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ASYMPTOTIC-NORMING AND MAZUR INTERSECTION PROPERTIES IN BOCHNER FUNCTION SPACES

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A Banach space X has the asymptotic-norming property if and only if the Lebesgue-Bochner function space $L^p(\mu, X)$ has the asumptotic-norming property for p with $1 < p < \infty$. It follows that a Banach space X is Hahn-Banach smooth if and only if $L^p(\mu, X)$ is Hahn-Banach smooth for p with $1 < p < \infty$. We also show that for p with $1 < p < \infty$, (1) if X has the compact Mazur intersection property then so does $L^p(\mu, X)$; (2) if the measure μ is not purely atomic, then the space $L^p(\mu, X)$ has the Mazur intersection property if and only if X is an Asplund space and has the Mazur intersection property.

Unless otherwise stated, we always assume X is a Banach space, $1 < p,\ q < \infty$ with $1/p + 1/q = 1$, and (Ω, Σ, μ) is a positive measure space so that Σ contains an element with finite positive measure. We use S_X and B_X to denote the unit sphere and the unit ball in X respectively.

The asymptotic-norming property (ANP) was introduced by James and Ho [**12**]. There are three different kinds of asymptotic-norming properties, and each of them implies the Radon-Nikodym property [**12**]. Ghoussoub and Maurey [**3**] proved that for separable Banach spaces the asymptotic-norming property is equivalent to the Radon-Nikodym property. However, in general, it is an open question whether the two properties are equivalent.

A set Φ in X^* is a *norming set* for X if $\Phi \subset B_{X^*}$ and Φ norms X, that is, $\|x\| = \sup_{f\in\Phi} f(x)$ for all x in X. A sequence $\{x_n\}$ is *asymptotically normed* by Φ if for each $\varepsilon > 0$, there are $N \geqslant 1$ and $f \in \Phi$ such that $f(x_n) > \|x_n\| - \varepsilon$, for all $n \geqslant N$. We say that X has the Φ-*ANP-I* (respectively Φ-*ANP-II*; or Φ-*ANP-III*) if every sequence $\{x_n\}$ in S_X that is asymptotically normed by Φ is convergent (respectively has a convergent subsequence; or $\bigcap_{n\geqslant 1} \overline{\mathrm{co}}\{x_k : k \geqslant n\} \neq \emptyset$). And we say that $(X, \|\ \|)$ has the *ANP-K* for $K =$I, II or III if there is a norming set Φ for $(X, \|\ \|)$ such that X has the Φ-ANP-K. The space X has the ANP-K for $K =$ I, II or III if there is an equivalent norm $\|\ \|$ on X such that $(X, \|\ \|)$ has the ANP-K.

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As was shown in [5] (see Lemma 1 for detail), the ANP is related to property (G) and the Kadec property which can be described respectively in terms of denting points and points of continuity. Let C be a subset of X. A point x is said to be a *point of continuity* of C if $x \in C$ and the relative weak and norm topologies on C coincide at x. And a point x is called a *denting point* of C if $x \in C$ and the family of all slices of C containing x is a neighbourhood base of x with respect to the relative norm topology on C, where a *slice* of C is given by

$$S(x^*, C, \delta) = \{x \in C : x^*(x) > \sup x^*(C) - \delta\}$$

for some functional x^* in X^* and $\delta > 0$. The space X is said to have the *property* (G) (respectively *the Kadec property*) if every unit vector in X is a denting point (respectively a point of continuity) of the unit ball of X.

LEMMA 1. [5] *Suppose X is a Banach space and Φ is a norming set for X.*

(1) *The space X has the Φ-ANP-III if and only if every sequence $\{x_n\}$ in S_X that is asymptotically normed by Φ has a weakly convergent subsequence.*
(2) *The space X has the Φ-ANP-III if and only if $X^{**} \setminus X = \{x^{**} : x^{**} \in X^{**}, \|x^{**}\| > \sup\{x^{**}(x^*) : x^* \in \Phi\}\}$.*
(3) *The space X has the Φ-ANP-I if and only if it has the Φ-ANP-III and the property (G).*
(4) *The space X has the Φ-ANP-II if and only if it has the Φ-ANP-III and the Kadec property.*
(5) *The space X has the ANP-II if and only if it has the ANP-III and it admits an equivalent norm with the Kadec property.*

REMARK. It is a result of James and Ho [12] that the ℓ_p-product $(1 \leqslant p < \infty)$ of spaces with ANP-II has the ANP-II. Using Lemma 1, it can be proved that the ℓ_p product $(1 \leqslant p < \infty)$ of spaces with ANP-K has the ANP-K for $K =$ I, II or III.

The concept of ANP can be used to characterise Hahn-Banach smoothness, the weak* Kadec-Klee property, and property (G^*) (see Lemma 2 for detail). A Banach space X is said to be *Hahn-Banach smooth* if in X^{***}, $x^* \in X^*$ and $\|x^* + x^\perp\| = \|x^*\| = 1$ imply $x^\perp = 0$, where $x^\perp \in X^{***}$ with $x^\perp|_X = 0$ [17]. A dual space X^* is said to have the *property* (G^*) if every unit vector in X^* is a *weak* denting point* of the unit ball of X^*, that is, for every unit vector x^* in X^*, the family of all weak* slices of B_{X^*} containing x^* is a neighbourhood base of x^* with respect to the relative norm topology on B_{X^*}, where a *weak* slice* of B_{X^*} is given by

$$S(x, B_{X^*}, \delta) = \{y^* \in B_{X^*} : y^*(x) > \sup x(B_{X^*}) - \delta\}$$

for some x in X and $\delta > 0$. We denote by w*-dent B_{X^*} the set of weak* denting points of B_{X^*}.

Lemma 2. [5, 6] *The following assertions are true for any Banach space X.*

(1) *X is Hahn-Banach smooth if and only if X^* has the Φ-ANP-III for some norming set Φ in B_X.*

(2) *X^* has the property (G^*) if and only if X^* has the Φ-ANP-I for some norming set Φ in B_X.*

Using techniques from [7] and results from [9], we show in Theorem 6 and Theorem 7 that ANP-K for $K =$ I, II, or III can be lifted from X to $L^p(\mu, X)$ with no restriction on X. As a consequence we obtain one of the main results of [7] in Corollary 8.

Lemma 3. *Suppose X has the Φ-ANP-K for $K =$ I, II or III. If Φ_1 is a norming set for X contained in $\overline{\mathrm{co}}(\Phi \cup -\Phi)$, then X has the Φ_1-ANP-K. In particular, if $\Phi_1 = \mathrm{co}(\Phi \cup \{0\}) \setminus S_X$, then X has the Φ_1-ANP-K.*

Proof: If X has the Φ-ANP-III, then by Lemma 1, for every x^{**} in $X^{**} \setminus X$ we have $\|x^{**}\| > \sup\{x^{**}(x^*) \colon x^* \in \Phi\}$ and $\|-x^{**}\| > \sup\{-x^{**}(x^*) \colon x^* \in \Phi\}$. Hence

$$\|x^{**}\| > \sup\{x^{**}(x^*) \colon x^* \in \overline{\mathrm{co}}(\Phi \cup -\Phi)\} \geqslant \sup\{x^{**}(x^*) \colon x^* \in \Phi_1\}.$$

So by Lemma 1, the space X has the Φ_1-ANP-III.

If X has the Φ-ANP-I (respectively Φ-ANP-II), then by Lemma 1, it has property (G) (respectively the Kadic property), and it also has the Φ_1-ANP-III because it has the Φ-ANP-III. By the same lemma, the space X has the Φ_1-ANP-I (respecitvely Φ_1-ANP-II), and the proof is complete. ∎

Note that $L^q(\mu, X^*)$ is a subspace of $L^p(\mu, X)^*$ and for any $f \in L^q(\mu, X^*)$ and $g \in L^p(\mu, X)$, the action of f on g is defined by $(f, g) = \int_\Omega (f(t), g(t))d\mu(t)$ [1]. A norming set Φ for X induces in a natural way a norming set for $L^p(\mu, X)$, namely the set

$$\left\{\sum_{i=1}^{m} \lambda_i x_i^* \chi_{E_i} : x_i^* \in \Phi,\ E_i \in \Sigma,\ E_i \cap E_j = \phi \text{ for } i \neq j,\ \lambda_i > 0 \text{ with } \sum_{i=1}^{m} \lambda_i^q \mu(E_i) = 1\right\}.$$

In order to prove our main result, we use a construction of norming sets for $L^p(\mu, X)$ which is a modification of the one just mentioned. Let $\Phi_1 = \mathrm{co}(\Phi \cup \{0\}) \setminus S_X$, and for each $n \geqslant 1$, let

$$\Delta_n = \left\{\sum_{i=1}^{m} \lambda_i x_i^* \chi_{E_i} : x_i^* \in \Phi_1,\ \frac{n-1}{n} \leqslant \|x_i^*\| < \frac{n}{n+1},\ E_i \in \Sigma,\ E_i \cap E_j = \phi \text{ for } i \neq j,\right.$$
$$\left.\lambda_i > 0 \text{ with } \sum_{i=1}^{m} \lambda_i^q \mu(E_i) = 1\right\}.$$

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Then $\Delta(\Phi, \mu, q) = \bigcup_{n\geqslant 1} \Delta_n$ is a norming set for $L^p(\mu, X)$. Let $g \in \Delta_n$; then in $L^q(\mu, X^*)$ the norm of g is less than or equal to $n/(n+1)$. If $t \in \operatorname{supp} g$, that is $g(t) \neq 0$, then $g(t) = \lambda x^*$ for some $\lambda > 0$ and $x^* \in \Phi_1$ such that $(n-1)/n \leqslant \|x^*\| < n/(n+1)$. Thus

$$\frac{n-1}{n}\frac{g(t)}{\|g(t)\|} = \frac{n-1}{n}\frac{x^*}{\|x^*\|} = \frac{n-1}{n\,\|x^*\|}x^* \in \Phi_1$$

because $0 \leqslant (n-1)/(n\,\|x^*\|) \leqslant 1$. So we have proved the following lemma.

Lemma 4. *Suppose Φ is a norming set for X. Then $\Delta(\Phi, \mu, q) = \bigcup_{n\geqslant 1} \Delta_n$ is a norming set for $L^p(\mu, X)$ such that if $g \in \Delta_n$ and $t \in \operatorname{supp} g$, then*

$$\frac{n-1}{n}\frac{g(t)}{\|g(t)\|} \in \Phi_1 \quad \text{and} \quad \|g\| \leqslant \frac{n}{n+1}.$$

Lemma 5. *Suppose X has the Φ-ANP-III, and V is a subspace of X^* which norms a closed subspace E of X. Let $\{x_n\}$ be a sequence in X such that for every x^* in V the sequence $\{x^*(x_n)\}$ converges. Let x^{**} be a weak* cluster point of $\{x_n\}$. If $\|x^{**}\| \geqslant \|x_n\|$ for $n \geqslant 1$, and $\|x^{**}\| = \sup\{x^{**}(x^*)\colon x^* \in \Phi \cap V\}$, then $x^{**} \in E$ and $\{x_n\}$ converges weakly to x^{**}.*

Proof: It is obvious that $\lim_n x^*(x_n) = x^{**}(x^*)$ for x^* in V. Since $\|x^{**}\| \geqslant \|x_n\|$ for $n \geqslant 1$, and $\|x^{**}\| = \sup\{x^{**}(x^*); x^* \in \Phi \cap V\}$, we have $\lim_n \|x_n\| = \|x^{**}\|$. For each $\varepsilon > 0$, there is $x^* \in \Phi \cap V$ such that $x^{**}(x^*) > \|x^{**}\| - \varepsilon$. Then there is $N \geqslant 1$ such that $x^*(x_n) > \|x^{**}\| - \varepsilon \geqslant \|x_n\| - \varepsilon$, for all $n \geqslant N$. Without loss of generality, we may assume that $x_n \neq 0$ for $n \geqslant 1$. Then $\{x_n/\|x_n\|\}$ is a sequence in S_X asymptotically normed by Φ. By Lemma 1 every subsequence of $\{x_n/\|x_n\|\}$ has a weakly convergent subsequence. Since $\lim_n \|x_n\| = \|x^{**}\| \neq 0$, every subsequence of $\{x_n\}$ also has a weakly convergent subsequence Because V norms E and the sequence $\{x^*(x_n)\}$ converges for every x^* in V, the sequence $\{x_n\}$ has to be weakly convergent. Therefore $\{x_n\}$ converges weakly to x^{**} and $x^{**} \in E$. This completes the proof. ∎

Theorem 6. *Let Φ be a norming set for X and $K = I$ or III. Then X has the Φ-ANP-K if and only if $L^p(\mu, X)$ has the $\Delta(\Phi, \mu, q)$-ANP-K.*

Proof: Choose $E \in \Sigma$ such that $0 < \mu(E) < \infty$. Define the function $J\colon X \to L^p(\mu, X)$ by $J(x) = \mu(E)^{-1/p} x \chi_E$. It is obvious that J is an isometry. If a sequence $\{x_n\}$ in S_X is asymptotically normed by Φ, then $\{Jx_n\}$ is asymptotically normed by $\Delta(\Phi, \mu, q)$. Thus X has the Φ-ANP-K if $L^p(\mu, X)$ has the $\Delta(\Phi, \mu, q)$-ANP-K for $K = $ I, II or III.

Now suppose X has the Φ-ANP-III. Let $\{f_n\}$ be a sequence in $S_{L^p(\mu, X)}$ which is asymptotically normed by $\Delta(\Phi, \mu, q)$. We need to show that $\bigcap_{n\geqslant 1} \overline{\mathrm{co}}\{f_k : k \geqslant n\} \neq \emptyset$. Let $\{g_n\}$ be a sequence in $\Delta(\Phi, \mu, q)$ such that for $k \geqslant 1$

$$\liminf_n \int_\Omega (g_k(t), f_n(t))d\mu(t) > 1 - \frac{1}{k}.$$

Then $\lim_{n,m} \|(f_n + f_m)/2\| = 1$. The uniform convexity of $L^p(\mu)$ implies that the sequence $\{\|f_n(\cdot)\|\}$ converges in norm to some f in $S_{L^p(\mu)}$. By Lemma 3 in [**8**] there is a sequence $\{f_n'\}$ in $L^p(\mu, X)$ such that $\|f_n'(t)\| = f(t)$ for all t in Ω, and $\lim_n \|f_n' - f_n\| = 0$. It is obvious that $\{f_n'\}$ is asymptotically normed by $\Delta(\Phi, \mu, q)$, and $\bigcap_{n\geqslant 1} \overline{\mathrm{co}}\{f_n' : k \geqslant n\} = \bigcap_{n\geqslant 1} \overline{\mathrm{co}}\{f_k : k \geqslant n\}$. Thus there is no loss of generality in assuming that $\|f_n(t)\| = f(t)$ for all t in Ω and $n \geqslant 1$. We may also assume that the subspace X_1 of X spanned by $\bigcup_{n\geqslant 1} f_n(\Omega)$ is separable. Hence there is a separable subspace V of X^* which contains $g_n(\Omega)$ for $n \geqslant 1$ and norms X. By Lemma 5 in [**7**] there are $h_n \in \mathrm{co}\{f_k : k \geqslant n\}$ for each $n \geqslant 1$, and $\Omega_1 \in \Sigma$ satisfying $\mu(\Omega \setminus \Omega_1) = 0$ such that $\{x^*(h_n(t))\}$ converges for every t in Ω_1 and x^* in V. Without loss of generality we may assume that $\Omega = \Omega_1$. For each t in Ω, let $h(t)$ be a weak* cluster point of $\{h_n(t)\}$. Then $\|h(t)\| \leqslant f(t)$ and $\lim_n x^*(h_n(t)) = h(t)(x^*)$ for all $t \in \Omega$ and $x^* \in V$. By the Lebesgue Convergence Theorem, for every $g \in L^q(\mu, V)$, we have $h(\cdot)(g(\cdot)) \in L^1(\mu)$ and

$$\int_\Omega (h(t), g(t))d\mu(t) = \lim_n \int_\Omega (g(t), h_n(t))d\mu(t).$$

In particular, for each $k \geqslant 1$, we have

$$\int_\Omega (h(t), g_k(t))d\mu(t) = \lim_n \int_\Omega (g_k(t), h_n(t))d\mu(t) > 1 - \frac{1}{k}.$$

Next we show that $\{h_n\}$ weakly converges to h, which would imply that $\bigcap_{n\geqslant 1} \overline{\mathrm{co}}\{f_k : k \geqslant n\} \neq \emptyset$.

CLAIM 1. For almost all t in Ω, we have

$$\|h(t)\| = f(t) = \sup\{h(t)(x^*) : x^* \in \Phi_1 \cap V\}.$$

By the definition of $\Delta(\Phi, \mu, q)$ there is $m_k \geqslant 1$ for each $k \geqslant 1$ with $g_k \in \Delta_{m_k}$. Since

$$1 - \frac{1}{k} < \int_\Omega (h(t), g_k(t))d\mu(t) \leqslant \int_\Omega f(t)\,\|g_k(t)\|\,d\mu(t) \leqslant \frac{m_k}{m_k + 1} < 1,$$

we have $\lim_k m_k = \infty$, and $\lim_k\{(h(\cdot), g_k(\cdot)) - f(\cdot)\|g_k(\cdot)\|\} = 0$ in $L^1(\mu)$, and $\lim_k \|g_k(\cdot)\| = f(\cdot)^{p-1}$ in $L^q(\mu)$. Passing to a subsequence if necessary, we may assume that there is $\Omega_2 \in \Sigma$ such that $\mu(\Omega \setminus \Omega_2) = 0$, and both $\lim_k\{(h(t), g_k(t)) - f(t)\|g_k(t)\|\} = 0$ and $\lim_k \|g_k(t)\| = f(t)^{p-1}$ for t in Ω_2. If $t \in \Omega_2 \cap \operatorname{supp} f$, then

$$\|h(t)\| \leqslant f(t) = \lim_k \left(h(t), \frac{g_k(t)}{\|g_k(t)\|}\right) = \lim_k \left(h(t), \frac{m_k}{m_k+1}\frac{g_k(t)}{\|g_k(t)\|}\right) \leqslant \|h(t)\|.$$

Since
$$\frac{m_k}{m_k+1}\frac{g_k(t)}{\|g_k(t)\|} \in \Phi_1 \cap V$$
whenever $g_k(t) \neq 0$, we have
$$\|h(t)\| = f(t) = \sup\{h(t)(x^*) \colon x^* \in \Phi_1 \cap V\}$$
whenever $t \in \Omega_2 \cap \operatorname{supp} f$. Thus the claim holds.

Without loss of generality, we may assume that for all t in Ω,

$$\|h(t)\| = f(t) = \sup\{h(t)(x^*) \colon x^* \in \Phi_1 \cap V\}.$$

Then $\|h_n(t)\| \leqslant \|h(t)\|$ for all t in Ω and $n \geqslant 1$. By Lemma 5, we have $h(t) \in X_1$ and $\{h_n(t)\}$ weakly converges to $h(t)$. Thus h is separably valued, and for every x^* in X^*, the real valued function $x^*(h(\cdot))$ is measurable. It follows from the Pettis Measurability Theorem and the fact that $\|h(t)\| = f(t)$ for all t in Ω, that $h \in L^p(\mu, X)$.

Claim 2. The sequence $\{h_n\}$ weakly converges to h.

To prove this, let $T \in L^p(\mu, X)^*$. There is a function g from Ω to X^* such that g is weak* measurable, $\|g(\cdot)\| \in L^q(\mu)$, and for any $\varphi \in L^p(\mu, X)$, $T(\varphi) = \int_\Omega (g(t), \varphi(t))d\mu(t)$ [**2, 11**]. Applying the Lebesgue Convergence Theorem to the sequence $\{g(\cdot)(h_n(\cdot))\}$ in $L^1(\mu)$, we have

$$T(h) = \int_\Omega (g(t), h(t))d\mu(t) = \lim_n \int_\Omega (g(t), h_n(t))d\mu(t) = \lim_n T(h_n).$$

Hence $\{h_n\}$ weakly converges to h as claimed.

Since $\{h_n\}$ weakly converges to h, we have $h \in \bigcap_{n\geqslant 1} \overline{\mathrm{co}}\{f_k \colon k \geqslant n\}$. Therefore $L^p(\mu, X)$ has the $\Delta(\Phi, \mu, q)$-ANP-III.

Finally suppose X has the Φ-ANP-I. Then $L^p(\mu, X)$ has the $\Delta(\Phi, \mu, q)$-ANP-III. By Lemma 1, the space X has the property (G). So $L^p(\mu, X)$ also has the property (G) [**13**]. By Lemma 1 again, we conclude that $L^p(\mu, X)$ has the $\Delta(\Phi, \mu, q)$-ANP-I, and hence the proof is complete. $\square$

It follows from Theorem 6 that $(X, \| \ \|)$ has the ANP-K if and only if $(L^p(\mu, X), \| \ \|)$ has the ANP-K for $K = \mathrm{I}$ or III. However if $(X, \| \ \|)$ has the ANP-II, the space $(L^p(\mu, X), \| \ \|)$ may not have the ANP-II. For example, the space ℓ_1 with the usual norm has the ANP-II [12], in fact it has the B_{c_0}-ANP-II, but if (Ω, Σ, μ) is not purely atomic, then since ℓ_1 is not strictly convex, $L^p(\mu, \ell_1)$ does not have the Kadec property [16], so by Lemma 1, $(L^p(\mu, X), \| \ \|)$ does not have the ANP-II.

THEOREM 7. *A Banach space X has the ANP-II if and only if $L^p(\mu, X)$ has the ANP-II.*

PROOF: Since X is isomorphic to a subspace of $L^p(\mu, X)$, if $L^p(\mu, X)$ has the ANP-II then X also has the ANP-II. Now suppose X has the ANP-II, then by Theorem 5, the space $L^p(\mu, X)$ has the ANP-III. And by Lemma 1, the space X admits an equivalent norm with the Kadec property. It can be proved that $L^p(\mu, X)$ also admits an equivalent norm with the Kadec property (see [9]). Thus by Lemma 1, the space $L^p(\mu, X)$ has the ANP-II and the proof is complete. ∎

COROLLARY 8. [7] *Let X be a Banach space.*

(1) *If X is Hahn-Banach smooth, then $L^p(\mu, X)$ is Hahn-Banach smooth.*
(2) *If X^* has the property (G^*), then $L^p(\mu, X)^*$ has the property (G^*).*

PROOF: (1) The space X is an Asplund space, because it is Hahn-Banach smooth [15]. So $L^q(\mu, X^*)$ is the dual of $L^p(\mu, X)$ [1]. By Theorem 6 and Lemma 2, the space $L^q(\mu, X^*)$ has the Φ-ANP-III for some norming set Φ in $B_{L^p(\mu, X)}$. Then by Lemma 2, we conclude that $L^p(\mu, X)$ is Hahn-Banach smooth. The proof of part (2) is similar. ∎

REMARK. For $K = \mathrm{I}$, II or III, a dual space X^* is said to have the *w^*-ANP-K* if there is an equivalent norm $\| \ \|$ on X such that $(X^*, \| \ \|)$ has the Φ-ANP-K for some norming set Φ in X (see [5]). It follows from Lemma 2 and Corollary 8 that if X^* has the w^*-ANP-I (respectively w^*-ANP-III), then $L^q(\mu, X^*)$ which is the dual of $L^p(\mu, X)$ has the w*-ANP-I (respectively w*-ANP-III). In [5] it was shown that X^* has the w^*-ANP-II if and only if there is an equivalent norm $\| \ \|$ on X such that the norm topology and the weak* topology on the unit sphere of $(X^*, \| \ \|)$ coincide, in other words, $(X^*, \| \ \|)$ has the weak*-Kadec property. Therefore, in terms of w^*-ANP-II, a result of [9] asserts that if X^* has the w^*-ANP-II, then $L^q(\mu, X^*)$ has the w^*-ANP-II.

A Banach space X is said to have the *Mazur intersection property (I)* if every bounded closed convex set in X is an intersection of balls in X. And X is said to have the *compact Mazur intersection property (CI)* if every compact convex set in X is an intersection of balls in X. In [4], Giles, Gregory and Sims proved that X has property (I) if and only if the weak* denting points of the unit ball of X^* are dense in

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the unit sphere. In [14], Sersouri proved that X has property (CI) if and only if the cone of extreme points of X^* is dense in X^* for the topology of uniform convergence on compact sets of X, where the *cone of extreme points* of X^* is the cone generated by the set of extreme points of B_{X^*}.

PROPOSITION 9. *Suppose the measure space space (Ω, Σ, μ) is not purely atomic. Then the following statements are equivalent:*

(1) *The Banach space X is an Asplund space with property (I).*
(2) *The Lebesgue-Bochner function space $L^p(\mu, X)$ has property (I).*

PROOF: Suppose X is an Asplund space with property (I). Then $L^q(\mu, X^*)$ is the dual of $L^p(\mu, X)$ [1] and w^*-dent B_{X^*} is dense in S_{X^*} [4]. Let D be the set

$$\left\{\sum_{i=1}^{m} \lambda_i x_i^* \chi_{E_i} : x_i^* \in w^* - \text{dent } B_{X^*},\ E_i \in \Sigma,\ E_i \cap E_j = \emptyset \text{ for } i \neq j,\right.$$
$$\left.\lambda_i > 0 \text{ with } \sum_{i=1}^{m} \lambda_i^q \mu(E_i) = 1\right\}.$$

It follows that the set D is dense in the set of norm-one simple functions in $L^q(\mu, X^*)$ which is dense in $S_{L^q(\mu,X^*)}$. On the other hand, D is a subset of w^*-dent $B_{L^q(\mu,X^*)}$ [8]. Thus w^*-dent $B_{L^q(\mu,X^*)}$ is dense in $S_{L^q(\mu,X^*)}$. Therefore $L^p(\mu, X)$ has property (I) [4].

Conversely, suppose $L^p(\mu, X)$ has property (I). Then w^*-dent $B_{L^p(\mu,X)^*}$ is dense in $S_{L^p(\mu,X)^*}$ [4]. Since $B_{L^q(\mu,X^*)}$ is weak* dense and norm closed in $B_{L^p(\mu,X)^*}$, it contains the set w^*-dent $B_{L^p(\mu,X)^*}$. It follows that $L^q(\mu, X^*)$ is the dual of $L^p(\mu, X)$. Hence X is an Asplund space [1]. Now choose $E \in \Sigma$ such that $0 < \mu(E) < \infty$. Without loss of generality, we may assume that $\mu(E) = 1$. Let x^* be a unit vector in X^*. Then the function $x^*\chi_E$ is a unit vector in $L^q(\mu, X^*)$. Hence for each ε with $1 > \varepsilon > 0$, there is a weak* denting point f of $B_{L^q(\mu,X^*)}$ wuch that

$$\|x^*\chi_E - f\| < \varepsilon.$$

Let A be the subset of E given by

$$\{t : t \in E \text{ and } \|x^* - f(t)\| < \varepsilon\}.$$

Then $\mu(A) > 0$ and E is a subset of the support of f. Since $f(t)/\|f(t)\|$ is a weak* denting point of B_{X^*} for almost all t in the support of f [8], there is t in A such that $f(t)/\|f(t)\| \in w^*$-dent B_{X^*}. Then

$$\left\|\frac{f(t)}{\|f(t)\|} - x^*\right\| \leqslant \left\|\frac{f(t)}{\|f(t)\|} - f(t)\right\| + \|f(t) - x^*\| \leqslant |1 - \|f\|| + \varepsilon \leqslant \|f(t) - x^*\| + \varepsilon \leqslant 2\varepsilon.$$

Thus w^*-dent B_{X^*} is dense in S_{X^*}, and hence X has property (I) [4]. ∎

REMARK. If (Ω, Σ, μ) is purely atomic, then X has property (I) if and only if $L^p(\mu, X)$ has property (I). In fact if $\{X_i, i \in I\}$ is a family of Banach spaces, then $\ell_p(X_i)$ has property (I) if and only if each X_i has property (I).

PROPOSITION 10. *If the Banach space X has property (CI), then so does $L^p(\mu, X)$.*

PROOF: Let T be a unit vector in $L^p(\mu, X)^*$ and K be a norm-compact set in $L^p(\mu, X)$. Since $B_{L^q(\mu,X^*)}$ is bounded and weak* dense in $B_{L^p(\mu,X)^*}$, and since K is norm-compact, for each $\varepsilon > 0$ there exists f in $B_{L^q(\mu,X^*)}$ such that for all g in K

$$|(f, g) - T(g)| < \varepsilon.$$

We may assume that f is a non-zero simple function, say, $f = \sum_{i=1}^{n} x_i^* \chi_{E_i}$, where $x_i^* \in X^*$, $E_i \in \Sigma$ with $E_i \cap E_j = \emptyset$ for $i \neq j$. We may assume that $\mu(E_i) < M$ for some finite number M. Let K_i be the subset of X given by

$$K_i = \left\{ \int_{E_i} g(t) d\mu(t) \colon g \in K \right\}.$$

Then for each i, the set K_i is norm-compact. Since X has property (CI) [4], there exists an element y_i^* which is a scalar multiple of an extreme point of B_{X^*} such that for all x in K_i

$$|y_i^*(x) - x_i^*(x)| < \frac{\varepsilon}{nM}.$$

Define the function h by

$$h = \sum_{i=1}^{n} y_i^* \chi_{E_i}.$$

Then for every g in K,

$$|(h, g) - T(g)| \leqslant |(h, g) - (f, g)| + |(f, g) - T(g)| < 2\varepsilon.$$

Since $h(t)/\,\|h(t)\|$ is an extreme point of B_{X^*} for all t in the support of h, the function $h(\cdot)/\,\|h\|$ is an extreme point of $B_{L^p(\mu, X)^*}$ [10]. Thus h belongs to the cone of extreme points of $L^p(\mu, X)^*$. Therefore $L^p(\mu, X)$ has property (CI) [14] and the proof is complete. $\square$

REMARK. We do not know whether the converse of Proposition 10 is true or not.

NOTE ADDED IN PROOF. After the paper was accepted, it came to the attention of the authors that both Proposition 9 and Proposition 10 had been obtained jointly by P. Bandyopadhyay and A.K. Roy in their paper published in Indag. Math. **1** (1990). In that paper they even proved that if μ is non-atomic then $L^p(\mu, X)$ has property (CI) for any Banach space X.

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