_v)$ by Corollary 3.11, we also get $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_\eta)\geq \mathbf{L}^{\mathrm{NA}}(\mathcal{G})$ by Lemma 3.8 and [XZ20, Lemma A.6]. Note that by Lemma 4.8, [BLX22, Theorem A.2] and (3) we have $\mathrm{Fut}_g\equiv 0$ on $N_{\mathbb{R}}$, thus $\mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}_\eta) = \mathbf{D}^{\mathrm{NA}}_g(\mathcal{F})$ and $\mathbf{D}^{\mathrm{NA}}_g(\mathcal{G}) = \mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}_v)$.

Since

$$\mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}) - c\cdot \mathbf{J}^{\mathrm{NA}}_g(\mathcal{F}) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}) - (1-c)\mathbf{E}^{\mathrm{NA}}_g(\mathcal{F}) - c\cdot\lambda_{\max}(\mathcal{F}),$$

we then obtain

$$\begin{aligned}\mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}) - c\cdot \mathbf{J}^{\mathrm{NA}}_{g,\mathbb{T}}(\mathcal{F}) &\geq \mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}_\eta) - c\cdot \mathbf{J}^{\mathrm{NA}}_g(\mathcal{F}_\eta)\\ &\geq \mathbf{D}^{\mathrm{NA}}_g(\mathcal{G}) - c\cdot \mathbf{J}^{\mathrm{NA}}_g(\mathcal{G}) = \mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}_v) - c\cdot \mathbf{J}^{\mathrm{NA}}_g(\mathcal{G}).\end{aligned}$$

As $\eta\in N_{\mathbb{R}}$ is arbitrary, this gives

$$\mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}) - c\cdot \mathbf{J}^{\mathrm{NA}}_{g,\mathbb{T}}(\mathcal{F}) \geq \mathbf{D}^{\mathrm{NA}}_g(\mathcal{F}_v) - c\cdot \mathbf{J}^{\mathrm{NA}}_{g,\mathbb{T}}(\mathcal{F}_v).$$

By (3), the right-hand side is ≥ 0. Thus, the same is true for the left-hand side. Since $\mathcal{F}$ is arbitrary, this proves that (3) implies (1). □

We get the following generalized version of Fujita–Li valuative criterion [Fuj19, Li17] which treat the case $g=1$ (see also [HL20, Theorem 5.18]).

Corollary 4.13. *A triple* (X,Δ,v_0) *is* $\mathbb{T}$*-equivariantly* g*-Ding semistable if and only if* $\delta_{g,\mathbb{T}}(X,\Delta,v_0)\geq 1$.

Proof. Assume that (X,Δ,v_0) is $\mathbb{T}$-equivariantly g-Ding semistable. If $\delta := \delta_{g,\mathbb{T}}(X,\Delta,v_0) < 1$, then we may choose some $\varepsilon>0$ such that $(1+\varepsilon)^2\delta<1$. Let $m\gg 0$ be such that $\delta_m := \delta_{g,\mathbb{T},m}(X,\Delta,v_0) < (1+\varepsilon)\delta$. By Lemma 4.2, we may also assume that $S_{g,m}(v_0;v)\leq (1+\varepsilon)S_g(v_0;v)$ for any $v\in\mathrm{Val}_X^\circ$. By definition, $\delta_m = \operatorname{lct}(X, \Delta; D)$ for some $\mathbb{T}$-invariant g-weighted (m, v_0)-basis type divisor D. Thus, if w is a $\mathbb{T}$-invariant divisorial valuation that computes the lct, we would have

$$A_{X,\Delta}(w) = \delta_m S_{g,m}(v_0; w) < (1+\varepsilon)^2 \delta \cdot S_g(v_0; w) < S_g(v_0; w).$$

On the other hand, since $\delta_m < 1$, we know that $-(K_X + \Delta + \delta_m D)$ is ample, thus $\operatorname{gr}_w R$ is finitely generated by [BCHM10, Corollary 1.4.3]. In other words, w induces a $\mathbb{T}$-equivariant test configuration of (X, Δ). As (X, Δ, v_0) is $\mathbb{T}$-equivariantly g-Ding semistable, Corollary 3.11 gives $A_{X,\Delta}(w) - S_g(v_0; w) \geq \mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_w) \geq 0$, a contradiction. Therefore, $\delta_{g,\mathbb{T}}(X, \Delta, v_0) \geq 1$ when (X, Δ, v_0) is $\mathbb{T}$-equivariantly g-Ding semistable.

Conversely, if $\delta_{g,\mathbb{T}}(X, \Delta, v_0) \geq 1$, then for any $v \in \operatorname{Val}_X^{\mathbb{T},\circ}$ we have $A_{X,\Delta}(v) \geq S_g(v_0; v)$. If v is an lc place of a $\mathbb{Q}$-complement, then we also have $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_v) = A_{X,\Delta}(v)$ (c.f. [XZ20, Proposition 4.2]). Thus, $\mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_v) \geq 0$ for any $v \in \operatorname{Val}_X^{\mathbb{T},\circ}$ that is an lc place of a $\mathbb{Q}$-complement. By Lemma 4.12 (with $c = 0$), this implies that (X, Δ, v_0) is $\mathbb{T}$-equivariantly g-Ding semistable. $\square$

4.3. *Existence of minimizer*

Throughout this section, let $\mathbb{T} < \operatorname{Aut}(X, \Delta)$ be a torus and let $\xi \in N_{\mathbb{R}}$. We aim to prove the following statement.

Proposition 4.14. *Assume that (X, Δ, ξ) is g-Ding semistable but not reduced uniformly g-Ding stable. Then there exists a $\mathbb{T}$-invariant quasi-monomial valuation v that is not of the form wt_η for any $\eta \in N_{\mathbb{R}}$ such that $1 = \delta_g(X, \Delta, \xi) (:= \delta_{g,\mathbb{T}}(X, \Delta, \mathrm{wt}_\xi))$ is computed by v.*

Proof. The argument is very similar to those in [XZ20, Appendix], so we only give a sketch. First, note that the assumption remains true if we enlarge the torus $\mathbb{T}$, and clearly if the conclusion holds for a maximal torus containing $\mathbb{T}$, then it also holds for $\mathbb{T}$. Thus, we may assume that $\mathbb{T}$ is a maximal torus.

By Lemma 4.12 and [BLX22, Theorem 3.5], we know that there exist some integer $N > 0$ and a sequence of divisorial lc places of N-complements $v_i \in \operatorname{Val}_X^{\mathbb{T},\circ}$ (not of the form wt_η) such that

$$\lim_{i\to\infty} \frac{\mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_{v_i})}{\mathbf{J}_{g,\mathbb{T}}^{\mathrm{NA}}(\mathcal{F}_{v_i})} = 0.$$

By the constructibility result [XZ20, Lemma A.11] and arguing as in the proof of [XZ20, Theorem A.5], we may assume that the v_i's are lc places of the same $\mathbb{Q}$-complement and after rescaling $v = \lim_i v_i \in \operatorname{Val}_X^{\mathbb{T},\circ}$ exists and $v \neq \mathrm{wt}_\eta$ for any $\eta \in N_{\mathbb{R}}$. By the following Lemma 4.15, we have $\lim_i S_g(v_0; v_i) = S_g(v_0; v)$. Since v_i are lc places of $\mathbb{Q}$-complements, it follows from Lemma 3.8 that $\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_{v_i}) = A_{X,\Delta}(v_i)$ and thus $\lim_i \mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_{v_i}) = \mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_v)$. Since the function $v \mapsto \lambda_{\max}(\mathcal{F}_v) = T_{X,\Delta}(v)$ is also continuous on $\mathrm{QM}(Y, E)$ by [BLX22, Proposition 2.4], we get $\lim_i \mathbf{J}_g^{\mathrm{NA}}(\mathcal{F}_{v_i}) = \mathbf{J}_g^{\mathrm{NA}}(\mathcal{F}_v)$ as well. Thus,

$$\frac{\mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_v)}{\mathbf{J}_{g,\mathbb{T}}^{\mathrm{NA}}(\mathcal{F}_v)} = \lim_{i\to\infty} \frac{\mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_{v_i})}{\mathbf{J}_{g,\mathbb{T}}^{\mathrm{NA}}(\mathcal{F}_{v_i})} = 0,$$

which implies $\mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_v) = A_{X,\Delta}(v) - S_g(v_0; v) = 0$. Since $\delta_g(X, \Delta, \xi) \geq 1$ by Corollary 4.13, we see that $\delta_g(X, \Delta, \xi) = 1$ and v computes $\delta_g(X, \Delta, \xi)$. $\square$

We have used the following statement in the above proof.

Lemma 4.15. *Let $Y \to X$ be a proper birational map with Y regular and $E := \sum_{i=1}^d E_i$ a reduced simple normal crossing divisor on Y. Then the function $S_g(v_0; \cdot)$ is continuous on $\mathrm{QM}(Y, E)$.*

We will deduce the result from the continuity of $S(\cdot)$ on $\mathrm{QM}(Y, E)$ shown in [BLX22].

Proof. Fix a point $\eta \in \operatorname{Supp}(E)$ and local coordinates $y_1, \ldots, y_r \in \mathcal{O}_{Y,\eta}$ so that each y_j cuts out an irreducible component of E at η. For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r_{\geq 0}$, write $v_\alpha \in \operatorname{Val}_X$ for the associated quasi-monomial valuation satisfying $v_\alpha(y_j) = \alpha_j$.

To prove the continuity of $S_g(v_0; \cdot)$ on $\mathbb{R}^r_{\geq 0}$, fix a convergent sequence α^i in $\mathbb{R}^r_{\geq 0}$ and set $\alpha := \lim_i \alpha^i$. We aim to show $\lim_i S_g(v_0; v_i) = S_g(v_0; v)$, where $v_i := v_{\alpha^i}$ and $v := v_\alpha$.

First, note that $S_g(v_0; cw) = cS_g(v_0; w)$ for all $w \in \operatorname{Val}^\circ_X$ and $c \in \mathbb{R}_{>0}$. Therefore, after rescaling the α^i and removing finitely many terms, we may assume the sequence (α^i) is nonincreasing. Hence, $v_i \geq v_{i+1} \geq v$ for all i. Next, consider the Okounkov body $\Sigma \subset \mathbb{R}^n$ induced by the good valuation form Lemma 4.4. Write G_i and G for the concave functions $\Sigma \to \mathbb{R}_{\geq 0}$ induced by the filtrations $\mathcal{F}_v$ and $\mathcal{F}_{v_i}$ (see [BJ20, Section 2.5]) and set $\operatorname{vol}_g(\Sigma) = \int_\Sigma g \, \mathrm{d}\rho$. Note that

$$|S_g(v_0; v_i) - S_g(v_0; v)| = \left| \frac{1}{\operatorname{vol}_g(\Sigma)} \int_\Sigma g G_i \, \mathrm{d}\rho - \frac{1}{\operatorname{vol}_g(\Sigma)} \int_\Sigma g G \, \mathrm{d}\rho \right| = \frac{1}{\operatorname{vol}_g(\Sigma)} \int_\Sigma g |G_i - G| \, \mathrm{d}\rho, \tag{4.3}$$

where the second equality uses that $G_i \leq G$. Using that $\lim_i S_1(v_0; v_i) = S_1(v_0; v)$ by [BLX22, Proposition 2.4] and equation (4.3), we see (G_i) converges to G a.e. Therefore, the dominated convergence theorem implies

$$\lim_{i\to\infty} |S_g(v_0; v_i) - S_g(v_0; v)| = \lim_{i\to\infty} \frac{1}{\operatorname{vol}_g(\Sigma)} \int_\Sigma g |G_i - G| \, \mathrm{d}\rho = 0,$$

which completes the proof. □

5. Finite generation

5.1. A valuative criterion for $\tilde{\beta}_{X,\Delta}$-minimizers

In this section, we give a valuative criterion for valuations computing h-invariant inspired by [XZ21] in terms of weighted stability thresholds. Let us recall that in Corollary 3.17, we know the valuation computing $h(X, \Delta)$ is quasi-monomial. Thus, we can apply the construction in Section 4.1. We use the following notation: for any quasi-monomial v_0 and $v \in \operatorname{Val}^\circ_X$, let $\widetilde{S}(v_0; v) := S_g(v_0; v)$, where $g(x) = e^{-x}$. By [HL20b] and Theorem 3.15, the unique valuation computing $h(X, \Delta)$ is trivial if and only if (X, Δ) is K-semistable. Hence, throughout this subsection, we assume that (X, Δ) is K-unstable, that is, $h(X, \Delta) < 0$.

Theorem 5.1. *A quasi-monomial valuation $v_0 \in \operatorname{Val}^\circ_X$ computes $h(X, \Delta)$ if and only if $A_{X,\Delta}(v) \geq \widetilde{S}(v_0; v)$ for any valuation $v \in \operatorname{Val}^\circ_X$ and $A_{X,\Delta}(v_0) = \widetilde{S}(v_0; v_0)$.*

Proof. We first show the 'if' part. It suffices to show that $\tilde{\beta}_{X,\Delta}(v) \geq \tilde{\beta}_{X,\Delta}(v_0)$ for any valuation $v \in \operatorname{Val}^\circ_X$. Let $\mathcal{F}_0 := \mathcal{F}_{v_0}$ and $\mathcal{F}_1 := \mathcal{F}_v$. Let $\boldsymbol{\nu}$ denote the compatible measure of $\mathcal{F}_0$ and $\mathcal{F}_1$ from Section 3.1.3. From the definitions, we know that

$$\widetilde{S}(v_0; v_0) = \frac{\int_{\mathbb{R}^2} x e^{-x} d\boldsymbol{\nu}}{\int_{\mathbb{R}^2} e^{-x} d\boldsymbol{\nu}}, \quad \text{and} \quad \widetilde{S}(v_0; v) = \frac{\int_{\mathbb{R}^2} y e^{-x} d\boldsymbol{\nu}}{\int_{\mathbb{R}^2} e^{-x} d\boldsymbol{\nu}}. \tag{5.1}$$

Consider the following function $f : [0, 1] \to \mathbb{R}$ given by

$$f(t) := (1-t) A_{X,\Delta}(v_0) + t A_{X,\Delta}(v) + \log \int_{\mathbb{R}^2} e^{-(1-t)x - ty} d\boldsymbol{\nu}.$$

It is clear that $f(0) = \tilde{\beta}_{X,\Delta}(v_0)$ and $f(1) = \tilde{\beta}_{X,\Delta}(v)$. By Hölder's inequality as in the proof of Proposition 3.14, we know that $f(t)$ is convex in t. Moreover, we have

$$f'(0) = A_{X,\Delta}(v) - A_{X,\Delta}(v_0) + \frac{\int_{\mathbb{R}^2}(x-y)e^{-x}d\boldsymbol{\nu}}{\int_{\mathbb{R}^2}e^{-x}d\boldsymbol{\nu}}$$
$$= (A_{X,\Delta}(v) - \widetilde{S}(v_0;v)) - (A_{X,\Delta}(v_0) - \widetilde{S}(v_0;v_0)) \geq 0.$$

Thus, $f(1) \geq f(0)$ and the 'if' part is proved.

Next, we show the 'only if' part. Let $(\mathcal{F}_t)_{t\in[0,1]}$ be the geodesic of filtrations connecting $\mathcal{F}_0$ and $\mathcal{F}_1$. Since v_0 computes $h(X,\Delta)$, we know that $\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_t) \geq \mathbf{H}^{\mathrm{NA}}(\mathcal{F}_0) = f(0)$ for any $t \in [0,1]$. Recall that

$$\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_t) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}_t) - \widetilde{S}(\mathcal{F}_t) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}_t) + \log\int_{\mathbb{R}^2} e^{-(1-t)x-ty}d\boldsymbol{\nu}.$$

By Proposition 3.12 and Corollary 3.11, we obtain

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_t) \leq (1-t)\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_0) + t\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_1) \leq (1-t)A_{X,\Delta}(v_0) + tA_{X,\Delta}(v).$$

Hence, we have $\mathbf{H}^{\mathrm{NA}}(\mathcal{F}_t) \leq f(t)$ which implies that $f(t) \geq f(0)$ for all $t \in [0,1]$ and thus $f'(0) \geq 0$, that is,

$$A_{X,\Delta}(v) - \widetilde{S}(v_0;v) \geq A_{X,\Delta}(v_0) - \widetilde{S}(v_0;v_0).$$

Since $v \in \mathrm{Val}_X^\circ$ is arbitrary, the above inequality remains true if we replace v by λv for any $\lambda \in \mathbb{R}_{>0}$, that is,

$$\lambda(A_{X,\Delta}(v) - \widetilde{S}(v_0;v)) \geq A_{X,\Delta}(v_0) - \widetilde{S}(v_0;v_0).$$

Thus, we have $A_{X,\Delta}(v) \geq \widetilde{S}(v_0;v)$ for any $v \in \mathrm{Val}_X^\circ$ and $A_{X,\Delta}(v_0) \leq \widetilde{S}(v_0;v_0)$, which implies that $A_{X,\Delta}(v_0) = \widetilde{S}(v_0;v_0)$. This finishes the proof. □

The previous theorem immediately implies the following corollary.

Corollary 5.2. *Let $g(x) = e^{-x}$, and let $v_0 \in \mathrm{Val}_X^\circ$ be the valuation computing $h(X,\Delta)$. Then $\delta_g(X,\Delta,v_0) = 1$ and is computed by v_0.*

5.2. δ_g-minimizers

In this section, we fix a continuous function $g\colon \mathbb{R} \to \mathbb{R}_{>0}$, a torus $\mathbb{T} < \mathrm{Aut}(X,\Delta)$ and a quasi-monomial valuation $v_0 \in \mathrm{Val}_X^{\mathbb{T},\circ}$. Let $N = \mathrm{Hom}(\mathbb{G}_m,\mathbb{T})$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ as before.

Question 5.3. Assume that $\delta_{g,\mathbb{T}}(X,\Delta,v_0) \leq 1$. Let $v \in \mathrm{Val}_X^{\mathbb{T},\circ}$ be a valuation that computes $\delta_{g,\mathbb{T}}(X,\Delta,v_0)$. Is the associated graded ring $\mathrm{gr}_v R := \mathrm{gr}_{\mathcal{F}_v} R$ finitely generated?

We give an affirmative answer in two special cases, which is enough for our applications.

Theorem 5.4. *Let $v \in \mathrm{Val}_X^{\mathbb{T},\circ}$ be a quasi-monomial valuation that computes $\delta_{g,\mathbb{T}}(X,\Delta,v_0)$. Assume that $\delta_{g,\mathbb{T}}(X,\Delta,v_0) \leq 1$ and that $v = (v_0)_\xi$ or $v_0 = \mathrm{wt}_\xi$ for some $\xi \in N_{\mathbb{R}}$. Then $\mathrm{gr}_v R$ is finitely generated.*

For the proof, we first recall a statement that can be extracted from the proof of [LXZ22, Lemma 3.1].

Lemma 5.5. *Let v be a quasi-monomial valuation on X. Assume that there exists a sequence of $\mathbb{Q}$-divisors D_m ($m \in \mathbb{N}$) such that $(X,\Delta+D_m)$ is lc and $-(K_X+\Delta+D_m)$ is semiample for all m and that $\lim_{m\to\infty} v(D_m) = A_{X,\Delta}(v)$. Then v is an lc place of $(X,\Delta+\Gamma)$ for some $\mathbb{Q}$-complement Γ of (X,Δ).*

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Proof. We only sketch the proof since the argument is almost the same as in [LXZ22, Lemma 3.1]. After rescaling, we assume that $A_{X,\Delta}(v) = 1$. Since v is quasi-monomial, we have $v \in \mathrm{QM}(Y, E)$ for some log smooth model $(Y, E) \to (X, \Delta)$. Let $\mathfrak{a}_m = \mathfrak{a}_m(v)$ $(m \in \mathbb{N})$ be the valuation ideals. By the proof of [LXZ22, (3.1)] (which only uses the fact that v is quasi-monomial), we know that, for any $\varepsilon \in (0, 1)$, there exists $\varepsilon_0 > 0$ and divisorial valuations $v_i = \mathrm{ord}_{F_i} \in \mathrm{QM}(Y, E)$ $(i = 1, \cdots, r)$ such that v is in the convex hull of $v_1, \cdots, v_r$ and $A_{X,\Delta+\mathfrak{a}_\bullet^{1-\varepsilon_0}}(F_i) < \varepsilon$ for all i. By assumption, we have $v(D_m) > 1 - \varepsilon_0$ for sufficiently large m and for such m we obtain

$$A_{X,\Delta+D_m}(F_i) \leq A_{X,\Delta+\mathfrak{a}_\bullet^{1-\varepsilon_0}}(F_i) < \varepsilon. \tag{5.2}$$

By [BCHM10, Corollary 1.4.3], we get a $\mathbb{Q}$-factorial birational model $p\colon \widetilde{X} \to X$ that extracts exactly the divisors F_i. By assumption, all $(X, \Delta + D_m)$ $(m \in \mathbb{N})$ have $\mathbb{Q}$-complements. Together with equation (5.2), this implies that $(\widetilde{X}, p_*^{-1}\Delta + (1 - \varepsilon)\sum_{i=1}^r F_i)$ has $\mathbb{Q}$-complements as well. Using [LXZ22, Lemma 3.2], we conclude that $\mathbb{Q}$-complements also exist for $(\widetilde{X}, p_*^{-1}\Delta + \sum_{i=1}^r F_i)$ as long as ε is sufficiently small. Since v is in the complex hull of ord_{F_i}, this yields a $\mathbb{Q}$-complement Γ of (X, Δ) that has v as an lc place. □

Lemma 5.6. *There exists some constant $c > 0$ such that $S_g(v_0; G) > c$ for all effective $\mathbb{Q}$-divisors $G \sim_{\mathbb{Q}} -(K_X + \Delta)$ on X. In particular, for any $m \gg 0$ and any g-weighted (m, v_0)-basis type divisor D that is compatible with G, we have $D \geq cG$.*

Proof. Let $T = T_{X,\Delta}(v_0) < \infty$, and let

$$c_0 = \frac{\inf_{x\in[0,T]} g(x)}{\sup_{x\in[0,T]} g(x)} > 0.$$

Let $\boldsymbol{\nu}$ denote the compatible DH measure associated to $\mathcal{F}_{v_0}$ and $\mathcal{F}_G$ as in Section 3.1.3. Then $\boldsymbol{\nu}$ is supported in $[0, T] \times \mathbb{R}$ and we have

$$S_g(v_0; G) = \frac{\int_{\mathbb{R}^2} y g(x) \mathrm{d}\boldsymbol{\nu}}{\int_{\mathbb{R}^2} g(x) \mathrm{d}\boldsymbol{\nu}} \geq c_0 \cdot \frac{\int_{\mathbb{R}^2} y \mathrm{d}\boldsymbol{\nu}}{\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{\nu}} = c_0 \cdot S_{X,\Delta}(G) = \frac{c_0}{n+1},$$

where the last equality is by [LXZ22, Lemma 2.20]. Thus, we may take, for example, $c = \frac{c_0}{3n}$. □

We are now ready to prove Theorem 5.4.

Proof of Theorem 5.4. The plan is to use Lemma 5.5 to show that v is a monomial lc place of a special complement (in the sense of [LXZ22, Definition 3.3]) and then apply [LXZ22, Theorem 4.2] to get the finite generation. To this end, let $\pi : (Y, E = \sum_{i=1}^r E_i) \to (X, \Delta)$ be a $\mathbb{T}$-equivariant log smooth model such that $\mathrm{QM}(Y, E)$ is a simplicial cone whose interior contains v, $C_Y(v) = \cap_{i=1}^r E_i$, and there is a π-exceptional and π-ample $\mathbb{Q}$-divisor $-F$ on Y.

Let G_Y be a $\mathbb{T}$-invariant $\mathbb{Q}$-divisor in the ample $\mathbb{Q}$-linear system $|-\pi^*(K_X + \Delta) - \varepsilon F|_{\mathbb{Q}}$ $(0 < \varepsilon \ll 1)$ whose support does not contain $C_Y(v_0)$ (such G_Y exists because there is some $\mathbb{T}$-invariant element Γ in $H^0(Y, m(-\pi^*(K_X + \Delta) - \varepsilon F))$ for sufficiently divisible m with $v_0(\Gamma) \neq 0$). Let $G := \pi_* G_Y$. For any $m \in \mathbb{N}$, let $D_m \in |-K_X - \Delta|_{\mathbb{R}}$ be a $\mathbb{T}$-invariant g-weighted (m, v_0)-basis type divisor that is also compatible with both v and G. Such divisors exist because:

- Both v and G are $\mathbb{T}$-invariant (so we can choose compatible basis in each individual piece in the weight decomposition), and
- By our assumption, any $\mathbb{T}$-invariant basis that is compatible with both v and G is automatically compatible with v_0.

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By Lemma 5.6, there exists some $c \in \mathbb{Q}_+$ such that $D_m \geq cG$ for all $m \gg 0$. Let $\delta_m = \min\{\delta_{g,\mathbb{T},m}(X,\Delta,v_0),1\}$. Then $(X,\Delta+\delta_m D_m)$ is lc and $\lim_{m\to\infty}\delta_m = \delta_{g,\mathbb{T}}(X,\Delta,v_0)$ by Lemma 4.5 and the assumption that $\delta_{g,\mathbb{T}}(X,\Delta,v_0) \leq 1$.

Since v computes $\delta_{g,\mathbb{T}}(X,\Delta,v_0)$, we also have

$$\lim_{m\to\infty}\delta_m v(D_m) = \delta_{g,\mathbb{T}}(X,\Delta,v_0)S_g(v_0;v) = A_{X,\Delta}(v).$$

Note that by construction $D_m = \sum \lambda_i D_m^{(i)}$ for some $\lambda \in \mathbb{R}_+$ and some effective $\mathbb{Q}$-divisors $D_m^{(i)} \sim_{\mathbb{Q}} -(K_X+\Delta)$. Thus, by perturbing the coefficients λ_i, for each $m \gg 0$ we get a $\mathbb{Q}$-divisor $D'_m = \sum \lambda'_i D_m^{(i)}$ such that $\delta_m D_m > D'_m \geq \frac{1}{2}cG$ and $\lim_{m\to\infty} v(D'_m) = A_{X,\Delta}(v)$. It follows that $(X,\Delta+cG+\widetilde{D}_m)$ is lc and $-(K_X+\Delta+cG+\widetilde{D}_m)$ is ample where $\widetilde{D}_m = D'_m - \frac{1}{2}cG$. By Lemma 5.5, we see that v is an lc place of $(X,\Delta+\frac{1}{2}cG+\Gamma)$ for some $\mathbb{Q}$-complement Γ of $(X,\Delta+\frac{1}{2}cG)$. Recall the $\pi_*^{-1}G$ is ample and does not contain $C_Y(v)$. By [LXZ22, Theorem 4.2], this implies that $\mathrm{gr}_v R$ is finitely generated. □

Corollary 5.7. *Assume that (X,Δ) is not K-semistable. Let $v_0 \in \mathrm{Val}_X^\circ$ be the unique valuation computing $h(X,\Delta)$. Then $\mathrm{gr}_{v_0} R$ is finitely generated.*

Proof. By [HL20b, Theorem 1.5], v_0 is quasi-monomial. So the result follows immediately from Corollary 5.2 and Theorem 5.4 (with $g(x) = e^{-x}$ and $\mathbb{T} = \{1\}$). □

Proof of Theorem 1.2. By Corollary 5.7, we know that v yields a special $\mathbb{R}$-test configuration in the sense of [HL20b, Definition 2.8]. Thus, by [HL20b, Theorem 1.6], we know that (X_0,Δ_0,ξ_v) is a K-semistable triple. □

Corollary 5.8. *Any quasi-monomial valuation $v \in \mathrm{Val}_X^{\mathbb{T},\circ}$ computing $\delta_g(X,\Delta,\xi)$, where $\xi \neq 0$, has a finitely generated associated graded ring.*

Proof. In view of Theorem 5.4, it suffices to show that $\delta_g(X,\Delta,\xi) \leq 1$. Indeed, we will prove a stronger statement:

$$\delta_{g,\mathbb{T},m}(X,\Delta,v_0) \leq 1 \tag{5.3}$$

for all $v_0 \in \mathrm{Val}_X^{\mathbb{T},\circ}$, $m \in \mathbb{N}$ and all positive functions $g \in C^0(\mathbb{R})$, as long as $\mathbb{T} \neq \{1\}$. To see this, let D be a $\mathbb{T}$-invariant g-weighted (m,v_0)-basis type divisor. If $(X,\Delta+D)$ is klt, then after perturbing the coefficients of D as in the proof of Theorem 5.4, we get a $\mathbb{T}$-invariant $\mathbb{Q}$-divisor $\widetilde{D} > D$ proportional to $-(K_X+\Delta)$ such that $(X,\Delta+\widetilde{D})$ is still klt and $K_X+\Delta+\widetilde{D}$ is ample. So such pairs have finite automorphism groups and this is a contradiction as $\mathbb{T} < \mathrm{Aut}(X,\Delta+\widetilde{D})$ by construction. Thus, $(X,\Delta+D)$ is not klt and $\mathrm{lct}(X,\Delta;D) \leq 1$. This proves equation (5.3). □

Corollary 5.9. *Any g-Ding polystable triple (X,Δ,ξ) is also reduced uniformly g-Ding stable. In particular, it admits a Kähler–Ricci g-soliton when $\mathbb{k} = \mathbb{C}$.*

Proof. The proof is very similar to that of [LXZ22, Theorem 5.2]. Let $\mathbb{T} < \mathrm{Aut}(X,\Delta,\xi)$ be a maximal torus such that $\xi \in N_{\mathbb{R}}$. Assume to the contrary that (X,Δ,ξ) is g-Ding polystable but not reduced uniformly g-Ding stable. Then by Proposition 4.14, we know that $\delta_g(X,\Delta,\xi) = 1$ is computed by some quasi-monomial valuation $v \in \mathrm{Val}_X^{\mathbb{T},\circ}$ that is not of the form wt_η. By Corollary 5.8, the associated graded ring $\mathrm{gr}_v R$ is finitely generated. Let $\pi\colon (Y,E) \to (X,\Delta)$ be a $\mathbb{T}$-equivariant log smooth model such that $\mathrm{QM}(Y,E)$ is a simplicial cone containing v and that its dimension is the same as the rational rank of v. As in [Xu21, Claim 3.10], this implies that in a neighbourhood of v in $\mathrm{QM}(Y,E)$, the function $w \mapsto S_g(v_0;w)$ is linear, and we have $\mathrm{gr}_w R \cong \mathrm{gr}_v R$.

Thus, $\delta_g(X,\Delta,\xi)$ is also computed by some $\mathbb{T}$-invariant divisorial valuation $w \in \mathrm{QM}(Y,E)$ that is sufficiently close to v. In particular, $\mathbf{D}_g^{\mathrm{NA}}(\mathcal{F}_w) \leq A_{X,\Delta}(w) - S_g(v_0;w) = 0$ (the first inequality is by Corollary 3.11) and $w \neq \mathrm{wt}_\eta$ for any $\eta \in N_{\mathbb{R}}$. It induces a nonproduct type $\mathbb{T}$-equivariant test

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configuration $(\mathcal{X}, \Delta_{\mathcal{X}}, \mathcal{L})$ such that $\mathbf{D}^{\mathrm{NA}}_g(\mathcal{X}, \Delta_{\mathcal{X}}, \mathcal{L}) \le 0$. This contradicts the g-Ding polystability of (X, Δ, ξ) and proves the first part of the corollary. The remaining part follows from Theorem 4.9. □

Proof of Theorem 1.3. By [BWN14, Theorem 1.5], (X, Δ, ξ) is K-polystable if it admits a Kähler–Ricci soliton. Thus, the result follows immediately from Theorem 4.9 and Corollary 5.9 by setting $g(x) = e^{-x}$. □

Remark 5.10. The above proof that reduced uniform Ding stability implies K-polystability uses Kähler–Ricci solitons, but it can be proved algebraically. Though, there is some subtlety, since the data of a triple (X, Δ, ξ) includes a torus $\mathbb{T}$ so that $\xi \in N_R := \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{T}$ is not necessarily maximal. While the K-polystability of (X, Δ, ξ) is with respect to $\mathbb{T}$, reduced uniform Ding stability is defined using a maximal torus $\mathbb{T} < \mathbb{T}^{\max} < \mathrm{Aut}(X, \Delta)$.

To prove the equivalence, observe that if (X, Δ, ξ) is reduced uniformly Ding stable, then it is K-polystable with respect to $\mathbb{T}^{\max}$ by [HL20, Proposition 5.16]. To show it is K-polystable with respect to $\mathbb{T}$, first by [HL20b, (168) or (189)], it follows that (X, Δ, ξ) is K-semistable with respect to $\mathbb{T}$. Then by verbatim applying the proof of [LWX21, Theorem 3.7] (see also [HL20b, Section 8]), we know the K-polystablity of (X, Δ, ξ) with respect to $\mathbb{T}^{\max}$ implies the K-polystability of (X, Δ, ξ) with respect to $\mathbb{T}$.

Proof of Theorem 1.1. It is a combination of Theorem 1.2, [HL20b, Theorem 1.3] and Theorem 1.3. □

As an application, we show the following theorem, which generalizes [WZ04] from the smooth case to general toric log Fano pairs. See also [HL20, Section 8].

Theorem 5.11. *For any toric log Fano pair (X, Δ) over $\mathbb{C}$, there exists a vector $\xi \in N_{\mathbb{R}} :=$ $\mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ where $\mathbb{T}$ is the maximal torus acting on (X, Δ), such that (X, Δ, ξ) admits a Kähler–Ricci soliton.*

Proof. By Corollary 3.17, there is a unique valuation $v \in \mathrm{Val}_X$ which computes $h(X, \Delta)$. By the uniqueness, v is $\mathbb{T}$-invariant, that is, $v = \mathrm{wt}_\xi$ for some $\xi \in N_{\mathbb{R}}$. Therefore, the K-semistable triple produced in Theorem 1.2 is (X, Δ, ξ).

Since (X, Δ, ξ) is K-semistable and toric, it is reduced uniformly Ding stable by Theorem 4.12. Indeed, condition (3) of the theorem holds trivially, since any $w \in \mathrm{Val}^{\mathbb{T},\circ}_X$ is of the form $w = \mathrm{wt}_\eta$ for some $\eta \in N_{\mathbb{R}}$ and, hence, satisfies $\mathbf{J}^{\mathrm{NA}}_{g,\mathbb{T}}(w) = 0$ where $g = e^{-x}$. Therefore, (X, Δ, ξ) admits a Kähler–Ricci soliton by [HL20]. □

6. Moduli stack

In this section, we will prove Theorem 1.5. It suffices to verify the boundedness and openness; see Theorem 6.3 and Theorem 6.4.

Theorem 6.1. *For a fixed dimension n, volume V, a positive integer N and a negative constant $h_0 < 0$, families of n-dimensional log Fano pairs (X, Δ) with $(-K_X - \Delta)^n = V$, $N\Delta$ integral and $h(X, \Delta) \ge h_0$ are parameterized by an Artin stack $\mathcal{M}^{\mathrm{Fano}}_{n,V,N,h_0^+}$ of finite type.*

The following result gives the boundedness.

Proposition 6.2. *Let (X, Δ) be a log Fano pair, and let $c \in \mathbb{R}$. Assume that $\tilde{\beta}_{X,\Delta}(v) \ge c$ for all divisorial valuations v on X. Then $\alpha(X, \Delta) \ge \alpha$ for some constant $\alpha > 0$ that only depends on c and $\dim(X)$.*

Here, $\alpha(X, \Delta)$ denotes Tian's α-invariant. In the proof, we use that $\alpha(X, \Delta)$ equals $\inf_v \frac{A_{X,\Delta}(v)}{T(v)}$, where the infimum runs through all divisorial valuations on X; see, for example, [BJ20].

Proof. It suffices to find some constant $M = M(c, n) > 0$ that only depends on c and $n = \dim(X)$ such that $T(v) \le M$ for all divisorial valuations v with $A_{X,\Delta}(v) = 1$. For now, fix any such v and let $\mathcal{F} = \mathcal{F}_v$, $T = T_{X,\Delta}(v)$, $\widetilde{S} = \widetilde{S}(v)$. We will apply a modified argument from [BJ20, Lemma 2.6]. Let

$f(\lambda) = \frac{\mathrm{vol}(V_\bullet^{(\lambda)})}{L^n}$. Note that $f(0) = 1$, $f(T) = 0$ and $\mathrm{DH}_{\mathcal{F}} = -f'(\lambda)\mathrm{d}\lambda$. By [Laz04, Theorem 11.4.9], the function $\lambda \mapsto f(\lambda)^{\frac{1}{n}}$ is concave on $(0, T)$, thus

$$f(\lambda) \geq \left(1 - \frac{\lambda}{T}\right)^n.$$

On the other hand, integration by parts yields

$$e^{-\widetilde{S}} = \int_0^T e^{-\lambda}\mathrm{DH}_{\mathcal{F}}(\mathrm{d}\lambda) = 1 - \int_0^T e^{-\lambda} f(\lambda)\mathrm{d}\lambda,$$

hence $e^{-\widetilde{S}} \leq 1 - \int_0^T e^{-\lambda}\left(1 - \frac{\lambda}{T}\right)^n \mathrm{d}\lambda$. We may further rewrite the right-hand side as

$$\begin{aligned}
\int_0^\infty e^{-\lambda}\mathrm{d}\lambda - \int_0^T e^{-\lambda}\left(1 - \frac{\lambda}{T}\right)^n \mathrm{d}\lambda &= \int_0^T e^{-\lambda}\left[1 - \left(1 - \frac{\lambda}{T}\right)^n\right]\mathrm{d}\lambda + \int_T^\infty e^{-\lambda}\mathrm{d}\lambda \\
&\leq \int_0^T \frac{n\lambda}{T} e^{-\lambda}\mathrm{d}\lambda + \int_T^\infty e^{-\lambda}\mathrm{d}\lambda \\
&\leq \frac{n}{T} + e^{-T}.
\end{aligned}$$

By assumption, $1 - \widetilde{S} = \tilde{\beta}(v) \geq c$, hence $e^{-\widetilde{S}} \geq e^{c-1}$. It follows that

$$\frac{n}{T} + e^{-T} \geq e^{c-1}.$$

From this, we deduce that T is bounded from above by some constant that only depends on c and n. The proof is now complete. □

Theorem 6.3. *Fixed positive integers n, N, a positive number V_0 and a constant h_0. Denote by $\mathcal{P}$ the set of n-dimensional log Fano pairs (X, Δ) with $N \cdot \Delta$ integral which satisfy $(-K_X - \Delta)^n \geq V_0$ and $h(X, \Delta) \geq h_0$. Then $\mathcal{P}$ is bounded.*

Proof. For fixed $\alpha_0 > 0$, the set of log Fano pairs (X, Δ) with $n = \dim(X)$, $N \cdot \Delta$ integral and $(-K_X - \Delta)^n \geq V_0$, and $\alpha(X, \Delta) > \alpha_0$ are bounded by [Che20] [Jia20] [XZ21]. Applying Proposition 6.2 then completes the proof. □

Next, we will prove the openness. It suffices to show the following theorem.

Theorem 6.4. *Let $(X, \Delta) \to B$ be a locally stable family of log Fano pairs over a scheme B of finite type. Then*

$$h\colon t \longrightarrow h(X_t, \Delta_t), \quad t \in B$$

is a constructible and lower semicontinuous.

Proof. By passing to a resolution of B_{red}, we may assume B is smooth. By the proof of Theorem 5.4, we know that the minimizer of $\tilde{\beta}_{X,\Delta}$ is an lc place of a $\mathbb{Q}$-complement. Then as showed in [BLX22, Theorem 3.5],

$$h(X_t, \Delta_t) = \min_v\{\tilde{\beta}_{X_t,\Delta_t} \mid v \text{ is an } N\text{-complement}\},$$

for some constant N which only depends on $\dim X$ and coefficients of Δ. Then the rest of the proof is similar to the one in [BL22, BLX22]. For the sake of completeness, we give a sketch here.

We know that there is a finite type variety $\phi\colon S \to B$ with a relative Cartier divisor $D \subset X \times_B S$ over S such that

1. For any $s \in S$, the fiber D_s is an N-complement of (X_t, Δ_t), where $t = \phi(s)$, and $(X_t, D_s + \Delta_t)$ is log canonical but not klt, and
2. For any N-complement Γ_t of (X_t, Δ_t), there is a point $s \in S$, such that $D_s \cong \Gamma_t$.

After resolving and stratifying S, as well as passing to a finite base change, we can assume S is a union of its smooth connected component S_i such that $(X \times_B S_i, \Delta \times_B S_i + D \times_S S_i)$ admits a fiberwise log resolution.

For a fixed i, we can identify the dual complex $\mathcal{CW}_i := \mathcal{DMR}(X_t, \Delta_t + D_t)$ for any $t \in S_i$. We claim $\tilde{\beta}_{X_t,\Delta_t}(v_t)$ does not depend on t, for different valuations v_t correspond to the same point of $\mathcal{CW}_i$. This is obvious for $A_{X_t,\Delta_t}(v_t)$. It also proved in [BLX22], using the invariance of plurigenera ([HMX13]), for v_t corresponding to the same point of $\mathcal{CW}$ over any $t \in S_i$, the induced DH-measure $\mathrm{DH}_{\mathcal{F}_{v_t}}$ on $\mathbb{R}$ is the same. Therefore,

$$\widetilde{S}(\mathcal{F}_{v_t}) = -\log \int_{\mathbb{R}} e^{-\lambda}\, \mathrm{DH}_{\mathcal{F}_{v_t}}(d\lambda)$$

does not depend on t.

Hence, for each i, we can define $a_i = \min\{\tilde{\beta}(v_t) \mid v_t \in \mathcal{CW}_i\}$, and we know that $h(X_t, \Delta_t) = \min\{a_i | \, t \in \phi(S_i)\}$, which implies that $h(X_t, \Delta_t)$ is constructible.

In light of the above constructibility result, to prove the lower semicontinuity of h it suffices to consider the case when B is the spectrum of a DVR R essentially of finite type over $\mathbb{k}$. Let K denote the fraction field of R and κ the residue field. By the properness of the flag variety, we know that any filtration $\mathcal{F}_K$ on $\bigoplus_m H^0(-mr(K_{X_K} + \Delta_{X_K}))$ extends to a filtration $\mathcal{F}_\kappa$ on $\bigoplus_m H^0(-mr(K_{X_\kappa} + \Delta_{X_\kappa}))$ (see [BL22]). By Lemma 3.8 and the lower semicontinuity of the log canonical threshold,

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{F}_K) = \mu(\mathcal{F}_K) \geq \mu(\mathcal{F}_\kappa) = \mathbf{L}^{\mathrm{NA}}(\mathcal{F}_\kappa).$$

Since $\mathrm{DH}_{\mathcal{F}_K}(d\lambda) = \mathrm{DH}_{\mathcal{F}_\kappa}(d\lambda)$, we also have $\widetilde{S}(\mathcal{F}_K) = \widetilde{S}(\mathcal{F}_\kappa)$. Therefore, h is lower semicontinuous. $\square$

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